The Effects of Monopolistic Pricing in a Network Resource Allocation Game

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1 Introduction

There is a recent body of literature that seeks to quantify the performance degradation of networks in the absence of centralized control via game theoretic techniques. Users are considered as selfish agents participating in a non-cooperative game who choose routes consistent with the notion of a Nash Equilibrium. As we saw last week [Khi04], results by Johari and Tsitsilikis have shown that equilibrium performance degrades to ¾ of the social optimum, which could be achieved with centralized planning [JoT03]. Other results by Roughgarden and Tardos [RoT00] show that if the latency of each edge is a linear function of its congestion, the total latency of the routes chosen by selfish network users is at most 4/3 times the minimum possible total latency. In the more general case where edge latency is assumed to be continuous and non-decreasing in the edge congestion, they also prove that latency is no more than the total latency incurred by optimally routing twice as much traffic.

While the former models captures the decentralized routing decisions of users, and the effects of those decisions on performance; the literature neglects that the networks are controlled by service providers whose objective isn't operating at an efficient social optimum, but rather is to maximize profits. The practically motivated nuance incorporated in the work of D. Acemoglu and A. Ozdaglar is the consideration of the effect of monopolistic pricing on congestion in unregulated networks. In "Flow Control, Routing, and Performance from Service Provider Viewpoint," the authors consider a two staged game where the single profit maximizing service provider sets prices anticipating the subsequent behavior of users, and then the users – taking prices and latency as given – game selfishly to optimize their own objectives. The authors develop results on the equilibrium behavior of such a network, characterize the equilibrium prices and develop insights into why monopoly pricing may improve the performance of the system. They also compare the performance of the Monopoly Equilibrium to the social optimum (full information and zero prices).

In the summary which follows, we first outline the network model in section 2. In Section 3, we characterize the Network Equilibrium. In Section 4, we show the derivation of the optimal monopoly pricing scheme and explain their effects on congestion. In Section 5 we compare the performance of the network under monopolized pricing to the performance in the centrally controlled, zero priced social optimum. We conclude with a summary of the key contributions of this work.

2. Network and Game Theoretic Model

The network is modeled as having I parallel links, and J users, where J >> 1. The following parameters are used to describe the network and users:

x_j^i	the flow of user j on link i
$\Gamma_j = \sum_{i=1}^{I} x_j^i$	the total flow rate of user j on all links
$\gamma^i = \sum_{j=1}^{J} x_j^i$	total flow on link i
$l^i(\gamma^i)$	flow dependent latency function which specifies the delays in transmission as a function of the link load. -Continuous and strictly increasing -$l^i(0) = 0$
$l^i(x) \to \infty \text{ as } x \to C^i$	Capacity constraint on the links
$u_j(\Gamma_j)$:	utility function
p^i	price of link i

The utility function is assumed to be strictly concave – that is, that we are considering elastic traffic (delay tolerant) where increased data rates has diminishing returns for users. We also assume that the utility function is non-decreasing, and continuously differentiable which allows the authors to characterize the equilibrium, and the equilibrium prices (shown here in sections 3 and 4).

For each user there is also an associated payoff function which depends not only on that user's own flows, but also on the flows of all the other users via the latency they all create on a given link.. Each user acts as a "price taker" and a "link load taker," taking prices and link loads as given and choosing flows to maximize their payoff function:

$$v_j(x_j;\lambda,p) = u_j\left(\sum_{i=1}^{I} x_j^i\right) - \sum_{i=1}^{I} l^i(\gamma^i)x_j^i - \sum_{i=1}^{I} p^i x_j^i$$

3. <u>Network Equilibrium</u>

To determine the equilibrium behavior if this network, we consider the game between selfish users who, given prices, anticipate the amount of congestion on all routes and choose their own routes according to the notion of the Nash Equilibrium (NE). Recall that a Nash Equilibrium is a steady state of the play of a strategic game in which no player can profitably deviate given the actions of the other players. More formally:

$$\text{where } a^* \in A \text{ of actions,} \quad (a_{-i}^*, a_i^*) \geq (a_{-i}^*, a_i) \qquad \forall a_i \in A_i$$

No player i has an action yielding an outcome that he prefers to that generated when he chooses a_i^*, given that every other player j chooses his equilibrium action a_j^*.

In the model described above, we assumed that the users were both price and latency takers – that is, they take the latency of the links as given. In the NE, users take into account the effect of their own flow on congestion. In the Wardrop Equilibrium (WE), each user's flow is assumed to be small relative the total flow on a given link so that when that user switches his flow, there is no substantial change in the link latencies (Wardrop Principle). As such, in the WE, users ignore the effects of their own flow on congestion. Formally, the WE says that a vector $r_l, l = 1,2,\ldots\ldots,L$ is called a Wardrop equilibrium if for each group l, the following holds:

> Let $K_*^l \subseteq K$ be the set of routes actually used, that is the set of classes such that:
> $\forall k \in K^l : k \in K_*^l \leftrightarrow p_{l,k} > 0$ such that $B_k(r) \le B_{k'}(r) \qquad \forall k \in K_*^l,\ k' \in K^l$

So the principles are fulfilled because in this state no group can reduce the costs by switching from its current paths to other ones connecting the same origin-destination pair.

Following HaM85, the authors prove that as the number of users becomes large, the NE converges to the WE (see pp 17-9). And so in this work, the equilibrium of the network is characterized in terms of the WE. For the model described above, the Generalized Wardrop Equilibrium GWE -- generalized because it is conditioned on a given vector of prices -- is defined as the vector of flows of all the users $x = \left[x_1',\ldots\ldots\ldots,x_J'\right]$ in the network such that for each x_j:

$$x_j = \arg\max_{0 \le y_j^i \le C_i,\ \forall i} \quad v_j(y_j;\gamma,p), \qquad \forall j \in J$$

The existence and the uniqueness of the GWE is proved by considering the optimization problem:

$$\text{maximize} \sum u_j(\Gamma_j) - \sum_{i \in I} \int_0^{\gamma^i} l^i(z)\,dz - \sum_{i \in I} p^i \gamma^i$$

$$\text{subject to:} \quad \Gamma_j = \sum_{i \in I} x_j^i, \quad \forall j$$

$$\gamma^i = \sum_{j \in J} x_j^i \quad \forall i$$

$$x_j^i \ge 0 \qquad \forall i, j$$

The proposition is proved as follows: Since the objective function is continuous and the feasible set is compact, this problem has an optimal solution (existence of the GWE). Since u_j is strictly concave, and l^i is strictly increasing – the objective function is a strictly concave function of Γ and γ (uniqueness of the flow rates and link loads at a GWE).

Using the essential uniqueness of the equilibrium proved above, the authors are able to prove the continuity and monotonicity of the flow rates and link loads which are used later on in the characterization of monopoly prices. The monotonicity results are quite intuitive. Flow rates being monotonic in prices says essentially that when prices are higher, users choose flow rates that are no higher than the rates chosen at a lower price. Similarly, the monotonicity of link loads (Prop 2.4) in prices says formally link loads are monotonically non-increasing in their own prices, and monotonically non-decreasing in other links' prices. Intuitively, the monotonicity of link loads tells us that higher prices reduce traffic on the link. And, a higher prices for one link increases the traffic on the others (implying that for users, links are perfect substitutes).

The authors go on to define two lemmas which will be useful to in the later characterization of the GWE. The first says that the effective cost of all links with positive flow must be equal, supporting the intuitive notion from the monotinicity of link loads that the links are perfect substitutes. The second characterizes the price and proves the existence of a GWE and the corresponding flows at that price vector.

<u>Lemma 2.1</u>: For a given $p \geq 0$, let γ^i be the load of link i at a GWE. Then for all i with $\gamma^i \geq 0$:

$$p^i + l^i(\gamma^i) = \min_{m \in I}\{p^m + l^m(\gamma^m)\}$$

Proof:
By the first order necessary optimality conditions,

$$u_j^i(\Gamma_j(p)) - l^i(\gamma^i(p)) - p^i \leq 0 \qquad if\ x_j^i \geq 0 \qquad eqn\,9$$
$$u_j^i(\Gamma_j(p)) - l^i(\gamma^i(p)) - p^i = 0 \qquad if\ x_j^i > 0 \qquad eqn\,10$$

so there exists some j such that $x_j^i \geq 0$ and satisfies eqn 10. Since eqn 9 also holds $\forall m$, combining eqn 9 and eqn 10 yields:

$$p^i + l^i(\gamma^i) \leq p^m + l^m(\gamma^m) \qquad \forall m$$

<u>Lemma 2.2</u>: For a given $p \geq 0$, let x be a GWE. Let $\bar{I} = \{i \mid \gamma^i > 0\}$ and $\bar{J} = \{j \mid \Gamma^j > 0\}$, then:

(a) For all $i \in \bar{I}$ and $j \in \bar{J}$, : $u_j(\Gamma_j) - l^i(\gamma^i) - p^i = 0$

(b) There exists a GWE $\tilde{x}$ at this price vector that satisfies $x_j^i > 0$ for all $i \in \bar{I}$ and $j \in \bar{J}$.

Proof:

(a) Since $\Gamma_j > 0$, then there exists some link s for which $x_j^s > 0$, and thus eqn 11 holds. By Lemma 2.1, we have that $p^i + l^i(\gamma^i) = p^m + l^m(\gamma^m)$ since $\gamma^i > 0$ and $\gamma^s > 0$. Substituting this into equation 11 proves part (a).

(b) The existence of a GWE at this price vector is established by the argument that starting at any GWE, we can always construct an alternative GWE with the same individual flow rates and link loads, in which all individual flows to all links with minimum effective cost are positive. As such, this alternative vector x satisfies the first order necessary and sufficient optimality conditions and is thus a GWE at the price vector p.

4. Monopolistic Pricing

The profit maximizing monopolists sets prices on each link of the network with the vector of prices which solve the following maximization problem:

$$\max \quad \sum_{i \in I} p^i \gamma^i(p)$$
$$\text{subject to: } p \geq 0$$

The existence of this optimal price vector follows from the continuity in p of $\gamma^i(p)$. The vector p* is termed the *monopoly equilibrium* price of the 2-stage game. The pair (p*, x*) defines as the monopoly equilibrium of the overall game.

Some characteristics of this optimal price vector are (proofs omitted):

1. $p^i > 0$
   otherwise, revenues would be decreased on that link, and reduce traffic and thus profits on all other links.
2. $\gamma^i > 0$
   otherwise, profit could be increased by adding traffic on this link; and this increase in profits outweighs the decrease in profits on the other links (see page 22 for the proof)

To help in the characterization of the equilibrium prices, establish Lemma 4.3 which shows the existence of an optimal solution (p, λ, Γ_j) to the following problem where (p,x) is an ME:

eqn 11

$$\max \sum_{i \in I} p^i \gamma^i$$

$$\text{subject to} \quad u_i^{'}(\Gamma_1) - l^i(\gamma^i) - p^i = 0 \qquad \forall i \in I$$
$$u_j^{i}(\Gamma_j) - l^1(\gamma^1) - p^1 = 0 \qquad \forall j \in J - \{1\}$$
$$\sum_{i \in I} \gamma^i = \sum_{j \in J} \Gamma_j$$

$$\gamma^i \geq 0, \Gamma_j \geq 0 \qquad \forall i \in I, j \in J$$

This Lemma is proved by contradiction, and shows that the existence of some other feasible solution with a greater cost violates the notion of the solution (p, λ, Γ_j) corresponding to a ME (p, x). (See pages 23-4 for the detailed proof).

One of the key results of this paper is the characterization of Equilibrium prices in Prop 4.8. We will first outline the proof, then discuss its implications. We assume that $u_j(*)$ is twice differentiable for each j, and l^i is continuously differentiable for each i, and (p,x) is an ME, and $\bar{J} = \{j \mid \Gamma_j > 0\}$. Then for $i \in I$ and, we have:

$$p^i = (l^i)'(\gamma^i)\gamma^i + \frac{\sum_{m\in I}\gamma^m}{-\sum_{j\in J}\frac{1}{u_j''(\Gamma_j)}} \qquad \text{eqn 12}$$

The proof of this Proposition uses Lagrange Multipliers to solve equation 11. From Lemma 4.3 (p, γ, Γ_j) is an optimal solution to this problem. The Lagrangian function for this problem assigns multipliers $\lambda_i \in I, \mu_j \in J/1, \mu_1$ to each of the constraints respectively:

$$L(p,x,\lambda) = \sum_{i\in I} p^i\gamma^i - \sum_{i\in I}\lambda^i(u_i'(\Gamma_1) - l^i(\gamma^i) - p^i) + \sum_{j\in J-1}\mu_j(u_j'(\Gamma_j) - l^1(\gamma^1) - p^1) + \mu_1(\sum_{i\in I}\gamma^i - \sum_{j\in J}\Gamma_j)$$

Taking partial derivatives with respect p^i *and* λ^i yield the following equations:

$$\gamma^1 = \lambda^1 + \sum_{j\in J-1}\mu_j \qquad \text{eqn 13}$$

$$\gamma^i = \lambda^i \qquad \forall i \neq 1 \qquad \text{eqn 14}$$

$$p^1 - \lambda^1(l^1)'(\gamma^1) - (\sum_{j\in J-1}u_j)(l^1)'(\gamma^1) + \mu_1 = 0 \qquad \forall i \neq 1 \qquad \text{eqn 15}$$

$$p^i - \lambda^i(l^i)'(\gamma^i) + u_1 = 0 \qquad \forall i \neq 1 \qquad \text{eqn 16}$$

When combined, they reduce to :

$$p^i - (l^i)'(\gamma^i)\gamma^i + \mu_1 = 0, \qquad \forall i \in I$$

Taking the partial derivative with respect to Γ yields:

$$u_1''(\Gamma_1)\sum\lambda^i - \mu_1 = 0 \qquad \text{eqn 17}$$

$$\mu_j\mu_j''(\Gamma_j) - \mu_1 = 0, \qquad \forall j \in J-1 \quad \text{eqn 18}$$

Summing eqn 13 and eqn 14 overall i, and combining these with eqn 18 and eqn 19 yields:

When these are substituted in to eqn 16, we arrive at eqn 12.

Equation 12 shows that the profit maximizing prices for the monopolist consists of two markups above the marginal cost[1] of the monopolist (which is zero in this model). Its this first term $(l^i)'(\gamma)\gamma^i$ which *may* work to improve the allocation of resources. Intuitively, since the monopolist is aware that the user payoff function decreases with both latency and price, this term can be interpreted as the monopolist's intervention to reduce congestion. The second term in the equation for the optimal monopoly pricing is similar to the standard markup term in the analysis of monopoly pricing in its consideration of the elasticity of the utility function of users (which can be translated into a demand curve)[2]. This term in standard economic analysis, is the source of inefficiencies, raising cost and lowering consumption below the social optimum.

5 Performance of Network under Monopolized Pricing versus Social Optimum

In the social problem, a network manager with centralized routing control and perfect knowledge of the user utility functions and latency on the links would route traffic according to the optimal solution of the following problem:

$$\begin{aligned} &\max \sum_{j\in J} u_j(\Gamma_j) - \sum_{i\in I} l^i(\gamma^i)\gamma^i \\ &\text{subject to: } \Gamma_j = \sum_{i\in I} x_j^i, \qquad \forall j\in J \\ &\gamma^i = \sum_{i\in I} x_j^i, \qquad \forall i\in I \\ &x_j^i \geq 0, \qquad \forall i\in I, j\in J \end{aligned}$$

Assuming the objective problem is concave, an equivalent characterization of the social problem is given by the first order conditions:

$$u_i'(\Gamma_1) - l^i(\gamma^i) - (l^i)'(\gamma^i)\gamma^i \begin{cases} \leq 0 & if\ x_j^i \geq 0 \\ = 0 & if\ x_j^i = 0 \end{cases}$$

Recalling the first order optimality conditions in eqn 10, it can been seen that if each user is charged $(l^i)'(\gamma)\gamma^i$ in the case where routing is decentralized, the allocation in that GWE will be equivalent to that of the social problem. The authors interpret this as the 'marginal congestion cost' which is the price charged to the users to induce less selfish behavior, modifying users' routing choices and flow selections. In Economics parlance, the 'marginal congestion cost' encourages the users to internalize the congestion externalities which they ignore when acting

[1] In a competitive market, firms set prices to equal the marginal cost at some level of output y. (See Varian, Chapter 21).
[2] See Varian C23.3

selfishly. Implicit in most economic models is the notion that agents make decisions without worrying about what the other agents are doing. All interaction takes place via the market, so that the agents only need to know the market prices – and it is in this case of the absence of externalizes, that a given market is capable of achieving efficient allocations. The internalization of an externality corresponds to some level of coordination which should serve the interests of the agents involved moving the market toward efficient allocations.[3]

The authors are then able to show the correspondence between user j's total flow rate Γ_j under the three control/pricing schemes discussed above: the monopolized pricing scheme w/ decentralized control, the GWE with zero prices, and the social optimum with centralized control. In Proposition 5.9, the authors prove (omitted here, see pp35-8) that where $\bar{x}$ corresponds to the GWE at 0 prices, $\tilde{x}$ corresponds to the social optimium, and x denotes the ME, For all j:

$$\Gamma_j \le \tilde{\Gamma}_j \le \bar{\Gamma}_j$$

This shows that when the prices are equal to 0, the users generate too much flow relative to the social optimum – consistent with the results from last week, and mentioned in the introduction. Monopoly pricing improves this behavior, but it may introduce distortion relative to the social optimum due to the flow control decisions of the users as a result of the monopolistic pricing. In the case of the routing problem where the user has a fixed amount of traffic to send, the authors demonstrate that ME achieves the social optimum (see 29-33).

6. Conclusions

Among the key contributions of this work has been the development of a more realistic framework within which the decentralized routing choices of selfish users as well as the profit maximizing pricing schemes of service providers can jointly be analyzed to characterize performance of these networks, and to develop methods to improve their efficiency. After developing that framework, Ozdaglar and Acemoglu then characterize the equilibrium behavior of such a network, and also characterize the monopoly prices, and the flow rates and link loads which result. In agreement with earlier results, when network equilibrium is analyzed without pricing, that network is proven to be inefficient relative to the social optimum. A key insight developed in their work is that monopolistic pricing methods produce an equilibrium that reduces flows below the social optimum, but which in some cases (routing problem) may achieve an allocation that is socially optimal.

[3] See Varian (C31.5)

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