LOWER BOUND FOR THE REDUNDANCY OF SELF-CORRECTING ARRANGEMENTS OF UNRELIABLE FUNCTIONAL ELEMENTS

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Arrangement of unreliable functional elements are considered. It is assumed that all the elements misfunction independently of one another with probability ε. The redundancy of a self-correcting arrangement that realizes some function is understood to mean the ratio of the number of elements (complexity) of a self-correcting arrangement of unreliable elements to the complexity of the arrangement of reliable elements that realizes the same function. It is shown that, for some functions, the redundancy of the self-correcting arrangements that realize them increases no more slowly than the logarithm of the complexity of the reliable-element arrangement.

1. INTRODUCTION

In this paper we will examine the redundancy of self-correcting arrangements of unreliable functional elements that realize Boolean functions. It is assumed that all the elements misfunction (issue incorrect results) independently of one another and of the values of the incoming signals, with probability ε. This problem was first taken up by von Neumann in his famous study [1], where he showed that for any prototype arrangement of reliable elements one can construct a self-correcting arrangement of unreliable elements that realizes the same Boolean function as the prototype. In a subsequent paper, the authors will demonstrate that, by somewhat refining von Neumann's construct, it can be shown that the redundancy of a self-correcting arrangement increases no more rapidly than the logarithm of the number of elements of the prototype. (It was von Neumann himself who proved this bound for arrangements with a multiply redundant output.) The following question then arises: Is this increase in redundancy of self-correcting arrangements necessary? This paper will offer a partial answer to the question. Specifically we will show that, at least for some Boolean functions, a logarithmic increase in the redundancy of the self-correcting arrangements that realize them is in fact necessary.

2. BASIC DEFINITIONS AND FORMULATION OF THE RESULT

Consider a directed graph G. Assume that n_a is the number of all edges going to vertex a of G. Assume that $b_1(a), \ldots, b_{n_a}(a)$ are edges going from vertex a from vertices $a_1(a), \ldots, a_{n_a}(a)$, respectively. Vertices $a_1, a_2, \ldots, a_n$ at which no edges arrive are called inputs of G. Assume we are given some set of vertices $\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_k$ from which no edges originate. We will call these vertices the outputs of G. Assume that A is the set of all vertices of G, while A_φ is the set of all vertices of G that are not inputs. Assume that Φ is some complete system of Boolean functions.

Definition 2.1. An arrangement S of functional elements (or simply an arrangement) in basis Φ is a finite directed graph G without loops, with fixed numbering of the inputs and outputs and of the edges leading to each vertex, such that each vertex $a \in A_\varphi$ is associated with a Boolean function $\varphi_a(x_1, \ldots, x_{n_a}) \in \Phi$, while each edge $b_\rho(a)$, $\rho = 1, \ldots, n_a$, leading to vertex a is associated with the argument x_ρ of this Boolean function.

Each vertex $a \in A_\varphi$ will be called a functional element that realizes Boolean function φ_a. The number of inputs of a functional element is the number of edges that lead to vertex a, i.e., the number n_a. The quantity L(S) – the number of functional elements of S – is the complexity of S.

Assume that X^n is the set of all n-component binary vectors $\mathbf{x} = (x_1, \ldots, x_n)$, $x_i \in \{0, 1\}$, $i = 1, 2, \ldots, n$. We will say that binary vector $\mathbf{z} = (z_1, z_2, \ldots, z_n)$ is the mod 2 sum of binary vectors $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and

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$y = (y_1, y_2, \ldots, y_n)$, and we will write $z = x \oplus y$, if $z_i = x_i \oplus y_i$, $i = 1, 2, \ldots, n$, where $\oplus$ will here and henceforth denote mod 2 addition.

We fix a number $\varepsilon \in [0, 1/2)$. Assume that $\eta(a, \varepsilon)$, $a \in A_\varphi$ are independent random variables that assume values 0 and 1, such that $\Pr\{\eta(a, \varepsilon) = 1\} = \varepsilon$.

Definition 2.2. The state of arrangement S corresponding to input vector $\mathbf{x} \in X^n$, for a probability of malfunction equal to ε, will be understood to be a system of random variables $\xi(a, \mathbf{x}, \varepsilon)$, $a \in A$, such that

1) $\xi(a_i, \mathbf{x}, \varepsilon) \equiv x_i$, $i = 1, 2, \ldots, n$,

2) $\xi(a, \mathbf{x}, \varepsilon) = \varphi_a(\xi(a, \mathbf{x}, \varepsilon)) \oplus \eta(a, \varepsilon)$, $a \in A_\varphi$, where $\xi(a, \mathbf{x}, \varepsilon) = (\xi(a_1(a), \mathbf{x}, \varepsilon), \ldots, \xi(a_{n_a}(a), \mathbf{x}, \varepsilon))$.

Note that since S is a directed graph without loops, Definition 2.2 uniquely specifies the joint distribution of all random variables $\xi(a, \mathbf{x}, \varepsilon)$.

A Boolean k-dimensional vector-valued function of n variables $\mathbf{f}(x_1, x_2, \ldots, x_n)$ is a k-component vector whose components are Boolean functions of n variables, i.e.,

$$\mathbf{f}(x_1, x_2, \ldots, x_n) = (f_1(x_1, \ldots, x_n), \ldots, f_k(x_1, \ldots, x_n)), \tag{2.1}$$

where $f_j(x_1, \ldots, x_n)$, $j = 1, \ldots, k$, are Boolean functions of n variables.

Definition 2.3. We will say that arrangement S realizes k-dimensional Boolean vector-valued function $\mathbf{f}(\mathbf{x})$, $\mathbf{x} \in X^n$, with error probability $P(S, \mathbf{f})$ for a malfunction probability equal to ε, if

$$P(S, \mathbf{f}) = \max_{\mathbf{x} \in X^n} \left\{ \frac{1}{k} \sum_{j=1}^{k} \Pr\{\xi(\tilde{a}_j, \mathbf{x}, \varepsilon) \neq f_j(\mathbf{x})\} \right\}. \tag{2.2}$$

Assume that $L_{p,\varepsilon}(\mathbf{f}, \Phi)$ is the minimum complexity of an arrangement in basis Φ that realizes Boolean vector-valued function $\mathbf{f}$ with error probability not greater than p, for a malfunction probability of ε. Let $L(\mathbf{f}, \Phi) = L_{0,0}(\mathbf{f}, \Phi)$; i.e., $L(\mathbf{f}, \Phi)$ is the minimum complexity of an arrangement of unreliable functional elements in basis Φ that realizes Boolean vector-valued function $\mathbf{f}$. Let

$$R_{p,\varepsilon}(\mathbf{f}, \Phi) = L_{p,\varepsilon}(\mathbf{f}, \Phi) / L(\mathbf{f}, \Phi). \tag{2.3}$$

The quantity $R_{p,\varepsilon}(\mathbf{f}, \Phi)$ will be called the redundancy of the minimum arrangement of unreliable elements in basis Φ that realizes Boolean vector-valued function $\mathbf{f}$ with error probability not greater than p, for a malfunction probability of ε. We introduce the quantity

$$R_{p,\varepsilon}(N, \Phi) = \max_{\mathbf{f}} R_{p,\varepsilon}(\mathbf{f}, \Phi), \tag{2.4}$$

where the maximum is taken over all Boolean vector-valued functions $\mathbf{f}$ such that $L(\mathbf{f}, \Phi) = N$. The basic result of this study is the following theorem.

THEOREM 2.1. Let $p \in (0, 1/3)$, $\varepsilon \in (0, 1/2)$. Then

$$R_{p,\varepsilon}(N) \geqslant \frac{\ln\{((N/C(\Phi))-1)(n(\Phi)-1)(1-3p)p^{-1}\}}{C(\Phi)\ln\{n(\Phi)/\varepsilon\}} - \frac{1}{N}\left(\frac{\ln\{((N/C(\Phi))-1)(n(\Phi)-1)(1-3p)p^{-1}\}}{\ln\{n(\Phi)/\varepsilon\}} + \frac{1}{n(\Phi)-1}\right), \tag{2.5}$$

where $n(\Phi) = \max_{\varphi \in \Phi} n(\varphi)$, while $n(\varphi)$ is the number of variables of Boolean function φ; $C(\Phi) = L(s, \Phi)$, where Boolean function $s = s(x_1, \ldots, x_{n(\Phi)}) = x_1 \oplus \ldots \oplus x_{n(\Phi)}$; i.e., $C(\Phi)$ is the minimum number of reliable functional elements needed to realize a binary adder of $n(\Phi)$ numbers using an arrangement in basis Φ.

It can be seen from (2.5) that as $N \to \infty$ we have

$$R_{p,\varepsilon}(N) \geqslant \frac{\ln N}{C(\Phi)\ln\{n(\Phi)/\varepsilon\}}. \tag{2.6}$$

3. DERIVATION OF LOWER BOUND FOR REDUNDANCY

The state of the arrangement as introduced above corresponds to the notion that errors in functional elements occur only at their outputs. In deriving a lower bound for the redundancy, it is convenient to imagine that errors in unreliable elements occur not only at the outputs but at the inputs as well. Below we will introduce a notion of state that corresponds to this view of unreliable elements; we will do so in such a way as to leave the error probability of the arrangement unaffected.

Consider an arbitrary arrangement S in basis Φ. Assume that B is the set of all edges of S. Let ζ_b, $b \in B$ and $\nu_a(\tau)$, $a \in A_\varphi$, $\tau \in X^{n_a}$ be independent random variables that assume values 0 and 1, where $\Pr\{\zeta_b = 1\} = \delta$, while $\Pr\{\nu_a(\tau) = 1\} = P_a(\tau, \delta)$, where $\delta \in [0, \varepsilon/n(\Phi)]$, while the probabilities $P_a(\tau, \delta)$ are chosen in such a way that for any $a \in A_\varphi$ and $t \in X^{n_a}$

$$\Pr\{\varphi_a(t \oplus \zeta_a) \oplus \nu_a(t \oplus \zeta_a) \neq \varphi_a(t)\} = \varepsilon, \tag{3.1}$$

where $\zeta_a = (\zeta_{b_1(a)}, \ldots, \zeta_{b_{n_a}(a)})$.

LEMMA 3.1. Let $\varepsilon \in (0, 1/2)$, $\delta \in [0, \varepsilon/n(\Phi)]$. Then for any vertex $a \in A_\varphi$ there exist unique values $P_a(\tau, \delta)$, $\tau \in X^{n_a}$, that satisfy (3.1) and are such that $P_a(\tau, \delta) \in [0, 1]$.

Proof. In the proof we will assume that vertex a is fixed and we will omit the subscript a in the quantities introduced above. Applying the total-probability formula to (3.1) and using the fact that $\nu(\tau)$ and ζ_b are independent random variables, we can rewrite (3.1) in the form

$$\sum_{\tau \in X^n} \Pr\{\nu(\tau) \oplus \varphi(\tau) \oplus \varphi(t) = 1\} \Pr\{\zeta = t \oplus \tau\} = \varepsilon, \; t \in X^n. \tag{3.2}$$

Let

$$c(t, \tau, \delta) = (-1)^{\varphi(\tau) \oplus \varphi(t)} \Pr\{\zeta = t \oplus \tau\} = (-1)^{\varphi(\tau) \oplus \varphi(t)} \delta^{w(t \oplus \tau)} (1-\delta)^{n - w(t \oplus \tau)}, \tag{3.3}$$

where $w(t)$, $t \in X^n$ is the number of 1's of binary vector t. Let us rewrite (3.2) allowing for (3.3):

$$\sum_{\tau \in X^n} P(\tau, \delta) c(t, \tau, \delta) = \varepsilon + \sum_{\substack{\tau \in X^n \\ \tau: \varphi(\tau) \neq \varphi(t)}} c(t, \tau, \delta), \; t \in X^n. \tag{3.4}$$

Thus to determine $P(\tau, \delta)$, $\tau \in X^n$ we have a system of 2^n linear equations with 2^n unknowns.

Now we will prove two assertions regarding the coefficients of this system. Specifically, for all $t \in X^n$ and $\delta \in [0, \varepsilon/n(\Phi)]$ we have

$$\sum_{\tau \in X^n, \, \tau \neq t} |c(t, \tau, \delta)| < \varepsilon \tag{3.5}$$

and

$$|c(t, t, \delta)| > 1 - \varepsilon. \tag{3.6}$$

Indeed, note that in accordance with (3.3)

$$\sum_{\tau \in X^n} |c(t, \tau, \delta)| = 1. \tag{3.7}$$

Then, allowing for (3.3), we have

$$\sum_{\tau \in X^n, \, \tau \neq t} |c(t, \tau, \delta)| = 1 - |c(t, t, \delta)| = 1 - (1-\delta)^n \leqslant 1 - (1 - (\varepsilon/n(\Phi)))^{n(\Phi)} < \varepsilon.$$

This expression and (3.7) yield (3.6).

We have from (3.5) and (3.6) that the Hadamard condition [2] (see Sec. 14.1) holds for the matrix of coefficients of system (3.4). Specifically, for all $t \in X^n$

$$|c(t, t, \delta)| > \sum_{\tau \in X^n, \, \tau \neq t} |c(t, \tau, \delta)|. \tag{3.8}$$

When this condition holds, the determinant of the system is nonzero. Consequently, there exists a unique group of values $P(\tau, \delta)$, $\tau \in X^n$, that satisfies (3.4).

Since $c(t, \tau, 0) = 0$ for $t \neq \tau$ and $c(t, t, 0) = 1$, we have $P(\tau, 0) = \varepsilon$, $\tau \in X^n$. We will show that for $\delta \in [0, \varepsilon/n(\Phi)]$ all the $P(\tau, \delta) \in (0, 1)$. Assume that this is not the case. Then, since $P(\tau, 0) = \varepsilon \in [0, 1]$ and $P(\tau, \delta)$ are continuous functions of δ for $\delta \in [0, \varepsilon/n(\Phi)]$, there exist a $\tau' \in X^n$ and $\delta' \in [0, \varepsilon/n(\Phi)]$ such that either $P(\tau', \delta'] = 0$ and $P(\tau, \delta') \in [0, 1]$, $\tau \in X^n$, or $P(\tau', \delta') = 1$ and $P(\tau, \delta') \in [0, 1]$, $\tau \in X^n$.

Consider the first case. Setting $t = \tau'$ and $\delta = \delta'$, we obtain from (3.4)

$$P(\tau', \delta') c(\tau', \tau', \delta') = \varepsilon + \sum_{\tau \in X^n, \, \tau: \varphi(\tau) \neq \varphi(\tau')} (1 - P(\tau, \delta')) c(\tau', \tau, \delta') - \sum_{\tau \in X^n, \, \tau: \varphi(\tau) = \varphi(\tau'), \, \tau \neq \tau'} P(\tau, \delta') c(\tau', \tau, \delta'). \tag{3.9}$$

Since $P(\tau, \delta') \in [0, 1]$, we have from (3.9) and (3.5) that

$$P(\tau', \delta'), c(\tau', \tau', \delta') \geqslant \varepsilon - \sum_{\tau \in X^n, \tau \neq \tau'} |c(\tau', \tau, \delta')| > 0.$$

Since $c(\tau', \tau', \delta') > 0$, we obtain a contradiction with the fact that $P(\tau', \delta') = 0$.

In the second case, allowing for the fact that $P(\tau, \delta') \in [0, 1]$, $c(\tau', \tau, \delta') \geq 0$ for $\varphi(\tau') = \varphi(\tau)$ and $c(\tau', \tau, \delta') \leq 0$ for $\varphi(\tau') \neq \varphi(\tau)$, we obtain from (3.9) that

$$P(\tau', \delta')c(\tau', \tau', \delta') \leqslant \varepsilon. \quad (3.10)$$

Since $c(\tau', \tau', \delta') > 1 - \varepsilon$ and $\varepsilon \in (0, 1/2)$, we have from (3.10) that $P(\tau', \delta') < \varepsilon/(1 - \varepsilon) < 1$. The resultant contradiction demonstrates that $P(\tau, \delta) \in (0, 1)$, $\tau \in X^n$ for $\delta \in [0, \varepsilon/n(\Phi)]$, and the lemma is thus proved.

Definition 3.1. A θ-state of arrangement S corresponding to input vector $\mathbf{x} \in X^n$, for a malfunction probability of ε, is a system of random variables $\theta(a, \mathbf{x}, \varepsilon)$, $a \in A$ such that

1) $\theta(a_i, \mathbf{x}, \varepsilon) \equiv x_i$, $i = 1, 2, \ldots, n$,

2) $\theta(a, \mathbf{x}, \varepsilon) = \varphi_a(\theta(a, \mathbf{x}, \varepsilon) \oplus \zeta_a) \oplus \nu_a(\theta(a, \mathbf{x}, \varepsilon) \oplus \zeta_a)$, $a \equiv A_{\varphi}$, $\theta(a, \mathbf{x}, \varepsilon) = (\theta(a_1(a), \mathbf{x}, \varepsilon), \ldots, \theta(a_{n_a}(a), \mathbf{x}, \varepsilon))$.
x

Thus a θ-state corresponds to the following conception of how unreliable elements operate. Signals arriving at the element inputs are distorted independently of one another with equal probability δ. Signals at element outputs are distorted with a probability that depends on the input-signal distortions in such a way that the probability that an element will issue an incorrect result is exactly equal to ε. The following lemma demonstrates that "state" and "θ-state" are equivalent in the sense that they yield an equal value for the error probability of the arrangement.

LEMMA 3.2. For any arrangement S in an arbitrary finite basis Φ and any $\mathbf{x} \in X^n$, $\varepsilon \in (0, 1/2)$, and vertex $a \in A$ we have

$$\Pr\{\theta(a, \mathbf{x}, \varepsilon) = 1\} = \Pr\{\xi(a, \mathbf{x}, \varepsilon) = 1\}. \quad (3.11)$$

Proof. We introduce the depth $\Gamma(a)$ of vertex a of S, this understood to mean the maximum path length from the inputs of the arrangement to vertex a. Assume that $a^1, a^2, \ldots, a^{L(S)+n}$ is the set of all vertices of S, numbered in some fashion in order of nondecreasing depth. Let $\xi_r = \xi(a^r, \mathbf{x}, \varepsilon)$, $\theta_r = \theta(a^r, \mathbf{x}, \varepsilon)$, $\xi_r = (\xi_1, \xi_2, \ldots, \xi_r)$, while $\theta_r = (\theta_1, \theta_2, \ldots, \theta_r)$, $r = 1, 2, \ldots, L(S) + n$. For an arbitrary binary vector $\mathbf{z} = (z_1, z_2, \ldots, z_{L(S)+n})$, assume that we have $\mathbf{z}^r = (z_1, z_2, \ldots, z_r)$.

To prove the lemma, it suffices to show that the probability distributions of the vector random variables $\xi_{L(S)+n}$ and $\theta_{L(S)+n}$ coincide, i.e., that for any $\mathbf{z} \in X^{L(S)+n}$

$$\Pr\{\xi_{L(S)+n} = \mathbf{z}\} = \Pr\{\theta_{L(S)+n} = \mathbf{z}\}. \quad (3.12)$$

Consider

$$\Pr\{\xi_{L(S)+n} = \mathbf{z}\} = \Pr\{\xi_n = \mathbf{z}^n\} \Pr\{\xi_{n+1} = z_{n+1} \mid \xi_n = \mathbf{z}^n\} \times \Pr\{\xi_{n+2} = z_{n+2} \mid \xi_{n+1} = \mathbf{z}^{n+1}\} \ldots \Pr\{\xi_{L(S)+n} = z_{L(S)+n} \mid \xi_{L(S)+n-1} = \mathbf{z}^{L(S)+n-1}\} \quad (3.13)$$

and

$$\Pr\{\theta_{L(S)+n} = \mathbf{z}\} = \Pr\{\theta_n = \mathbf{z}^n\} \Pr\{\theta_{n+1} = z_{n+1} \mid \theta_n = \mathbf{z}^n\} \times \ldots \times \Pr\{\theta_{L(S)+n} = z_{L(S)+n} \mid \theta_{L(S)+n-1} = \mathbf{z}^{L(S)+n-1}\}. \quad (3.14)$$

Since vertices $a^1, a^2, \ldots, a^n$ are inputs of S, it follows immediately from the definition of state and θ-state that

$$\Pr\{\xi_n = \mathbf{z}^n\} = \Pr\{\theta_n = \mathbf{z}^n\}. \quad (3.15)$$

Thus, in accordance with (3.13)-(3.15), to prove (3.12) it suffices to show that for any $r = n + 1, \ldots, n + L(S)$ we have

$$\Pr\{\xi_r = z_r \mid \xi_{r-1} = \mathbf{z}^{r-1}\} = \Pr\{\theta_r = z_r \mid \theta_{r-1} = \mathbf{z}^{r-1}\}. \quad (3.16)$$

Assume that edges from vertices $a^{r_\rho} = a_\rho(a^r)$, $\rho = 1, 2, \ldots, n_a$, arrive at vertex a^r. Since $\Gamma(a^r) \leq \Gamma(a^r) - 1$, we have $r_\rho < r$, $\rho = 1, 2, \ldots, n_a$, and the definitions of state and θ-state yield that

$$\Pr\{\xi_r = z_r \mid \xi_{r-1} = \mathbf{z}^{r-1}\} = \Pr\{\theta_r = z_r \mid \theta_{r-1} = \mathbf{z}^{r-1}\} = \begin{cases} \varepsilon, & \text{if} \quad z_r \neq \varphi_{a^r}(z_{r_1}, \ldots, z_{r_{n_a}}), \\ 1-\varepsilon, & \text{if} \quad z_r = \varphi_{a^r}(z_{r_1}, \ldots, z_{r_{n_a}}). \end{cases} \quad (3.17)$$

Thus we have proved (3.12) and hence the lemma as well.

For what follows we will require the following auxiliary lemma.

LEMMA 3.3. Let $p \in (0, 1/3)$, $\delta \in (0, 1/2)$. Assume that Q is an arbitrary set of natural numbers for which the number of elements $|Q| < \infty$, while m_l, $l \in Q$, are nonnegative integers. Assume that H_l, $l \in Q$, are independent events such that

$$\Pr\{H_l\} \geqslant \exp\{-m_l \ln(1/\delta)\}. \tag{3.18}$$

$$p \geqslant (1-p)\Pr\{\widetilde{\bigcup_{l\in Q}} H_l\}, \tag{3.19}$$

where event $\{\widetilde{\bigcup_{l\in Q}} H_l\}$ consists in that exactly one of the events H_l, $l \in Q$, has occurred; then

$$\sum_{l\in Q} m_l \geqslant \frac{|Q|}{\ln(1/\delta)} \ln\left\{\frac{|Q|(1-3p)}{p}\right\}. \tag{3.20}$$

Proof. First, by induction on the cardinality of set Q, we can show that for any $\gamma \in (0, 1/2)$ the fact that $\Pr\{\widetilde{\bigcup_{l\in Q}} H_l\} \leqslant \gamma$ implies that

$$\Pr\{\widetilde{\bigcup_{l\in Q}} H_l\} \geqslant (1-2\gamma)\sum_{l\in Q}\Pr\{H_l\}. \tag{3.21}$$

If $|Q| = 1$, inequality (3.21) holds.

Assume that (3.21) holds for any Q such that $|Q| = M$. Let $Q' \equiv Q \cup l'$, where $l' \notin Q$. Then $|Q'| = |Q| + 1 = M + 1$. Consider

$$\Pr\{\widetilde{\bigcup_{l\in Q'}} H_l\} = \Pr\{\widetilde{\bigcup_{l\in Q}} H_l\} + \Pr\{H_{l'}\} - 2\Pr\{\widetilde{\bigcup_{l\in Q}} H_l\}\Pr\{H_{l'}\} \geqslant (1-2\gamma)\sum_{l\in Q'}\Pr\{H_l\}.$$

Thus inequality (3.21) has been proved.

Using (3.19), (3.21), and (3.18), we have

$$\Pr\{\widetilde{\bigcup_{l\in Q'}} H_l\} \geqslant \left(1 - 2\frac{p}{1-p}\right)\sum_{l\in Q}\Pr\{H_l\} \geqslant \frac{1-3p}{1-p}\sum_{l\in Q}\exp\left\{-m_l \ln\frac{1}{\delta}\right\}. \tag{3.22}$$

Using the inequality between the arithmetic mean and the geometric mean [3], we obtain from (3.19) and (3.22) that

$$p \geqslant (1-3p)|Q|\exp\left\{-\frac{\ln(1/\delta)}{|Q|}\sum_{l\in Q} m_l\right\}.$$

Taking the logarithm of this inequality, we obtain (3.20). Lemma 3.3 has thus been proved.

Consider k-dimensional Boolean vector-valued function **f** of n variables. We fix a variable $\mathbf{x} \in X^n$. Assume that $\mathbf{x}^l = (x_1^l, \ldots, x_n^l)$ is a binary vector that differs from **x** only with respect to the l-th component; i.e., $x_i^l = x_i$ for all $i \neq l$ and $x_l^l = \bar{x}_l$. Assume that $Q_j(\mathbf{f}, \mathbf{x})$, $j = 1, 2, \ldots, k$, is the set of all natural numbers $l \leq n$ such that $f_j(\mathbf{x}) \neq f_j(\mathbf{x}^l)$. Let $W_j(\mathbf{f}, \mathbf{x}) = |Q_j(\mathbf{f}, \mathbf{x})|$, i.e., $W_j(\mathbf{f}, \mathbf{x})$ is the number of variables of Boolean vector-valued function **f** such that for fixed vector x a change in any of these variables results in a change in the j-th component of vector-valued function **f**. Let

$$W(\mathbf{f}) = \max_{j=1,\ldots,k} \max_{\mathbf{x}\in X^n} W_j(\mathbf{f}, \mathbf{x}). \tag{3.23}$$

THEOREM 3.1. Assume that S is the arrangement of minimal complexity in basis Φ that realizes k-dimensional Boolean vector-valued function **f** with an error probability not greater than $p \in (0, 1/3)$, for a malfunction probability $g \in (0, 1/2)$. Then

$$L(S) \geqslant \frac{W(\mathbf{f})\ln\{W(\mathbf{f})(1-3p)p^{-1}\}}{(n(\Phi)-1)\ln\{n(\Phi)/\varepsilon\}} - \frac{k}{n(\Phi)-1}. \tag{3.24}$$

Proof. Assume that the maximum value in (3.23) is attained for $j = j_0$ and $\mathbf{x} = \mathbf{x}_0$. We introduce events E and E_l, setting $E = \{\theta(\tilde{a}_{j_0}, \mathbf{x}_0, \varepsilon) \neq f_{j_0}(\mathbf{x}_0)\}$, $E_l = \{\theta(\tilde{a}_{j_0}, \mathbf{x}_0^l, \varepsilon) = f_{j_0}(\mathbf{x}_0^l)\}$. Let B_l, $l = 1, \ldots, n$, be the set of all edges originating from input a_l of S. For any set $\beta \subset B_l$ we introduce the event

$$H_l(\beta) = \{(\xi_b = 1, \text{ if } b \in \beta) \cap (\xi_b = 0, \text{ if } b \in B_l \setminus \beta)\}.$$

Assume that the set $\beta_l \subset B_l$ is such that

$$\Pr\{E_l|H_l(\beta_l)\}=\max_{\beta\subset B_l}\Pr\{E_l|H_l(\beta)\}. \tag{3.25}$$

We introduce the notation $H_l = H_l(B_l \setminus \beta_l)$ and $\tilde{H}_l = H_l(\beta_l)$. Then

$$\Pr\{E\}\geqslant\Pr\{E|\tilde{\bigcup_{l\in Q}} H_l\}\Pr\{\tilde{\bigcup_{l\in Q}} H_l\}, \tag{3.26}$$

where $Q \equiv Q_{j_0}(f, x_0)$. Lemma 3.2 and the condition of the theorem yield that

$$\Pr\{E\}\leqslant p;\ \Pr\{E_l\}\geqslant 1-p. \tag{3.27}$$

The definition of θ-state implies that for all $l \in Q$

$$\Pr\{E|H_l\}=\Pr\{E_l|\tilde{H}_l\}. \tag{3.28}$$

In accordance with (3.25) and (3.27),

$$\Pr\{E_l|\tilde{H}_l\}\geqslant\Pr\{E_l\}\geqslant 1-p. \tag{3.29}$$

From (3.29) and (3.28) we have that

$$\Pr\{E|\tilde{\bigcup_{l\in Q}} H_l\}\geqslant 1-p. \tag{3.30}$$

Then (3.26), (3.27), and (3.30) yield

$$p\geqslant(1-p)\Pr\{\tilde{\bigcup_{l\in Q}} H_l\}. \tag{3.31}$$

The fact that ξ_b are independent random variables implies that events H_l are independent. Note that

$$\Pr\{H_l\}\geqslant\exp\{-|B_l|\ln(1/\delta)\}. \tag{3.32}$$

Comparing (3.31) and (3.19), and (3.32) and (3.18), we see that the conditions of Lemma 3.3 hold for $m_l = |B_l|$. Using the result of this lemma, we obtain

$$\sum_{l\in Q}|B_l|\geqslant\frac{|Q|}{\ln(1/\delta)}\ln\{|Q|(1-3p)p^{-1}\}. \tag{3.33}$$

If N_B edges originate from the inputs of arrangement of S in basis Φ, while N_A is the number of vertices from which no edges originate, then

$$N_A\geqslant N_B-(n(\Phi)-1)L(S). \tag{3.34}$$

In the minimal arrangement, all vertices from which no edges originate are outputs of the arrangement. Since S is minimal and $N_B\geqslant\sum_{l\in Q}|B_l|$, we have, allowing for (3.33) and (3.34),

$$k\geqslant\frac{|Q|}{\ln(1/\delta)}\ln\{|Q|(1-3p)p^{-1}\}-(n(\Phi)-1)L(S). \tag{3.35}$$

Since $|Q| = W(f)$, by setting $\delta = \varepsilon / n(\Phi)$ we obtain (3.24) from (3.35). Theorem 3.1 has thus been proved.

<u>Proof of Theorem 2.1.</u> Consider one-dimensional Boolean vector-valued function $\tilde{f}(x_1, x_2, \ldots, x_n) = x_1 \oplus x_2 \oplus \ldots \oplus x_n$. Let $N = L(\tilde{f}, \Phi)$. If basis Φ includes Boolean function $s = s(x_1, \ldots, x_{n(\Phi)}) = x_1 \oplus \ldots \oplus x_{n(\Phi)}$, then setting up arrangement S that realizes $\tilde{f}$ from unreliable elements that realize function s in the form of a tree, we obtain

$$L(S)\leqslant(n-1)/(n(\Phi)-1)+1. \tag{3.36}$$

Then, given an arbitrary basis,

$$L(\tilde{f},\Phi)=N\leqslant C(\Phi)L(S)\leqslant C(\Phi)((n-1)/(n(\Phi)-1)+1). \tag{3.37}$$

From this we have

$$n\geqslant(N/C(\Phi)-1)(n(\Phi)-1). \tag{3.38}$$

Assume that $\tilde{S}$ is the arrangement of minimal complexity in basis Φ that realizes function $\tilde{f}$ with error probability not greater than p for a malfunction probability equal to ε. Since $W(\tilde{f}) = n$, we obtain, using the result of Theorem 3.1,

$$L(\tilde{S}) \geqslant \frac{n \ln\{n(1-3p)p^{-1}\}}{(n(\Phi)-1)\ln\{n(\Phi)/\varepsilon\}} - \frac{1}{n(\Phi)-1}. \tag{3.39}$$

In accordance with our definition of redundancy (2.4),

$$R_{p,\varepsilon}(N) \geqslant L(\tilde{S})/N. \tag{3.40}$$

We now obtain (2.5) from (3.38)-(3.40), and the theorem is thus proved.

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SINGLE-LINE QUEUING SYSTEM WITH ABSOLUTE PRIORITIES AND A LIMITED NUMBER OF INTERRUPTIONS

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The article considers a single-line queuing system with many incoming flows that have absolute priority relative to one another. Only a restricted number of interruptions of low-priority requests by high-priority requests are permitted. Once the limiting number of interruptions is exceeded, an interrupted request leaves the system.

Recent years have witnessed a considerable growth of interest in queuing systems with priorities. There have been a number of studies, both domestic [1-6] and foreign [7, 8], of this problem. In such studies, high-priority requests interrupt the servicing of low-priority ones either an unrestricted number of times [7, 8], or an interrupted request is lost after the first interruption [1, 2]. In this paper we will investigate absolute priority with a restricted number of interruptions. The problem can be formulated as follows.

A single-line queuing system receives l elementary flows of requests. The intensity of the k-th and overall flows will be denoted by a_k and $\sigma_l = \sum_{i=1}^{l} a_i$, respectively. Assume that the servicing durations for the requests of all flows are independent random variables in the aggregate with distribution function $B_k(x)$ for the k-th flow. The number of waiting places in queue is unrestricted. The discipline for selecting requests from queue for servicing is such that a request of the k-th flow has absolute priority over all requests of the j-th flow for k < j. We will consider two arrangements for servicing requests after interruption: 1) servicing for interrupted request begun anew; 2) servicing of interrupted request resumed. However, the number of interruptions of a request of flow k (we will also speak of k-th priority or k-request) by higher-priority requests is bounded by a quantity ν that is the same for all k, $k = \overline{2, l}$. After ν interruptions, an interrupted request leaves the system.

The aim of this paper is to obtain an expression for the distribution functions of the waiting time and dwell time in the system for requests of priority k.

We introduce the following notation: $H_k(x)$ is the distribution function (DF) of the length of the time interval beginning at the instant that a request of priority k first arrives for servicing until the subsequent instant

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