

Anytime Capacity of Stabilization of a Linear System over Noisy Channel

Graduate Seminar in Area I (6.454)
October 26, 2011

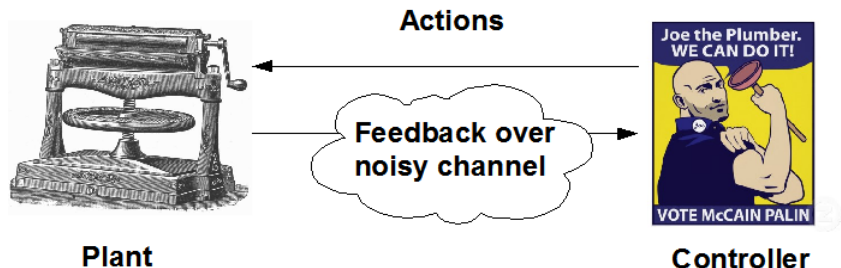
Outline

- 1 Introduction
- 2 A Counter Example
- 3 Necessity of Anytime Capacity
- 4 Conclusions

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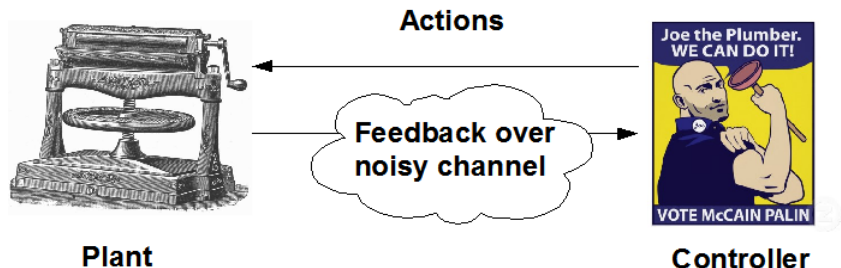
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Control and Communications



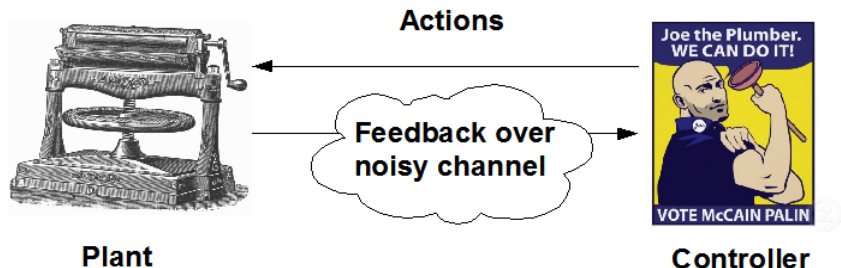
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- How much “information” do we need?
- What is the correct measure of “information”?

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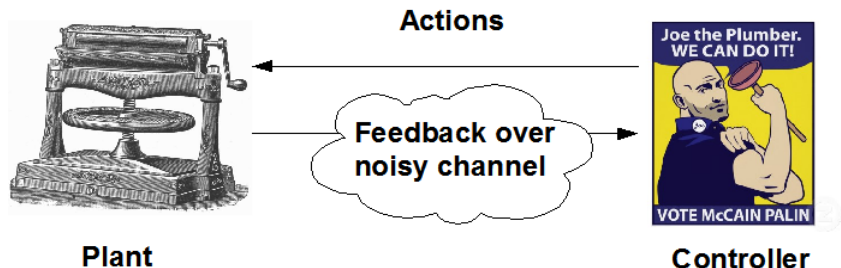
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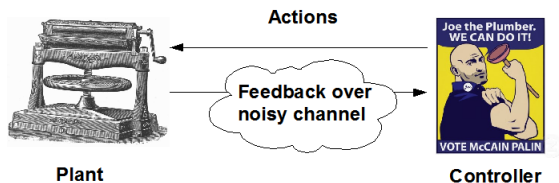
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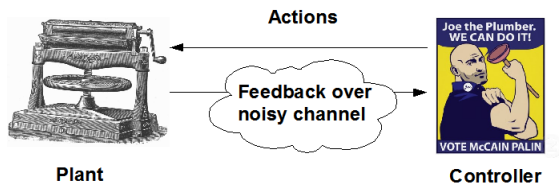
Control and Communications



Main insights

- How much “information” do we need?
 - ▶ No single answer. It depends on the degree of “stability” desirable.
- What is the correct measure of “information”?
 - ▶ *Shannon capacity* may not be adequate for stronger notions of stability. Need *anytime capacity*.

Control and Communications



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 - ▶ *Shannon capacity* may not be adequate for stronger notions of stability. Need *anytime capacity*.

Plan of This talk

- A simple example to illustrate that Shannon capacity is not strong enough for control applications.
 - ▶ In particular, a plant can be *unstable* even if the Shannon capacity of the channel is *infinite*.
- A necessary condition for stability in terms of anytime capacity.

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 - ▶ In particular, a plant can be *unstable* even if the Shannon capacity of the channel is *infinite*.
- A necessary condition for stability in terms of anytime capacity.

Main Reference

A. Sahai, S. K. Mitter, “The Necessity and Sufficiency of Anytime Capacity for Stabilization of a Linear System Over a Noisy Communication Link. Part I: Scalar Systems,” *IEEE Trans. Inform. Th.*, vol. 52, no. 8, pp. 3369-3395, Aug. 2006.

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The Control Problem

$$X_{t+1} = \lambda X_t + U_t + W_t, \quad t \in \mathbb{Z}^+.$$

- Time (discrete): $t \in \mathbb{Z}^+$.
- State: $X_t \in \mathbb{R}$.
- control: $U_t \in \mathbb{R}$.
- *Bounded* disturbance: $|W_t| < \frac{\Omega}{2}$, with probability 1.

To make things interesting:

- *unstable* gain: $\lambda > 1$.

The Control Problem

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- **Goal:** choose good U_t to keep X_t “small”.
- If feedback is perfect, simply set $U_t = -\lambda X_t$.
- What if feedback is sent through a noisy channel?

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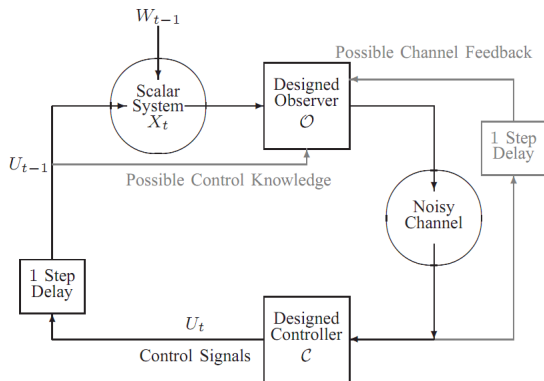
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Definition of Stability



- **Observer \mathcal{O}** : sees X_t and generates channel input a_t .
- **Controller \mathcal{C}** : observes channel output B_t and generates control signal U_t .

The Control Problem

Definition: η -stability

A closed-loop system is η -stable if there exists $K < \infty$, such that

$$\mathbb{E} [|X_t|^\eta] < K$$

for all $t \geq 0$.

(More general notions of stability can be defined, but we will focus on η -stability for now.)

Counter Example in Real-Erasure Channel

When is Shannon capacity not sufficient in describing communications in control systems?

Real Erasure Channel (REC)

The *real packet erasure channel* has

- Input alphabet: $\mathcal{A} = \mathbb{R}$.
- Output alphabet: $\mathcal{B} = \mathbb{R}$.
- Transition probabilities

$$p(x|x) = 1 - \delta,$$

$$p(0|x) = \delta.$$

I.e., a symbol is either received *perfectly*, or received as *zero*.

Counter Example in Real-Erasure Channel

- What is the Shannon capacity of the channel?
- It is *infinite*, because a real number can carry as many bits as we want.

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Counter Example in Real-Erasure Channel

- What is the optimal communication / control policy?
- **Communication:** set

$$a_t = X_t.$$

- **Control:** set

$$U_t = -\lambda B_t.$$

- **Resulting dynamics:** X_t is reset to 0 every $Geo(\delta)$ steps.

Counter Example in Real-Erasure Channel

- Is the system η -stable under optimal control?
- It is 1-stable,

$$\mathbb{E}[|X_t|] = \left(\frac{3}{2}\right) \left(\frac{1}{2}\right)^t < 1,$$

for all t .

- However, it is *not* η -stable, for $\eta \geq 2$,

$$\mathbb{E}[|X_t|^2] > \frac{4\sigma^2}{5} \sum_{i=0}^t \left(\left(\frac{9}{8}\right)^{i+1} - \left(\frac{1}{2}\right)^{i+1} \right)$$

which diverges as $t \rightarrow \infty$.

Counter Example in Real-Erasure Channel

Lesson learned: notion of information depends on the strength of stability required (e.g., values of η).

- Why was Shannon capacity insufficient?
- Need good information about the system state at *all times*, not just the end of a large block.
- Fix: define a stronger notion of capacity to guarantee good estimation of system state at any point in time (“anytime capacity”).

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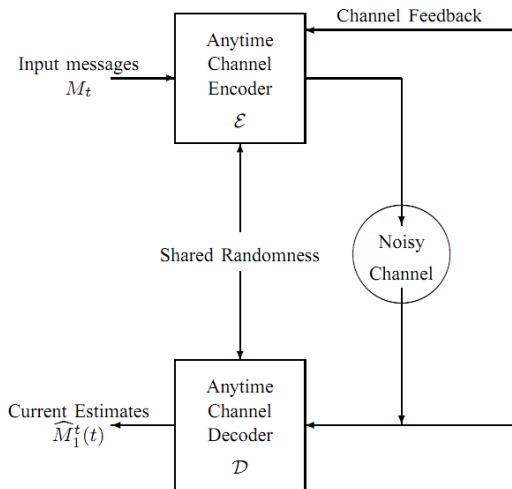
Anytime Reliability and Capacity

Communication System

A rate R communication system is

- Encoder receives R -bit message M_t in slot t . (details on whiteboard)
- Encoder produces channel input based on all past messages and possible feedback $B_1^{t-1-\theta}$ (with delay $1 + \theta$).
- Decoder updates estimates of *all* past messages, $\hat{M}_i(t)$, for all $i \leq t$, based on *all* channel outputs till time t .

Anytime Reliability and Capacity



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Anytime Reliability

A rate R communication system achieves *anytime reliability* α if there exists constant K such that

$$\mathbb{P}\left(\hat{M}_1^i(t) \neq M_1^i\right) \leq K2^{-\alpha(t-i)}.$$

The system is *uniformly anytime reliable* if the above holds for all messages M .

- Comparing to Shannon reliability? Block versus sequential?
- Exercise: fix t or i and vary the other.

Anytime Reliability and Capacity

α -anytime Capacity

$C_{any}(\alpha)$ of a channel is the highest rate R , at which the channel can achieve uniform anytime reliability α .

More stringent than Shannon capacity, C :

$$C_{any}(\alpha) \leq C,$$

for any $\alpha > 0$.

Necessity of Anytime Capacity

Theorem: Necessity of Anytime Capacity

If there exists an observer / controller pair that achieves η -stability under bounded disturbance, then the channel's *feedback anytime capacity* satisfies

$$C_{any}(\eta \log_2 \lambda) \geq \log_2 \lambda,$$

Necessity of Anytime Capacity: Proof

Use the control system as a **black box** to construct a communication system with good anytime reliability. (sketch on white board)

- 1 Encoder sits with the plant; decoder with the controller.
- 2 Encode messages in the *disturbance*, W_t .
- 3 Controller must somehow know the disturbances, otherwise there is no way to stabilize the plant.
- 4 Decoder then reads off the *control actions* chosen by the controller to decode message.

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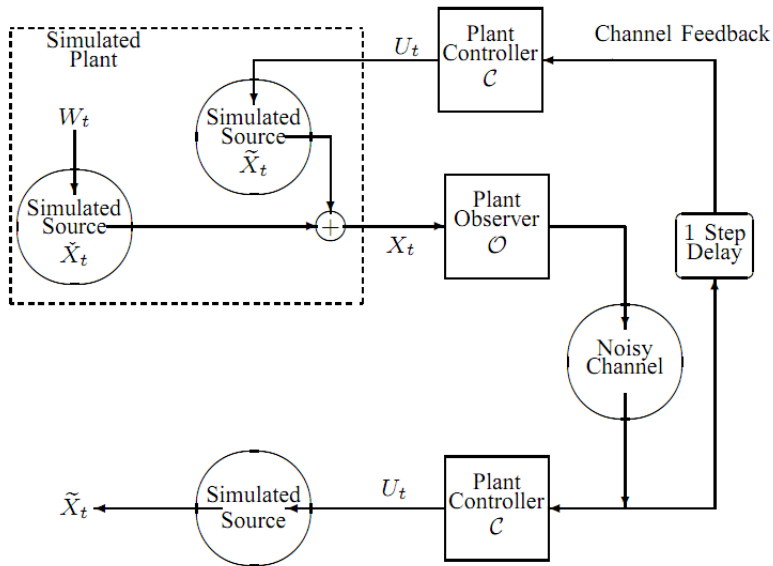
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- But what do you mean by “knowing the disturbances”?
- Alright, let's be more concrete here. Write

$$X_t = Y_t + Z_t,$$

such that

$$X_0 = Y_0 = Z_0 = 0.$$

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Necessity of Anytime Capacity: Proof

- Y_t is the **control branch**

$$Y_{t+1} = \lambda Y_t + W_t.$$

- Z_t is the **disturbance branch**

$$Z_{t+1} = \lambda Z_t + U_t.$$

- Can easily verify by recursion

$$X_t = Y_t + Z_t.$$

Necessity of Anytime Capacity: Proof

- Key idea: encoder can control W_t (hence Y_t), while decoder knows Z_t perfectly.
- The plant is η -stable, so $|X_t|$ must be *small at all times*.
- Therefore, we must have

$$Y_t \approx -Z_t.$$

- Voila! Decoder should be able to extract good information of W_t by looking at Z_t .

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Part 1: Encoding

- Now, down to business: step 1, encoding.
- Let each message M_t be a collection of R bits.
- Let $S_i \in \{-1, 1\}$ be the i th bit in the system.
- Write

$$\begin{aligned} Y_t &= \lambda Y_{t-1} + W_{t-1} \\ &= \lambda^{t-1} \sum_{j=0}^{t-1} \lambda^{-j} W_j. \end{aligned}$$

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- Encoding: choose W_t to be the value of the fractional representation of $\{S_i\}$, $\lfloor Rt \rfloor + 1 \leq i \leq \lfloor R(t+1) \rfloor$,
- In particular, set

$$W_t = \gamma \lambda^{t+1} \sum_{k=\lfloor Rt \rfloor + 1}^{\lfloor R(t+1) \rfloor} (2 + \epsilon_1)^{-k} S_k.$$

- Need the right constants to make things work

$$\epsilon_1 = 2^{\frac{\log_2 \lambda}{R}} - 2,$$

$$\gamma = \frac{\Omega}{2\lambda^{1+\frac{1}{R}}}.$$

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Part 2: Decoding

- Sent...fingers crossed...how much separation did we get?
- Main technical lemma:

Technical Lemma

Let $\hat{S}_i(t)$ be the estimate of bit S_i at time t . For all $0 \leq j \leq t$,

$$\left\{ \omega \mid \exists i \leq j, \hat{S}_i(t) \neq S_i(t) \right\} \subset \left\{ \omega \mid |X_t| \geq \lambda^{t-\frac{j}{R}} \left(\frac{\gamma \epsilon_1}{1 + \epsilon_1} \right) \right\}$$

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- Intuition: if early disturbances (S_t) were guessed incorrectly, control will blow up exponentially fast!
- Hence small $|X_t|$ must imply good estimates of early disturbances.

Part 2: Decoding

Proof:

- If two messages differ in the first bit, S_1 , how much will they differ on resulting Y_t ?

$$\begin{aligned} & \inf_{\bar{S}: \bar{S}_1 \neq S_1} |Y_1(S) - Y_1(\bar{S})| \\ & \geq \gamma \lambda^t \left(\left(1 - \sum_{k=1}^{\lfloor Rt \rfloor} (2 + \epsilon_1)^{-k} \right) - \left(-1 - \sum_{k=1}^{\lfloor Rt \rfloor} (2 + \epsilon_1)^{-k} \right) \right) \\ & > \gamma \lambda^t 2 \left(1 - \sum_{k=1}^{\lfloor Rt \rfloor} (2 + \epsilon_1)^{-k} \right) \\ & = \lambda^t \left(\frac{2\epsilon_1 \gamma}{1 + \epsilon_1} \right) \end{aligned}$$

- Note the exponential dependence on λ (will force controller to report good estimates).

Part 2: Decoding

- More generally,

$$\inf_{\bar{S}:\bar{S}_i \neq S_i} |Y_t(S) - Y_t(\bar{S})| > \lambda^{t-\frac{i}{R}} \left(\frac{2\epsilon_1\gamma}{1+\epsilon_1} \right),$$

if $i \leq \lfloor Rt \rfloor$.

- We will decode to get the \hat{S}_i by pretending that $-Z_t$ is Y_t .
- Complete the proof of Lemma by noting that

$$|Z_t(S) - Z_t(\bar{S})| \geq |Y_t(S) - Y_t(\bar{S})| - |X_t(S) - X_t(\bar{S})|.$$

- In other words, smallness of X_t guarantees the closeness of $-Z_t$ and Y_t .

Part 3: Probability of Error

$$\begin{aligned}\mathbb{P}(|X_t| > m) &= \mathbb{P}(|X_t|^\eta > m^\eta) \\ &\leq \mathbb{E}(|X_t|^\eta) m^{-\eta} \\ &< K m^{-\eta} \text{ (definition of } \eta\text{-stability).}\end{aligned}$$

Combine this with the Technical Lemma

$$\begin{aligned}\mathbb{P}\left(\hat{S}_1^i(t) \neq S_1^i(t)\right) &\leq \mathbb{P}\left(|X_t| > \lambda^{t-\frac{j}{R}} \left(\frac{\gamma\epsilon_1}{1+\epsilon_1}\right)\right) \\ &< \left(K \left(\frac{1}{\gamma} + \frac{1}{\gamma\epsilon_1}\right)^\eta\right) 2^{-(\eta \log_2 \lambda)(t-\frac{j}{R})}.\end{aligned}$$

This proves the theorem.

Review: we just proved...

Theorem: Necessity of Anytime Capacity

If there exists an observer / controller pair that achieves η -stability under bounded disturbance, then the channel's *feedback anytime capacity* satisfies

$$C_{any}(\eta \log_2 \lambda) \geq \log_2 \lambda.$$

- Sufficient conditions for anytime reliability in stabilizing a plant is in the paper, but will not be covered here.
- Is the necessary condition tight? Are there simpler ways to interpret / proof this result?

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Conclusions

Concluding Remarks

- Thinking about the required information rate for particular application: may need different (or stronger) notion of capacity / reliability.
- Exponentially unstable nature of linear control system underlies the higher information barrier.
- Put in an adversarial way, *instability* and *noise* are both our enemies in communications. Any other major adversaries that we should consider?