Convergent LMI relaxations for nonconvex quadratic programs

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Abstract

We consider the general nonconvex quadratic programming problem and provide a series of convex positive semidefinite programs (or LMI relaxations) whose sequence of optimal values is monotone and converges to the optimal value of the original problem. It improves and includes as a special case the well-known Shor’s LMI formulation. Often, the optimal value is obtained at some particular early relaxation as shown on some nontrivial test problems from Floudas and Pardalos [9].

1 Introduction

In this paper we consider the general nonconvex quadratic programming problem

\[ \min_{x \in \mathbb{R}^n} g_0(x) \mid \sum_{i=1}^{m} g_i(x) \geq 0, \quad i = 1, 2, \ldots, m. \]  

(1)

where the \( g_i(x) \) are all quadratic polynomials, \( i = 0 \ldots m \), and where neither the criterion nor the constraint set are assumed to be convex. We also allow equality constraints since they can be written with two opposite inequalities. In fact, any optimization problem with polynomials can be put in this form (see Shor [13], [14] and Ferrier [3]). This problem is very difficult in general, for several NP-hard problems can be put in this form. However, there is a well-known relaxation of this problem, known as Shor’s relaxation [13], which replaces \( \mathbb{P} \) by a convex positive semidefinite (psd) program (or LMI relaxation). This relaxation has been proved very useful for computing estimates of global minima, even in combinatorial optimization. For instance, for the well-known MAX-CUT problem (a special case of (1)), very good approximations of a global minimum have been obtained in Goemans and Williamson [4] using Shor’s relaxation, followed by a randomized rounding procedure.

In this paper, we propose a family \( \{Q_i\} \) of convex LMI relaxations (or psd programs) that contains Shor’s relaxation as its first member \( Q_1 \) and with an associated increasing sequence of lower bounds \( \{\inf Q_i\} \).

When the constraint set is compact, the nondecreasing sequence of these lower bounds converges to the global optimal value \( p^* \) in (1). In fact, in many cases, the global optimal value is reached exactly in a few steps. The approach is based on the theory of moments and recent results on the representation of polynomials that are strictly positive on a compact semi algebraic set. For results on the theory of moments and the representation of positive polynomials, the reader is referred to Curto and Fialkow [2], Berg [1], Schmüdgen [11], Putinar [10], Jacobi [5], Jacobi and Prestel [6]. It turns out that this theory is a natural and appropriate tool for global optimization since \( g_0(x) - p^* \) is precisely a (non strictly) positive polynomial, a feature that distinguishes \( p^* \) from the other (local) minima. Moreover, the LMI relaxations are well-suited since both primal and dual psd programs perfectly match both sides of the same theory (moments and positive polynomials). Indeed, when the optimal value \( p^* \) is obtained at some relaxation, say \( Q_i \), the primal psd program provides a global minimizer whereas the dual psd program \( Q^*_i \) provides the coefficients of polynomials in the decomposition of \( g_0(x) - p^* \) into a weighted sum of squares.

Of course, there is a price to pay for these refined relaxations. The number of variables in the LMI relaxation \( Q_i \) is the dimension of the vector space of real-valued polynomials in \( n \) variables of degree \( 2i \), that is, \( O(n^{2i}) \).

However, the \( Q_2 \) relaxation is already interesting. On a sample of 50 randomly generated MAX-CUT problems in \( \mathbb{R}^{10} \), the \( Q_2 \) relaxation always provided the optimal solution, in contrast to only a few cases for the \( Q_1 \) relaxation. The results obtained on a sample of nontrivial test problems taken from Floudas and Pardalos [9] are also promising.

2 Notation, Definitions and Preliminaries

For all \( i = 0, 1, \ldots, m \), let \( A_i \) be a real-valued symmetric \((n, n)\)-matrix, \( c_i \in \mathbb{R}^n \), and consider the general nonconvex quadratic programming problem (1) with \( g_i(x) = x' A_i x + c_i' x + b_i, \quad i = 1, \ldots, m \) and \( g_0(x) = x' A_0 x + c_0' x \). Let \( \mathbb{K} \) be the feasible set of \( \mathbb{P} \), that is, \( \mathbb{K} = \{g_i(x) \geq 0, \quad i = 1, \ldots, m\} \).

Given any two real-valued symmetric matrices \( A, B \) let \( A \preceq B \) (resp. \( A \succeq B \)) denote the usual scalar product trace(\( AB \)) and let \( A \succeq B \) (resp. \( A \preceq B \)) stand for \( A - B \) psd (resp. \( A - B \) positive definite). Let

\[ 1, x_1, x_2, \ldots, x_n, x_1^2, x_1 x_2, \ldots, x_n^2, \ldots, x_n^m \]  

(2)
be a basis for the $r$-degree real-valued polynomials and let $s(r)$ be its dimension. Therefore, a $r$-degree polynomial $p(x) : \mathbb{R}^n \to \mathbb{R}$ is written
\[
p(x) = \sum_{a \in \delta_r} p_a x^a, \quad x \in \mathbb{R}^n,
\]
where
\[
x^a = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}, \quad \text{with } \sum_{i=1}^n a_i = k;
\]
is a monomial of degree $k$ with coefficient $p_a$. Denote by $p = \{p_a\} \in \mathbb{R}^{s(r)}$ the coefficients of the polynomial $p(x)$ in the basis (2).

Hence, the respective vectors of coefficients of the polynomials $g_i(x), i = 0, 1, \ldots, m$, are denoted $\{(g_i)_a\} := g_i \in \mathbb{R}^{s(q)}, i = 0, 1, \ldots, m$.

Given a $s(2r)$-sequence $(1, y_1, \ldots, y_r)$, let $M_r(y)$ be the moment matrix of dimension $s(r)$, with rows and columns labelled by (2). For instance, for illustration purposes, and for clarity of exposition, consider the 2-dimensional case. The moment matrix $M_r(y)$ is the block matrix $\{M_{ij}(y)\}_{0 \leq i, j \leq r}$ defined by
\[
M_{ij}(y) = \begin{bmatrix}
y_{i+j,0} & y_{i+j-1,1} & \cdots & y_{i,j} \\
y_{i+j-1,1} & y_{i+j-2,2} & \cdots & y_{i-1,j+1} \\
\cdots & \cdots & \cdots & \cdots \\
y_{i,j} & y_{i+j-1,1} & \cdots & y_{0,i+j}
\end{bmatrix}.
\]

To fix ideas, when $n = 2$ and $r = 2$ one obtains
\[
M_2(y) = \begin{bmatrix}
1 & y_{10} & y_{01} & y_{20} & y_{11} & y_{02}
\end{bmatrix}
\]
For the 3-dimensional case, $M_r(y)$ is defined via blocks $\{M_{ij}(y)\}_{0 \leq i, j \leq r}$. Let $M_r(\theta y)$ be the matrix defined by
\[
M_r(\theta y)(i, j) = \sum_\alpha \theta^\alpha y_{[i,j]+\alpha}.
\]

For instance, with
\[
M_1(y) = \begin{bmatrix}
1 & y_{10} & y_{01} \\
y_{10} & y_{20} & y_{11} \\
y_{01} & y_{11} & y_{02}
\end{bmatrix}
\]
and $x \mapsto \theta(x) = a - x_1^2 - x_2^2$.

$M_r(\theta y)$ reads
\[
\begin{bmatrix}
-a - y_{20} - y_{02} & ay_{10} - y_{30} - y_{12} & ay_{01} - y_{21} - y_{03} \\
ay_{10} - y_{30} - y_{12} & ay_{20} - y_{40} - y_{22} & ay_{11} - y_{31} - y_{13} \\
ay_{01} - y_{21} - y_{03} & ay_{11} - y_{31} - y_{13} & ay_{02} - y_{22} - y_{04}
\end{bmatrix}.
\]

$M_r(y)$ define a bilinear form $\langle \cdot, \cdot \rangle_{y}$ on the space $\mathcal{A}_r$ of $r$-degree real-valued polynomials by
\[
\langle q(x), v(x) \rangle_{y} := \langle q, M_r(y)v \rangle, \quad q(x), v(x) \in \mathcal{A}_r,
\]
and if $y$ is a sequence of moments of some measure $\mu_y$, then
\[
\langle q, M_r(y)q \rangle = \int q(x)^2 \mu_y(dx) \geq 0,
\]
so that $M_r(y) \succeq 0$, and similarly,
\[
\langle q, M_r(\theta y)q \rangle = \int \theta(x)q(x)^2 \mu_y(dx),
\]
so that $M_r(\theta y) \succeq 0$ whenever $\mu_y$ has its support contained in $\{\theta(x) \geq 0\}$. The theory of moments identifies those sequences $y$ with $M_r(y) \succeq 0$, which are moment-sequences. For details and recent results, the interested reader is referred to Berg [1], Curto and Fialkow [2], Simon [12], Schmüdgen [11] and the many references therein.

3 A family of convex LMI relaxations

Consider the problem $\mathbb{P}$ with feasible set $\mathcal{K} = \{g_i(x) \geq 0, i = 0, 1, \ldots, m\}$ and criterion $g_0(x)$. When needed below, the vectors $g_i \in \mathbb{R}^{s(q)}$, are extended to vectors of $\mathbb{R}^{s(2q)}$ by completing with zeros, i.e., the quadratic polynomials $g_i(x), i = 0, \ldots, m$, are considered as $2r$-degree polynomials with null coefficients for terms of degree larger than $2$. As we minimize $g_0$, we assume that its constant term is zero, that is, $g_0(0) = 0$.

For $i = 1, 2, \ldots$, consider the following family of convex LMI problems
\[
\mathbb{Q}_i \begin{cases}
\min_\alpha \sum_\alpha (g_0)_\alpha y_\alpha \\
M_{i-1}(y) & \succeq 0, \quad k = 1, \ldots, m.
\end{cases}
\]
with respective dual problems
\[
\mathbb{Q}_i^* \begin{cases}
\max_{x, \alpha \geq 0} -X(1,1) - \sum_{k=1}^m g_k(0)Z_k(1,1) \\
\langle X, B_\alpha \rangle + \sum_{k=1}^m \langle Z_k, C_\alpha^k \rangle = (g_0)_\alpha, \forall \alpha \neq 0,
\end{cases}
\]
where we have written
\[
M_i(y) = \sum_\alpha B_\alpha y_\alpha; \quad M_{i-1}(y) = \sum_\alpha C_\alpha^k y_\alpha, \quad k = 1, \ldots, m.
\]
(with $y_0 = 1$) for appropriate real-valued symmetric matrices $B_\alpha, C_\alpha^k, k = 1, \ldots, m$.

At this stage, it is worthy to write down the LMI program $\mathbb{Q}_1$, that is, when $i = 1$.

\[
\mathbb{Q}_1 \begin{cases}
\min_\alpha \sum_\alpha (g_0)_\alpha y_\alpha \\
\sum_\alpha M_1(y) & \succeq 0, \quad k = 1, \ldots, m
\end{cases}
\]
The family of LMI problems

Proposition 3.1

and remembering that

Observe that, if we write

Proof: Next, consider an LMI problem

for appropriate matrices

\( M \), corresponding a solution

\( (1, y_1) \) of dimensions \( 2(2i − 1) \), we have

\[ M_i(y) = \begin{bmatrix} M_{i-1}(y_1) & M \\ M' & N \end{bmatrix}, \]

and

\[ M_{i-1}(g_k y) = \begin{bmatrix} M_{i-2}(g_k y_1) & V \\ V' & \text{NS} \end{bmatrix}, \quad k = 1, \ldots, m \]

for appropriate matrices \( M, N, V, S \). Therefore,

\[ M_i(y) \succeq 0 \Rightarrow M_{i-1}(y_1) \succeq 0, \]

and

\[ M_{i-1}(g_k y) \succeq 0 \Rightarrow M_{i-2}(g_k y_1) \succeq 0, \quad k = 1, \ldots, m. \]

so that \( y_1 \) is admissible for \( Q_{i-1} \). Moreover, the value of \( y \) in \( Q_i \) is the same as the value of \( y_1 \) in \( Q_{i-1} \), and the result follows.

Proposition 3.1 ensures that better and better lower bounds on \( \mathbb{P} \) can be obtained by solving the relaxations \( Q_i, i = 1, \ldots \). The next result shows that in fact, whenever \( \mathbb{K} \) is compact, one may approach as closely as desired, the optimal value \( p^* = \inf \mathbb{P} \). We will use the fact that under a certain condition on the feasible set \( \mathbb{K} \), every polynomial \( p(x) \), strictly positive on \( \mathbb{K} \), has the following representation:

\[ p(x) = \sum_{j=1}^{r_0} q_j(x)^2 + \sum_{k=1}^{m} g_k(x) \left( \sum_{j=1}^{r_k} q_{kj}(x)^2 \right) \]

for a finite family of polynomials \( \{ q_j(x) \}, j = 1, \ldots, r_0 \), and \( \{ q_{kj}(x) \}, j = 1, \ldots, r_k, k = 1, \ldots, m \). In fact, a necessary and sufficient condition for the representation (13) to exist is that there exists a polynomial \( u(x) \) of the form (13) such that \( \{ u(x) \succeq 0 \} \) is compact (see Putinar [10] and Jacobi [5]).

For instance, the representation (13) holds whenever \( \{ g_k(x) \succeq 0 \} \) is compact for some \( k \), or when all the \( g_k(x) \) are linear and \( \mathbb{K} \) is compact. In particular, it holds for every 0-1 program. Indeed, write the integral constraints as \( x_i^2 - x_i \succeq 0 \) for all \( i = 1, \ldots, n \). Then, consider the polynomial \( u(x) := \sum_{x_i} (x_i - x_i^2) \). Its level set \( \{ u(x) \succeq 0 \} \) is compact. Moreover, if one knows that a global minimizer is contained in some ball \( \| s \|^2 \leq M \), for some large enough, then one may add the redundant constraint \( g_{m+1}(x) := M - \sum_{x_i} x_i^2 \geq 0 \), and the set \( \mathbb{K} \) (defined as previously with the latter additional constraint \( g_{m+1}(x) \succeq 0 \)) has the required property. And, we have:

\[ \inf Q_i \leq \inf \mathbb{P}, \quad i = 1, 2, \ldots. \]

Proposition 3.1. The family of LMI problems \( \{ Q_i \} \) satisfies

Next, consider an LMI problem \( Q_i \), with \( i > 1 \), and let \( y = \{ y_k \} \) a feasible sequence for \( Q_i \), that is, \( (1, y) \) is a vector of dimension \( 2i \), the dimension of the vector space of real-valued polynomials \( p(x) : \mathbb{R}^n \to \mathbb{R} \), of degree \( 2i \). If we write \( y = (y_1, y_2) \) with \( (1, y_1) \) of dimension \( 2(i − 1) \), we have

\[ y_k := (x_1, x_2, \ldots, x_{2i}, \ldots, x_{2n}) \]

admissible for \( Q_i \), and thus \( \inf Q_i \leq \inf \mathbb{P} \) for every \( i = 1, \ldots \).

Theorem 3.2. Assume that there is some polynomial \( u(x) : \mathbb{R}^n \to \mathbb{R} \) of the form (13) and with \( \{ u(x) \succeq 0 \} \) compact. Then, as \( i \to \infty \),

\[ \inf Q_i \uparrow \min \mathbb{P}. \]

If \( \mathbb{K} \) has a nonempty interior, then as \( i \to \infty \),

\[ \max Q_i^* = \inf \mathbb{P} \uparrow \min \mathbb{P}. \]

The equality max \( Q_i^* \) occurs for all \( i \geq i_0 \) for some index \( i_0 \) if and only if \( g_0(x) - p^* \) is of the form (13).

Proof: Let \( \varepsilon > 0 \) be fixed arbitrary. Then, the polynomial \( x \mapsto G_0(x) := g_0(x) - p^* + \varepsilon \) is strictly positive on \( K \). From the assumption on \( \mathbb{K} \), it follows that \( G_0(x) \) has the representation

\[ G_0(x) = \sum_{j=1}^{r_0} q_j(x)^2 + \sum_{k=1}^{m} g_k(x) \left( \sum_{j=1}^{r_k} q_{kj}(x)^2 \right) \]

for some polynomials \( q_j(x), j = 1, \ldots, r_0 \), and \( q_{kj}(x), j = 1, \ldots, r_k, k = 1, \ldots, m \) (see e.g. Putinar [10], Jacobi [5]). Now, let \( i_0(\varepsilon) \) (that for notational convenience we simply write \( i \) for this maximum degree of \( q_j(x), j = 1, \ldots, r_0 \), and
\[ i_2(\varepsilon) \] (noted \( i_2 \)) be the maximum degree of the polynomials \( \{ q_{kj}(x) \} \), so that the polynomials \( \{ g_t(x) q_{kj}(x)^2 \} \) have maximum degree \( 2i_2 + 2 \) (as the \( g_t(x) \)'s are quadratic polynomials). Let \( i := \max\{ i_1, i_2 + 1 \} \). Let \( q_j \in \mathbb{R}^{R(n)} \) and \( q_{kj} \in \mathbb{R}^{R(n-1)} \) be the respective vectors of coefficients of the polynomials \( \{ q_j(x) \} \) and \( \{ q_{kj}(x) \} \), and write

\[
X := \sum_{j=1}^{\infty} q_j d_j, \quad Z_k := \sum_{j=1}^{\infty} q_{kj}^2, \quad k = 1, \ldots, m.
\] (17)

Observe that with

\[
y^\ast = (x_1, \ldots, x_i^2, \ldots, x_n^2),
\]
then,

\[
g_t(x) q_{kj}(x) = \langle q_{kj}, M_{1-1}(g_t y^\ast) \rangle = \langle q_{kj}^2, M_{1-1}(g_t y^\ast) \rangle,
\]
so that

\[
g_t(x) \sum_{j=1}^{i} q_{kj}^2 = \langle Z_k, M_{1-1}(g_t y^\ast) \rangle
\] (18)

and similarly,

\[
\sum_{j=1}^{i} q_{kj}^2 = \sum_{\alpha} x^{\alpha} \langle X, B_{\alpha} \rangle,
\] (19)

with \( B_{\alpha}, C_{\alpha}^k \) as in \( \mathbb{Q} \).

Therefore, from the representation (16), it follows that

\[
\sum_{\alpha} x^{\alpha} \left[ \langle X, B_{\alpha} \rangle + \sum_{k=1}^{i} \langle Z_k, C_{\alpha}^k \rangle \right] = \sum_{\alpha} x^{\alpha} (g_0)_{\alpha} - p^\ast + \varepsilon.
\] (20)

Identifying terms of same power yields

\[
\langle X, B_{\alpha} \rangle + \sum_{k=1}^{i} \langle Z_k, C_{\alpha}^k \rangle = (g_0)_{\alpha} \quad \forall \alpha \neq 0,
\]
and, for the constant term,

\[
X(1, 1) + \sum_{k=1}^{i} Z_k(1, 1) g_t(0) = -p^\ast + \varepsilon,
\]
which proves that \( \{ X, Z_1, \ldots, Z_m \} \) is admissible for \( \mathbb{Q}^i \) with value \( p^\ast - \varepsilon \). Therefore, \( p^\ast - \varepsilon \leq \inf \mathbb{Q}_i \leq p^\ast \), and from the monotonicity and the fact that \( \varepsilon \) is arbitrary, \( \inf \mathbb{Q}_i \uparrow p^\ast = \min \mathbb{P} \).

The statement (15) follows from a standard result in convexity. Indeed, we have just proved that \( \mathbb{Q}^i \) has a feasible solution \( \{ X, Z_1, \ldots, Z_m \} \). Moreover, if \( \mathbb{K} \) has a nonempty interior, let \( \mu \) be a probability measure with support \( \mathbb{K} \) and with a uniform distribution. Denote by \( y^\mu \) the vector of all the (well-defined) moments of \( \mu \). From (5)-(6) and the fact that \( \mu \) has a uniform distribution on \( \mathbb{K} \), it follows easily that \( M_t(y^\mu) \geq 0 \) and \( M_{1-1}(g_t y^\mu) \geq 0 \) for all \( k = 1, \ldots, m \), so that \( y^\mu \) is a strictly admissible solution. Therefore, there is no duality gap between \( \mathbb{Q}_t^i \) and \( \mathbb{Q}_i \), and (15) follows.

Finally, consider the if part of the final statement. From the representation (13) of \( g_0(x) - p^* \) and using the same above arguments, the corresponding matrices \( X, Z_1, \ldots, Z_m \) form an admissible solution for \( \mathbb{Q}_t^i \) with value \( p^* = \min \mathbb{P} \), hence proving \( \max \mathbb{Q}_t^i = \min \mathbb{P} \).

Conversely, the only if part of the final statement also follows easily from the above proof. Let \( \{ X, Z_1, \ldots, Z_m \} \) be an optimal solution of \( \mathbb{Q}_t^i \) with \( \max \mathbb{Q}_t^i = p^* = \min \mathbb{P} \). As \( X, Z_1, \ldots, Z_m \geq 0 \), write them as in (17). The admissibility of \( \{ X, Z_1, \ldots, Z_m \} \) implies (20) with \( p^* \) in lieu of \( p^* - \varepsilon \), which in turn, using (18)-(19) yields the desired result.

The representation (13) has a nice interpretation as a global optimality condition à la Karush-Kuhn-Tucker. Indeed,

**Proposition 3.3** Assume that \( g_0(x) - p^* \) has the representation (13). Then, at a global minimizer \( x^\ast \), we have

\[
\nabla g_0(x^\ast) = m \sum_{k=1}^{i} \nabla g_t(x^\ast) \left[ \sum_{j=1}^{i} q_{kj}(x^\ast)^2 \right] = \sum_{k=1}^{i} \sum_{j=1}^{i} q_{kj}(x^\ast)^2.
\] (21)

and

\[
g_t(x^\ast) \left[ \sum_{j=1}^{i} q_{kj}(x^\ast)^2 \right] = 0, \quad k = 1, \ldots, m.
\] (22)

**Proof:** The proof is immediate. As \( g_0(x^\ast) - p^* = 0 \), the representation (13) leads

\[
\sum_{j=1}^{i} q_{kj}(x^\ast)^2 = 0; \quad \sum_{k=1}^{i} g_t(x^\ast) \left[ \sum_{j=1}^{i} q_{kj}(x^\ast)^2 \right] = 0,
\]
so that (22) follows. Differentiating in (13) and using the above relationship also yields (21).

Hence, the theory of representation of polynomials, positive on the feasible set \( \mathbb{K} \), can be viewed as a global optimality condition. The polynomials (sums of squares) \( \sum_{j=1}^{i} q_{kj}(x)^2 \) in the representation (13) of \( g_0(x) - p^* \) (when it holds) are nothing less than Lagrange Karush-Kuhn-Tucker “multipliers”\(^a\). In contrast to the usual Karush-Kuhn-Tucker necessary optimality condition with “scalar multipliers”\(^b\) \( \lambda_k \), a non-active constraint \( g_k(x) \) at a global minimizer \( x^\ast \) may have a nontrivial associated “polynomial” multiplier (when this constraint plays a role to eliminate some better (non feasible) solutions. However, this polynomial multiplier vanishes at \( x^\ast \), as in the usual Karush-Kuhn-Tucker conditions where \( \lambda_k = 0 \) at a non-active constraint \( g_k(x^\ast) > 0 \).  

**4 Examples**

For illustration purposes we will consider the MAX-CUT problem and some nontrivial nonconvex quadratic test problems from Floudas and Pardalos [9].
4.1 The max-cut problem

Roughly speaking, the MAX-CUT problem is a special case of (1) with

- \( g_0(x) = x^T A_0 x \) and \( \text{diag}(A_0) = 0 \).
- \( g_i(x) = x_i^2 - 1 \) and the constraint is an equality constraint.

As for 0-1 programs, the feasible set \( \mathbb{R}^n \) satisfies the condition required in Theorem 3.2. Indeed, write the integral constraint \( x_i^2 = 1 \) as \( x_i^2 - 1 \geq 0 \) and \( 1 - x_i^2 \geq 0 \) for all \( i = 1, \ldots, n \), and consider the polynomial \( u(x) := \sum_{i=1}^{n} (1 - x_i^2) \). Its level set \( \{ u(x) \geq 0 \} \) is compact. Shor’s relaxation (equivalently, \( Q_1 \)), is the convex LMI problem:

\[
Q_1 \begin{cases} \min \langle Y, A_0 \rangle \\ \begin{bmatrix} 1 & y' \\ -y & Y \end{bmatrix} \succeq 0; \quad \text{diag}(Y) = e \end{cases} \tag{23}
\]

where \( e \) is a vector of ones. To visualize the two relaxations \( Q_1 \) and \( Q_2 \), take an example in \( \mathbb{R}^3 \). In that case, \( Q_1 \) reads

\[
Q_1 \begin{cases} \min q_{110}y_{110} + q_{101}y_{101} + q_{011}y_{011} \\ \begin{bmatrix} 1 & y_{100} & y_{010} & y_{001} \\ y_{100} & 1 & y_{110} & y_{011} \\ y_{010} & y_{110} & 1 & y_{011} \\ y_{001} & y_{111} & y_{011} & 1 \end{bmatrix} \succeq 0, \end{cases}
\]

whereas \( Q_2 \) reads

\[
Q_2 \begin{cases} \min q_{110}y_{110} + q_{101}y_{101} + q_{011}y_{011} \\ \begin{bmatrix} M_1(y) & B \\ -B' & C \end{bmatrix} \succeq 0, \end{cases}
\]

with

\[
B = \begin{bmatrix} 1 & y_{110} & y_{101} & 1 & y_{011} & 1 \\ y_{110} & 1 & y_{010} & y_{011} & y_{100} & y_{110} \\ y_{101} & y_{010} & y_{111} & y_{110} & y_{001} & y_{010} \\ y_{011} & y_{111} & y_{110} & y_{100} & y_{001} & 1 \end{bmatrix}
\]

and

\[
C = \begin{bmatrix} 1 & y_{110} & y_{101} & 1 & y_{011} & 1 \\ y_{110} & 1 & y_{010} & y_{011} & y_{101} & y_{110} \\ y_{101} & y_{010} & y_{111} & y_{110} & y_{001} & y_{010} \\ y_{011} & y_{111} & y_{110} & y_{100} & y_{001} & 1 \end{bmatrix}
\]

where we have used the fact that the equality \( x_{100}^2 = 1 \) translates into \( y_{200} = 1 \) and therefore, a term like \( y_{200} \) is replaced by 1, a term like \( y_{110} \) is replaced by \( y_{100}, y_{111} \) by \( y_{011} \), etc. so that only the variables \( y_\alpha \) with \( \alpha \leq 1 \) and \( \sum \alpha_i \leq 4 \) are present.

Hence, the \( Q_2 \) relaxation of a MAX-CUT problem of dimension \( n \) is an LMI problem with \( \left( \binom{n}{4} \right) + \left( \binom{n}{2} \right) \) variables and one LMI constraint of dimension \( (n+1)(n+2)/2 \) in comparison with \( \left( \binom{n}{2} \right) \) variables and one LMI constraint of dimension \( n+1 \) for the \( Q_1 \) relaxation. Of course, as \( i \) increases, then so does the computational burden, a price to pay to obtain better and better lower bounds.

For a sample of 50 randomly generated problems in \( \mathbb{R}^4 \) with positive and negative weights, the \( Q_2 \) relaxation provided the optimal solution in all cases whereas the \( Q_1 \) relaxation in less than 20% of the cases only.

In both relaxations \( Q_1 \) and \( Q_2 \), Slater’s interior point condition fails, and we have to mention that in using the MATLAB LMI toolbox, the running time for \( Q_2 \) was surprisingly very high compared to \( Q_1 \), despite the fact that \( Q_2 \) contains relatively few variables (15 for \( \mathbb{R}^4 \) and 30 for \( \mathbb{R}^5 \) and 375 for \( \mathbb{R}^{10} \)).

4.2 Nonconvex quadratic problems

The nonconvex quadratic test problems below are from Floudas and Pardalos [9].

4.2.1 : Problem 2.2 in Floudas and Pardalos [9].

\[
\begin{align*}
\min f(x, y) & := c^T x - 0.5x^T Q x + d^T y \\
6 x_1 + 3 x_2 + 3 x_3 + 2 x_4 + x_5 & \leq 6.5 \\
10 x_1 + 10 x_3 + y & \leq 20 \\
0 & \leq x_i \leq 1, \ i = 1, \ldots, 5 \\
0 & \leq y
\end{align*}
\]

with \( Q := I \) and \( c = [-10.5, -7.5, -3.5, -2.5, -1.5] \). The optimal value \(-213 \) is obtained at the \( Q_2 \) relaxation.

4.2.2 : Problem 2.6 in Floudas and Pardalos [9].

\[
\begin{align*}
\min f(x) & := c^T x - 0.5x^T Q x \\
Ax & \leq b \\
0 & \leq x_i \leq 1, \ i = 1, \ldots, 10 \\
y_5 & \leq 2
\end{align*}
\]

with \( A \) being the matrix

\[
\begin{bmatrix}
-2 & -6 & -1 & 0 & -3 & -3 & -2 & -6 & -2 & -2 \\
6 & -5 & 8 & -3 & 0 & 1 & 3 & 8 & 9 & 3 \\
-5 & 6 & 5 & 3 & 8 & -8 & 9 & 2 & 0 & -9 \\
9 & 5 & 0 & -9 & 1 & -8 & 3 & -9 & -9 & -3 \\
-8 & 7 & -4 & -5 & -9 & 1 & -7 & -1 & 3 & 2
\end{bmatrix}
\]

\( c = [48, 42, 48, 45, 44, 41, 47, 42, 45, 46] \). \( Q = 100I \) and \( b = [-4.82, -6.23, -12.4] \). The optimal value \(-39 \) is obtained at the \( Q_2 \) relaxation.

4.2.3 : Problem 2.9 in Floudas and Pardalos [9].

\[
\begin{align*}
\max f(x) & := \sum_{i=1}^{9} x_i x_{i+1} + \sum_{i=1}^{8} x_i x_{i+2} \\
& + x_1 x_7 + x_1 x_9 + x_3 x_{10} + x_2 x_{10} + x_4 x_7 \\
\sum_{i=1}^{10} x_i & = 1 \\
x_i & \geq 0, \ i = 1, \ldots, 10.
\end{align*}
\]
The optimal value 0.375 is obtained at the $Q_2$ relaxation.

**4.2.4 :** Problem 3.3 in Floudas and Pardalos [9].

\[
\begin{align*}
\min & \quad f(x) := -25(x_1 - 2)^2 - (x_2 - 2)^2 \\
& \quad - (x_3 - 1)^2 - (x_4 - 4)^2 - (x_5 - 1)^2 - (x_6 - 4)^2 \\
& \quad (x_3 - 3)^2 + x_4 \geq 4; (x_3 - 3)^2 + x_6 \geq 4 \\
& \quad x_1 - 3x_2 \leq 2; -x_1 + x_2 \leq 2 \\
& \quad x_1 + x_2 \leq 6; x_1 + x_2 \geq 2 \\
& \quad 1 \leq x_3 \leq 5; 0 \leq x_4 \leq 6 \\
& \quad 1 \leq x_5 \leq 5; 0 \leq x_6 \leq 10 \\
& \quad x_1, x_2 \geq 0
\end{align*}
\]

The optimal value −310 is obtained at the $Q_2$ relaxation.

**4.2.5 :** Problem 3.4 in Floudas and Pardalos [9].

\[
\begin{align*}
\min & \quad f(x) := -2x_1 + x_2 - x_3 \\
& \quad x_1 + x_2 + x_3 \leq 4 \\
& \quad x_1 \leq 2; x_3 \leq 3; 3x_2 + x_3 \leq 6 \\
& \quad x_i \geq 0, i = 1, 2, 3 \\
& \quad x^TBx - 2x^TBx + ||v||^2 - 0.25||b - v||^2 \geq 0
\end{align*}
\]

with $r = [1.5, -0.5, -5]$ and

\[
B = \begin{bmatrix}
0 & 0 & 1 \\
0 & -1 & 0 \\
-2 & 1 & -1
\end{bmatrix}; b = \begin{bmatrix}
3 \\
0 \\
-4
\end{bmatrix}; v = \begin{bmatrix}
0 \\
-1 \\
-6
\end{bmatrix}.
\]

The optimal value −4 is obtained at the $Q_4$ relaxation whereas $\inf Q_3 = -4.0685$.

**5 Conclusion**

We have proposed a sequence of LMI relaxations \{$Q_k$\} and an associated sequence of nondecreasing lower bounds that converges to the global minimum $p^*$, and in many cases, $p^*$ is obtained at a particular relaxation. It has been shown that the primal and dual psd programs perfectly match both sides of the same theory (moments and positive polynomials). Moreover, the representation of polynomials, positive on the feasible set $K$, is interpreted as a Karush-Kuhn-Tucker global optimality condition with “polynomial” Lagrange multipliers instead of scalar multipliers. However, although efficient LMI software packages are now available, high order relaxations require a lot of variables. It is hoped that for a large class of problems, low order relaxations like $Q_2$, $Q_3$ or $Q_4$ will provide the optimal value, or at least, a good lower bound that could be exploited in other optimization methods.

**References**


