A Comparison of Universal and Mean-Variance Efficient Portfolios

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Universal and MVE Portfolios

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Investment Problem

Fixed parameters:
- $M = \text{number of assets}$
- $N = \text{number of periods to invest}$

Market determines:
- $p_{n,m} = \text{price of asset } m \text{ at beginning of period } n$

You choose:
- $a_{n,m} = \text{number of asset } m \text{ owned at beginning of per. } n$

To control:
- $h_n = \text{wealth at beginning of per. } n$
Wealth

Assume:

- No new wealth added to portfolio
- Prices at the end of a period are equal those at beginning of next

Wealth at beginning of period $n$:

$$h_n = a_{n,1}p_{n,1} + \cdots + a_{n,M}p_{n,M}$$

Wealth at beginning of period $n + 1$:

$$h_{n+1} = a_{n,1}p_{n+1,1} + \cdots + a_{n,M}p_{n+1,M}$$
Period Returns

Return for period $n$:

$$R_n = \frac{h_{n+1}}{h_n}$$

$$= \frac{a_{n,1} p_{n,1}}{h_n} \frac{p_{n+1,1}}{p_{n,1}} + \cdots + \frac{a_{n,M} p_{n,M}}{h_n} \frac{p_{n+1,M}}{p_{n,M}}$$

$$= b_{n,1} X_{n,1} + \cdots + b_{n,M} X_{n,M}$$

$$= b_n^t X_n$$

where

$$b_{n,m} = \frac{a_{n,m} p_{n,m}}{h_n}, \quad b_n = (b_{n,1} \cdots b_{n,M})^t$$

$$X_{n,m} = \frac{p_{n+1,m}}{p_{n,m}}, \quad X_n = (X_{n,1} \cdots X_{n,M})^t$$
Cumulative Returns

Return from period 1 to $N$:

$$S_N = \frac{h_{N+1}}{h_1} = \left( \frac{h_{N+1}}{h_N} \right) \left( \frac{h_N}{h_{N-1}} \right) \ldots \left( \frac{h_2}{h_1} \right)$$

$$= \prod_{n=1}^{N} R_n$$

$$= \prod_{n=1}^{N} b_n^t X_n$$
Doubling Rate

Assuming that $b_n^t X_n > 0$ for $n = 1, \ldots, N$:

$$S_N = \exp \left( \sum_{n=1}^{N} \log b_n^t X_n \right)$$

$$= \exp (NW_N)$$

where

$$W_N = \frac{1}{N} \sum_{n=1}^{N} \log b_n^t X_n$$

is called the “doubling” rate of the portfolio.
Constant Rebalanced Portfolios

- **Constant Rebalanced Portfolio (CRP):**
  \[ b_1 = b_2 = \cdots = b_N = b \]

- **Buy-and-Hold (B&H) Portfolio:**
  \[ b = e_m = (0, 0, \ldots, 0, 1, 0, \ldots, 0)^t \]

- **Best CRP:**
  \[ b_N^* = \arg\max_{b \in B} S_N(b) = \arg\max_{b \in B} W_N(b) \]

where

\[ B = \{ b \in \mathcal{R}^M : b_m \geq 0, m = 1, 2, \ldots, M; \sum_{m=1}^M b_m = 1 \} \]
CRP Example [Blum97]

Two Stocks:
- Stock #1: \( X_{n,1} = 1 \), \( n = 1, \ldots, N \)
- Stock #2: \( X_{n,2} = \begin{cases} 
\frac{1}{2} & n = \text{even} \\
2 & n = \text{odd}
\end{cases} \)

Portfolios:
- \( S_N \leq 2 \) for any B&H portfolio
- Return of \( b = \left( \frac{1}{2}, \frac{1}{2} \right)^t \) increases exponentially by \( 9/8 \) every two days:

\[
R_n = \begin{cases} 
\left( \frac{1}{2} \right) (1) + \left( \frac{1}{2} \right) \left( \frac{1}{2} \right) = \frac{3}{4} & n = \text{even} \\
\left( \frac{1}{2} \right) (1) + \left( \frac{1}{2} \right) (2) = \frac{3}{2} & n = \text{odd}
\end{cases}
\]
Equally Weighted CRP [Wang00]

- Equally weighted CRP: \( b = \left( \frac{1}{M}, \cdots, \frac{1}{M} \right)^t \)

- Assume \( X_n \) are IID random vectors with \( E[X_n] = \bar{X} \) and \( V = E[(X_n - E(X_n))(X_n - E(X_n))^t] \):
  - Expected return: \( E(R_n) = \frac{1}{M} \sum_{m=1}^{M} \bar{X}_m \)
  - Variance of return:

\[
\text{var}(R_n) = \frac{1}{M} \left( \frac{1}{M} \sum_{m=1}^{M} v_{m,m} \right) \\
+ \left( \frac{M^2 - M}{M^2} \right) \left( \frac{1}{M^2 - M} \sum_{m=1}^{M} \sum_{k \neq m} v_{m,k} \right) \\
= \left( \frac{1}{M} \right) \text{(avg. var.)} + \left( \frac{M^2 - M}{M^2} \right) \text{(avg. cov.)}
\]

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Universal Portfolios


- **Objective:** Track return of best CRP chosen after asset returns revealed ($b^*_N$)

- **Algorithm:** Return-weighted average of all CRPs

- **Performance:**
  - Yields average return of all CRPs
  - Return exceeds geometric mean of asset returns
  - Asymptotically “tracks” best CRP return
Universal Portfolio Algorithm

Return-weighted average of all CRPs:

Initialize: \( \hat{b}_1 = \left( \frac{1}{M}, \ldots, \frac{1}{M} \right)^t \)

For \( n > 1 \): \( \hat{b}_n = \frac{\int_B b S_n(b) db}{\int_B S_n(b) db} \)

where

\( B = \{ b \in \mathcal{R}^M : b_m \geq 0, m = 1, 2, \ldots, M; \sum_{m=1}^M b_m = 1 \} \)
Performance

Regardless of the stock return sequence:

- Portfolio return is the average of all CRPs’ returns

\[ \hat{S}_N = \prod_{n=1}^{N} \hat{b}_n^t X_n = \frac{\int_B S_n(b) \, db}{\int_B db} \]

- Return is greater than geometric mean

\[ \hat{S}_N \geq \left( \prod_{m=1}^{M} S_N(e_m) \right)^{1/M} \]
Asymptotic Performance

Under many assumptions:

\[ \frac{\hat{S}_N}{S^*_N} \sim \left(\sqrt{\frac{2\pi}{N}}\right)^{M-1} \frac{(M - 1)!}{|J^*|^{1/2}} \]

where

\( J^* = \) Hessian of \( W_N(b) \) at its maximum for large \( N \)

\( S^*_N = \) return of best CRP \( b^*_N \)

and \( \sim \) means the ratio of left- and right-hand sides equals one for large \( N \)
Assumptions for Asymptotic Perf.

The stock sequence $X_1, X_2, \ldots$, satisfies:

- $a \leq X_{n,m} \leq q$, $m = 1, \ldots, M, n = 1, 2, \ldots$, for some $0 < a \leq q < \infty$
- $J_N^* \to J^*$ for some positive definite matrix $J^*$
- $b_N^* \to b^*$ for some $b^*$ in the interior of the $B$

and there exists a function $W(b)$ such that

- $W_N(b) \nearrow W(b)^*$ for any portfolio $b \in B$
- $W(b)$ is strictly concave
- $W(b)$ has bounded third partial derivatives
- $W(b)$ achieves its maximum at $b^*$ in the interior of the $B$
Outline of Proof

Show LHS $\geq$ RHS:

Expand $W_N(b)$ in Taylor series about the best CRP $b^*_N$ to three terms:

- First term is zero because gradient is zero at $b^*_N$
- Second term is related to $J^*_N$
- Third term is bounded

Manipulate expression to look like a Gaussian CDF

Bound the Gaussian CDF

Show LHS $\leq$ RHS:

Follows from Laplace’s method of integration
Mean-Variance Efficient Portfolios

Origins: Markowitz (1952) and Sharpe (1963) (1990 Nobel Laureates)

Objective: To minimize the variance of portfolio returns, while maintaining a desired average return

Algorithm:
- A solution to a constrained optimization problem
- Requires estimates of asset return mean and covariance matrix

Performance:
- Tracks objective portfolio if asset returns are IID
- Otherwise, no guarantee
Objective of MVE Portfolio

Minimize: \( b_n^t V_n b_n \)

Subject to: 1) \( \sum_{m=1}^{M} b_{n,m} = 1 \)

2) \( b_n^t \bar{X}_n = \bar{R}_n \)

where

\( \bar{X}_n \) = estimate of asset return mean
\( V_n \) = estimate of asset return covariance matrix
\( \bar{R}_n \) = desired average return
The MVE Portfolio

Mean and covariance estimators:

\[
\bar{X}_n = \frac{1}{\Lambda} \sum_{k=1}^{n-1} \lambda^{n-k-1} X_k
\]

\[
V_n = \frac{1}{\Lambda} \sum_{k=1}^{n-1} \lambda^{n-k-1} (X_k - \bar{X}_n)(X_k - \bar{X}_n)^t
\]

where \(0 < \lambda \leq 1\) and \(\Lambda = \sum_{k=1}^{n-1} \lambda^{n-k-1}\)

The MVE Portfolio:

\[
b_n = w_0 (1 - \bar{R}_n) + w_1 \bar{R}_n
\]

where \(w_0\) and \(w_1\) are portfolios that yield on average \(0\%\) and \(100\%\) returns, respectively.
The 0% and 100% Avg. Ret. Ports.

\[
\begin{align*}
    w_0 &= \frac{1}{D} \left( c_1 V_n^{-1} 1 - c_2 V_n^{-1} \bar{X}_n \right) \\
    w_1 &= \frac{1}{D} \left( c_1 V_n^{-1} 1 - c_2 V_n^{-1} \bar{X}_n \right) + \frac{1}{D} \left( c_3 V_n^{-1} \bar{X}_n - c_2 V_n^{-1} 1 \right)
\end{align*}
\]

where

\[
\begin{align*}
    1^t &= (1, \ldots, 1) \\
    c_1 &= \bar{X}_n^t V_n^{-1} \bar{X}_n \\
    c_2 &= \bar{X}_n^t V_n^{-1} 1 \\
    c_3 &= 1^t V_n^{-1} 1 \\
    D &= c_1 c_3 - c_2^2
\end{align*}
\]
Performance

- If asset returns are IID then:
  - Estimators converge to their true values by LLN
  - MVE portfolio asymptotically achieves objective

- Otherwise, no guarantee
Examples

Data:
- Daily adjusted closing prices (yahoo.finance.com)
- January 2, 1991 to August 8, 2001 (2,679 days)
- Missing data interpolated

Examples:
- Universal and constant rebalanced portfolios
- Mean-Variance efficient portfolios

Comparison:
- MVE portfolio allows short sales, but univ. portfolio does not
- No transaction costs
- Investor preferences
Univ. and CRPs of C & MSFT

Cumulative Returns for CRPs of C and MSFT

Portfolio Weight in C

Cum. Ret. from 19910102 to 20010808

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Mean–Variance Efficient Portfolios

Return Mean

Return Std. Dev.

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Comparison of Portfolio Returns

Parameters:
- Num. Rand. Port. = 5000
- Des. MVE Port. Ret. = 0.12%
- Forgetting Fact. = 1.00

Daily Return Statistics: mean (std) [hi lo]
- Univ. Port. = 0.12 (1.38) [8.25 –7.52]%
- Best Port. = 0.22 (3.34) [31.94 –26.52]%
- MVE Port. = 0.10 (1.23) [6.35 –7.05]%
- VFINX B&H = 0.06 (0.97) [5.15 –6.91]%
- Eq. Wt. Port. = 0.12 (1.32) [7.51 –7.47]%

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Conclusions

Universal portfolio:
- Attempts to track return of best CRP given knowledge of future asset returns
- Yields average return of all CRPs
- Return greater than geometric mean
- Asymptotically “tracks” return of best CRP

MVE portfolio:
- Attempts to minimize return variance while maintaining a desired average return
- Obtains objective in IID markets