A Comparison of Universal and Mean-Variance Efficient Portfolios

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Universal and MVE Portfolios

- Investment Dynamics
- Universal Portfolios
- Mean-Variance Efficient (MVE) Portfolios
- Examples
- Conclusions

Investment Problem

Fixed parameters:

- M = number of assets
- N = number of periods to invest

Market determines:

- $p_{n,m}$ = price of asset m at beginning of period n

You choose:

- $a_{n,m}$ = number of asset m owned at beginning of per. n

To control:

- h_n = wealth at beginning of per. n

Wealth

Assume:

- No new wealth added to portfolio
- Prices at the end of a period are equal those at beginning of next

Wealth at beginning of period n:

$$h_n = a_{n,1}p_{n,1} + \cdots + a_{n,M}p_{n,M}$$

Wealth at beginning of period $n+1$:

$$h_{n+1} = a_{n,1}p_{n+1,1} + \cdots + a_{n,M}p_{n+1,M}$$

Period Returns

Return for period n:

$$\begin{aligned} R_n &= h_{n+1}/h_n \\ &= \frac{a_{n,1}p_{n,1}}{h_n}\frac{p_{n+1,1}}{p_{n,1}} + \cdots + \frac{a_{n,M}p_{n,M}}{h_n}\frac{p_{n+1,M}}{p_{n,M}} \\ &= b_{n,1}X_{n,1} + \cdots + b_{n,M}X_{n,M} \\ &= b_n^t X_n \end{aligned}$$

where

$$b_{n,m} = \tfrac{a_{n,m}p_{n,m}}{h_n}, \quad b_n = (b_{n,1} \ldots b_{n,M})^t$$
$$X_{n,m} = \tfrac{p_{n+1,m}}{p_{n,m}}, \quad X_n = (X_{n,1} \ldots X_{n,M})^t$$

Cumulative Returns

Return from period 1 to N:

$$
\begin{aligned}
S_N &= \frac{h_{N+1}}{h_1} \\
&= \left(\frac{h_{N+1}}{h_N}\right)\left(\frac{h_N}{h_{N-1}}\right)\cdots\left(\frac{h_2}{h_1}\right) \\
&= \prod_{n=1}^{N} R_n \\
&= \prod_{n=1}^{N} b_n^t X_n
\end{aligned}
$$

Doubling Rate

Assuming that $b_n^t X_n > 0$ for $n = 1, \ldots, N$:

$$\begin{aligned} S_N &= \exp\left(\sum_{n=1}^{N} \log b_n^t X_n\right) \\ &= \exp\left(N W_N\right) \end{aligned}$$

where

$$W_N = \frac{1}{N}\sum_{n=1}^{N} \log b_n^t X_n$$

is called the “doubling” rate of the portfolio.

Constant Rebalanced Portfolios

- Constant Rebalanced Portfolio (CRP):

$$b_1 = b_2 = \cdots = b_N = b$$

- Buy-and-Hold (B&H) Portfolio:

$$b = e_m = (0, 0, \ldots, 0, 1, 0, \ldots, 0)^t$$

- Best CRP:

$$b_N^* = \underset{b \in B}{\text{argmax}}\, S_N(b) = \underset{b \in B}{\text{argmax}}\, W_N(b)$$

where

$$B = \{b \in \mathcal{R}^M : b_m \geq 0, m = 1, 2, \ldots, M; \textstyle\sum_{m=1}^M b_m = 1\}$$

CRP Example [Blum97]

- Two Stocks:
  - Stock #1: $X_{n,1} = 1$, $n = 1, \ldots, N$
  - Stock #2: $X_{n,2} = \begin{cases} \frac{1}{2} & n = \text{even} \\ 2 & n = \text{odd} \end{cases}$
- Portfolios:
  - $S_N \leq 2$ for any B&H portfolio
  - Return of $b = (\frac{1}{2}, \frac{1}{2})^t$ increases exponentially by $9/8$ every two days:

$$R_n = \begin{cases} \left(\frac{1}{2}\right)(1) + \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{3}{4} & n = \text{even} \\ \left(\frac{1}{2}\right)(1) + \left(\frac{1}{2}\right)(2) = \frac{3}{2} & n = \text{odd} \end{cases}$$

Equally Weighted CRP [Wang00]

- Equally weighted CRP: $b = (\frac{1}{M}, \cdots, \frac{1}{M})^t$
- Assume X_n are IID random vectors with $E[X_n] = \bar{X}$ and $V = E[(X_n - E(X_n))(X_n - E(X_n))^t]$:
  - Expected return: $E(R_n) = \frac{1}{M} \sum_{m=1}^{M} \bar{X}_m$
  - Variance of return:

$$
\begin{aligned}
\text{var}(R_n) &= \frac{1}{M} \left(\frac{1}{M} \sum_{m=1}^{M} v_{m,m} \right) \\
&\quad + \left(\frac{M^2 - M}{M^2} \right) \left(\frac{1}{M^2 - M} \sum_{m=1}^{M} \sum_{k \neq m}^{M} v_{m,k} \right) \\
&= \left(\frac{1}{M} \right) (\text{avg. var.}) + \left(\frac{M^2 - M}{M^2} \right) (\text{avg. cov.})
\end{aligned}
$$

Universal Portfolios

- **Origin:** T. Cover, "Universal Portfolios", *Mathematical Finance*, 1991
- **Objective:** Track return of best CRP chosen after asset returns revealed (b_N^*)
- **Algorithm:** Return-weighted average of all CRPs
- **Performance:**
  - Yields average return of all CRPs
  - Return exceeds geometric mean of asset returns
  - Asymptotically “tracks” best CRP return

Universal Portfolio Algorithm

Return-weighted average of all CRPs:

Initialize: $\hat{b}_1 = \left(\frac{1}{M}, \cdots, \frac{1}{M}\right)^t$

For $n > 1$: $\hat{b}_n = \frac{\int_B bS_n(b)db}{\int_B S_n(b)db}$

where

$$B = \{b \in \mathcal{R}^M : b_m \geq 0, m = 1, 2, \ldots, M; \sum_{m=1}^{M} b_m = 1\}$$

Performance

Regardless of the stock return sequence:

- Portfolio return is the average of all CRPs' returns

$$\hat{S}_N = \prod_{n=1}^{N} \hat{b}_n^t X_n = \frac{\int_B S_n(b)db}{\int_B db}$$

- Return is greater than geometric mean

$$\hat{S}_N \geq \left(\prod_{m=1}^{M} S_N(e_m) \right)^{1/M}$$

Asymptotic Performance

Under many assumptions:

$$\frac{\hat{S}_N}{S_N^*} \sim \left(\sqrt{\frac{2\pi}{N}}\right)^{M-1} \frac{(M-1)!}{|J^*|^{1/2}}$$

where

$$\begin{aligned} J^* &= \text{Hessian of } W_N(b) \text{ at its maximum for large N} \\ S_N^* &= \text{return of best CRP } b_N^* \end{aligned}$$

and $\sim$ means the ratio of left- and right-hand sides equals one for large N

Assumptions for Asymptotic Perf.

The stock sequence $X_1, X_2, \ldots,$ satisfies:

- $a \le X_{n,m} \le q$, $m = 1, \ldots, M$,$n = 1, 2, \ldots$, for some $0 < a \le q < \infty$
- $J_N^* \to J^*$ for some positive definite matrix J^*
- $b_N^* \to b^*$ for some b^* in the interior of the B

and there exists a function $W(b)$ such that

- $W_N(b) \nearrow W(b)^*$ for any portfolio $b \in B$
- $W(b)$ is strictly concave
- $W(b)$ has bounded third partial derivatives
- $W(b)$ achieves its maximum at b^* in the interior of the B

Outline of Proof

- Show LHS $\geq$ RHS:
  - Expand $W_N(b)$ in Taylor series about the best CRP b_N^* to three terms:
    - First term is zero because gradient is zero at b_N^*
    - Second term is related to J_N^*
    - Third term is bounded
  - Manipulate expression to look like a Gaussian CDF
  - Bound the Gaussian CDF
- Show LHS $\leq$ RHS:
  - Follows from Laplace's method of integration

Mean-Variance Efficient Portfolios

- **Origins:** Markowitz (1952) and Sharpe (1963) (1990 Nobel Laureates)
- **Objective**: To minimize the variance of portfolio returns, while maintaining a desired average return
- **Algorithm**:
  - A solution to a constrained optimization problem
  - Requires estimates of asset return mean and covariance matrix
- **Performance:**
  - Tracks objective portfolio if asset returns are IID
  - Otherwise, no guarantee

Objective of MVE Portfolio

Minimize: $b_n^t V_n b_n$

Subject to: 1) $\sum_{m=1}^{M} b_{n,m} = 1$

2) $b_n^t \bar{X}_n = \bar{R}_n$

where

$\bar{X}_n$ = estimate of asset return mean

V_n = estimate of asset return covariance matrix

$\bar{R}_n$ = desired average return

The MVE Portfolio

- Mean and covariance estimators:

$$\bar{X}_n = \frac{1}{\Lambda}\sum_{k=1}^{n-1}\lambda^{n-k-1}X_k$$

$$V_n = \frac{1}{\Lambda}\sum_{k=1}^{n-1}\lambda^{n-k-1}(X_k - \bar{X}_n)(X_k - \bar{X}_n)^t$$

  where $0 < \lambda \leq 1$ and $\Lambda = \sum_{k=1}^{n-1}\lambda^{n-k-1}$

- The MVE Portfolio:

$$b_n = w_0(1 - \bar{R}_n) + w_1\bar{R}_n$$

  where w_0 and w_1 are portfolios that yield on average 0% and 100% returns, respectively

The 0% and 100% Avg. Ret. Ports.

$$
\begin{aligned}
w_0 &= \frac{1}{D}\left(c_1 V_n^{-1} 1 - c_2 V_n^{-1} \bar{X}_n\right) \\
w_1 &= \frac{1}{D}\left(c_1 V_n^{-1} 1 - c_2 V_n^{-1} \bar{X}_n\right) + \frac{1}{D}\left(c_3 V_n^{-1} \bar{X}_n - c_2 V_n^{-1} 1\right)
\end{aligned}
$$

where

$$
\begin{aligned}
1^t &= (1, \ldots, 1) \\
c_1 &= \bar{X}_n^t V_n^{-1} \bar{X}_n \\
c_2 &= \bar{X}_n^t V_n^{-1} 1 \\
c_3 &= 1^t V_n^{-1} 1 \\
D &= c_1 c_3 - c_2^2
\end{aligned}
$$

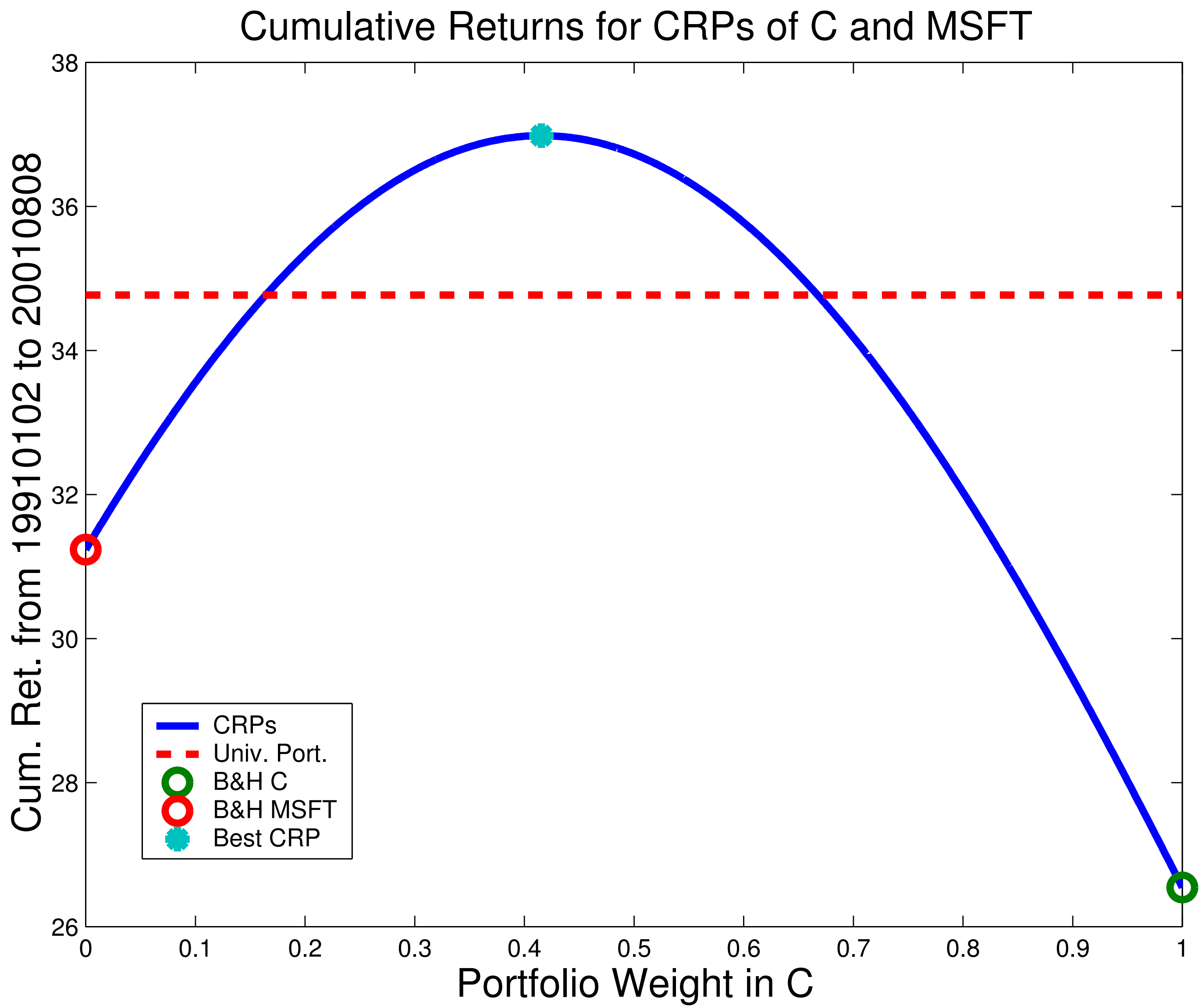
Performance

- If asset returns are IID then:
  - Estimators converge to their true values by LLN
  - MVE portfolio asymptotically achieves objective
- Otherwise, no guarantee

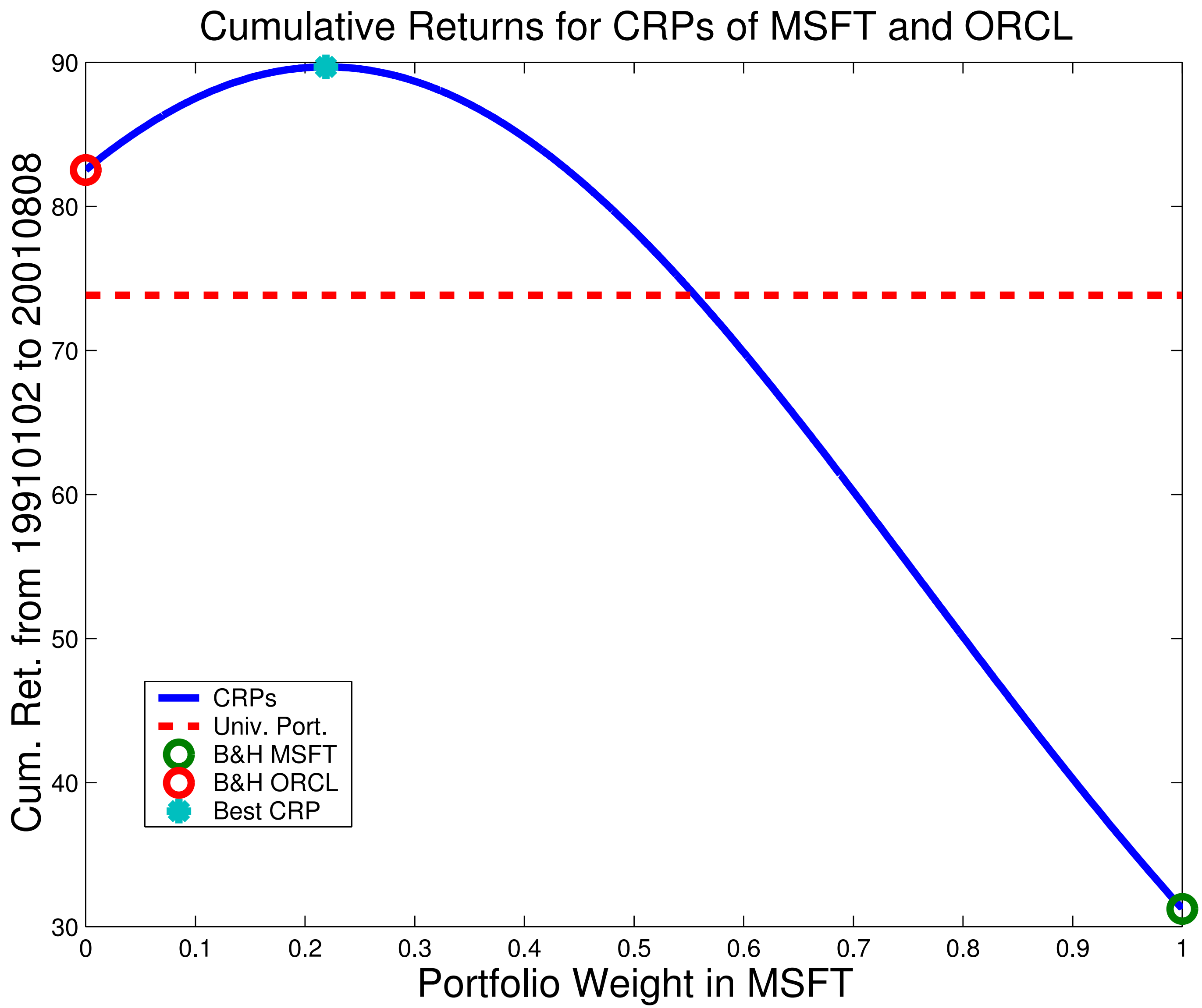
Examples

- Data:
  - Daily adjusted closing prices (yahoo.finance.com)
  - January 2,1991 to August 8, 2001 (2,679 days)
  - Missing data interpolated
- Examples:
  - Universal and constant rebalanced portfolios
  - Mean-Variance efficient portfolios
- Comparison:
  - MVE portfolio allows short sales, but univ. portfolio does not
  - No transaction costs
  - Investor preferences

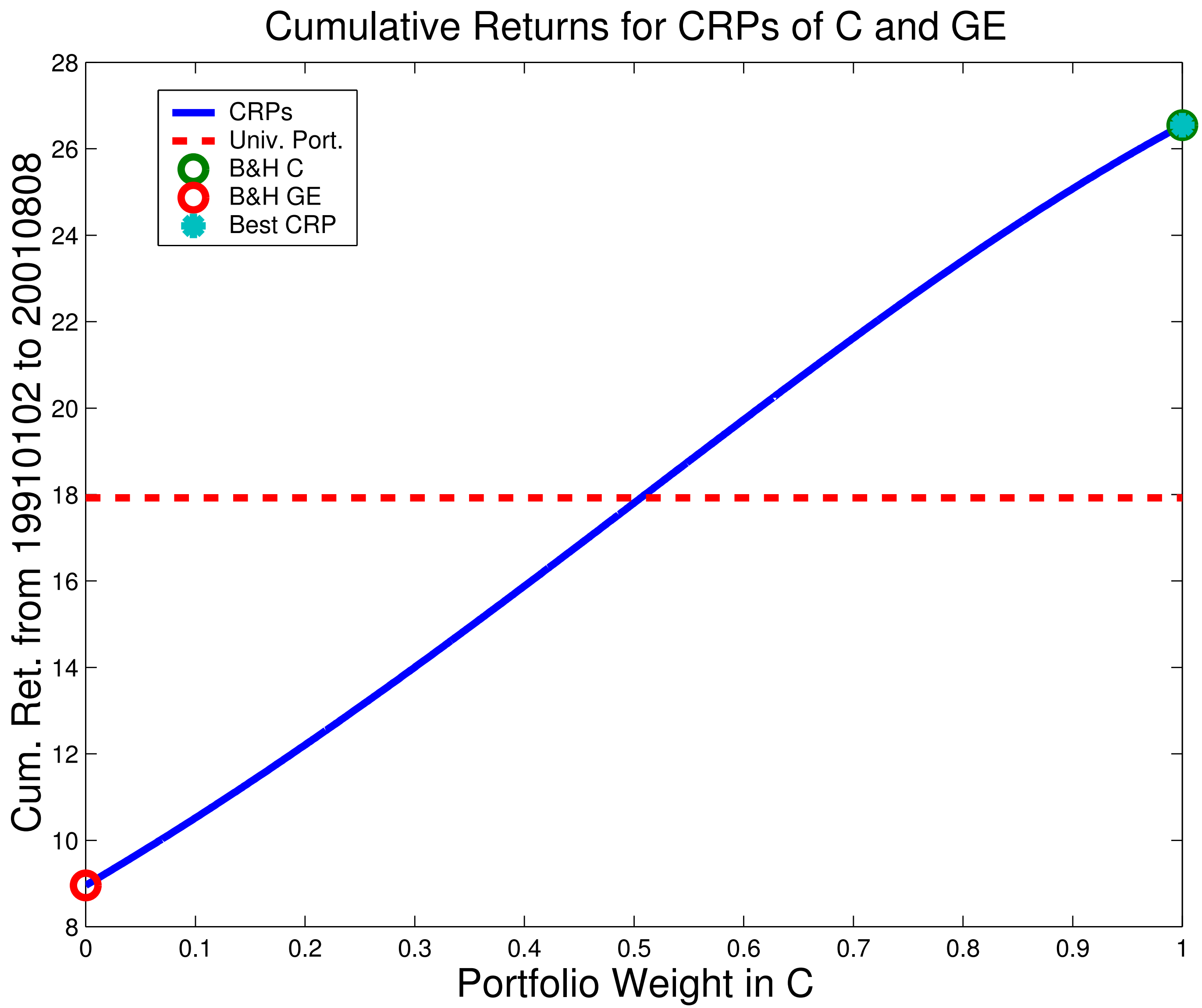
Univ. and CRPs of C & MSFT



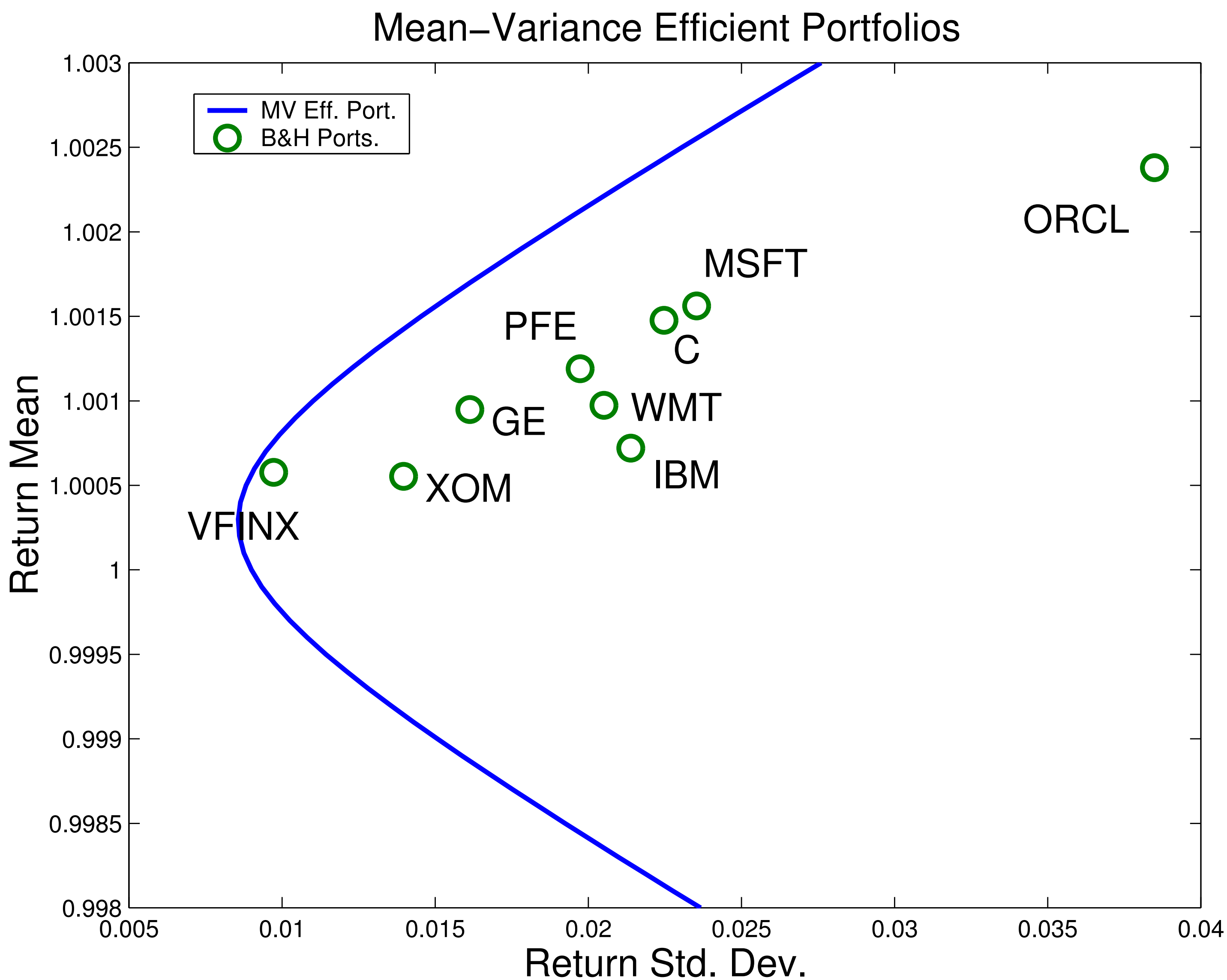
Univ. and CRPs of MSFT & ORCL



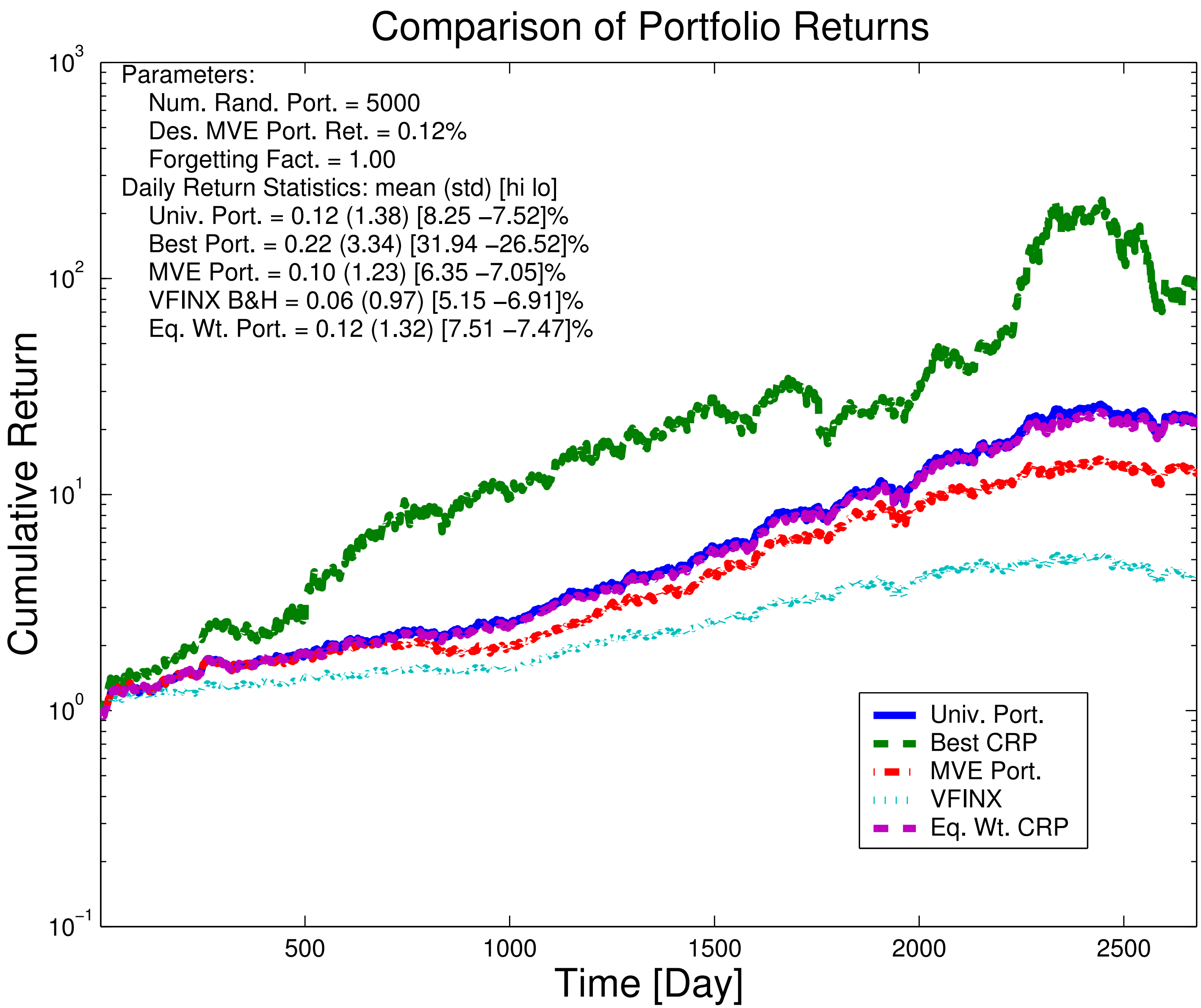
Univ. and CRPs of C & GE



MVE Portfolios



Comparison



Comparison of Portfolio Returns
Parameters:
Num. Rand. Port. = 5000
Des. MVE Port. Ret. = 0.12%
Forgetting Fact. = 1.00
Daily Return Statistics: mean (std) [hi lo]
Univ. Port. = 0.12 (1.38) [8.25 –7.52]%
Best Port. = 0.22 (3.34) [31.94 –26.52]%
MVE Port. = 0.10 (1.23) [6.35 –7.05]%
VFINX B&H = 0.06 (0.97) [5.15 –6.91]%
Eq. Wt. Port. = 0.12 (1.32) [7.51 –7.47]%
Univ. Port.
Best CRP
MVE Port.
VFINX
Eq. Wt. CRP
Cumulative Return
Time [Day]
10^{3}
10^{2}
10^{1}
10^{0}
10^{-1}
500
1000
1500
2000
2500

Conclusions

- Universal portfolio:
  - Attempts to track return of best CRP given knowledge of future asset returns
  - Yields average return of all CRPs
  - Return greater than geometric mean
  - Asymptotically “tracks” return of best CRP
- MVE portfolio:
  - Attempts to minimize return variance while maintaining a desired average return
  - Obtains objective in IID markets