LQG Control with Communication Constraints

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Introduction

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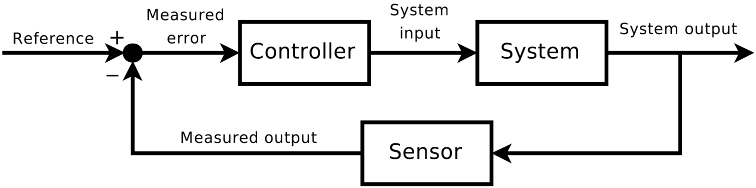

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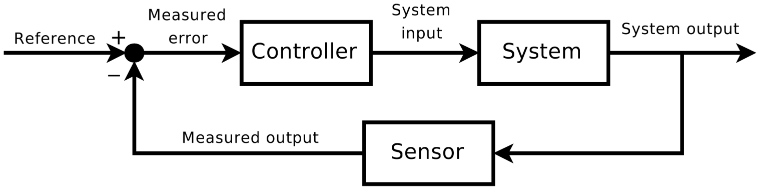

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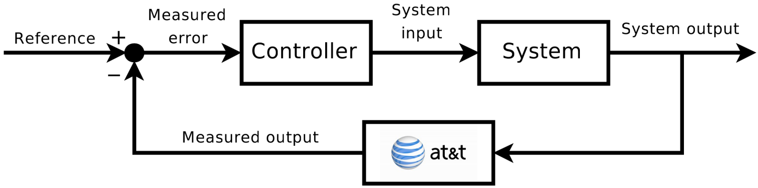

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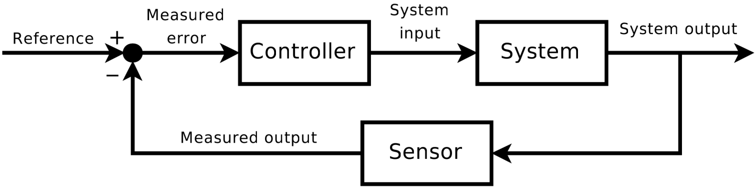

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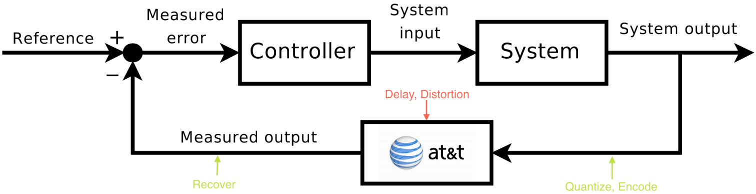

Linear Quadratic Gaussian Control

Linear: Dynamical System

$X_{k+1} = AX_k + Bu_k + v_k$

$X_k \in \mathbb{R}^d, u_k \in \mathbb{R}^m, A \in \mathbb{R}^{d\times d}, B \in \mathbb{R}^{m\times d}$

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Quadratic: Cost Criterion

Determine a control process $\{u_k\}$

$\arg\min_{\{u_k\}} \limsup_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n} E[X^T GX + u_k^T F u_k]$

$F \geq 0, G \geq 0; F, G \in \mathbb{R}^{d\times d}$

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Gaussian: Noise

$v_k \overset{\text{i.i.d}}{\sim} \mathcal{N}(0, Q)$

"Non-anticipative" noise

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- Caveat! We don’t encode and transmit the state/observation information, but rather, an innovations process, which is related to them
  - The innovations process is i.i.d. Gaussian, statistically independent of control- so can use a fixed, optimal vector quantizer.
  - This optimal quantizer has a centroid property- view quantized variable as a conditional expectation
  - Controller need not remember the past states.

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Delay, $N = \psi(M) \geq 1$

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where

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The sequence $\{\tilde{v}_k\}$ is termed the innovation sequence, and this is what we quantize, encode and transmit to the controller.

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- Quantization - Optimal Vector Quantizer
  - Maps $\tilde{v}_k \in A_i \to a_i$ in a particular way. Output is $\hat{v}_k$
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The Issue- Delay + Distortion

The controller only see all this distorted information at time $(k+1)N$. What's the optimal control?

“Separated” Control Problem

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So we can reformulate the controller's problem as finding $\{u_k\}$ that satisfies

$$\arg\min_{\{u_k\}} \limsup_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n} E[\hat{X}^T G \hat{X} + u_k^T F u_k]$$

Setting up the Control Problem

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This is like the LQG problem, except the statistics on noise. Solution also very similar.

Solving it

Theorem

The optimal (steady state) value of the controller $\{u_k\}$ for the separated control problem (and, it follows, for the original) is given by $u_k = \mu(\hat{X}_k), k \geq 0$ where

$$\mu(x) = -(B^T KB + R)^{-1} B^T KAx$$

K is the solution to the algebraic Ricatti equation

$$K = A^T (K - KB(B^T KB + GF)^{-1} B^T K)A + G$$

Also, with this controller, the cost as $n \to \infty$ is given by

$$\lambda(N) = \frac{1}{N}\text{tr}[(A^T)^N KA^N E]$$

Optimal Codelength

The total cost of the original cost function is given by

$$J(N) = \underbrace{\lambda(N)}_{\text{Cost for the new criterion}} + \underbrace{\frac{1}{N}\sum_{i=0}^{N-1} \mathrm{tr}[GA^i\bar{R}(A^T)^i] + \mathrm{tr}[GQ]}_{\text{Error due to variance}}$$

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The idea of the optimization is:
Shorter codes mean less delay, more distortion
Longer codes mean more resolution, but more delay
as we'd expect.

Partial Observations

- If we have partial observations, i.e., an observation procedure $Y_k = HX_k + \eta_k, k \geq 0,$ where $\eta_k \sim \mathcal{N}(0, S)$.

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- Nothing really changes- the solution becomes a modified Kalman filter. The results (costs, etc.) become modified versions of the corresponding results for the Kalman filter.

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- Other applications of analysis?