# Cooperative Strategies and Capacity Theorems for Relay Networks

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# What are relay networks?

- Relay network:
  - multi-terminal network;
  - single pair of terminals wish to communicate (source and destination);
  - all other terminals assist (relays).
- Special case of single relay $\longrightarrow$ relay channel.
- Relay channel:
  - first studied by van der Meulen (1971);
  - capacity of the relay channel is an open problem.

# What problem is addressed?

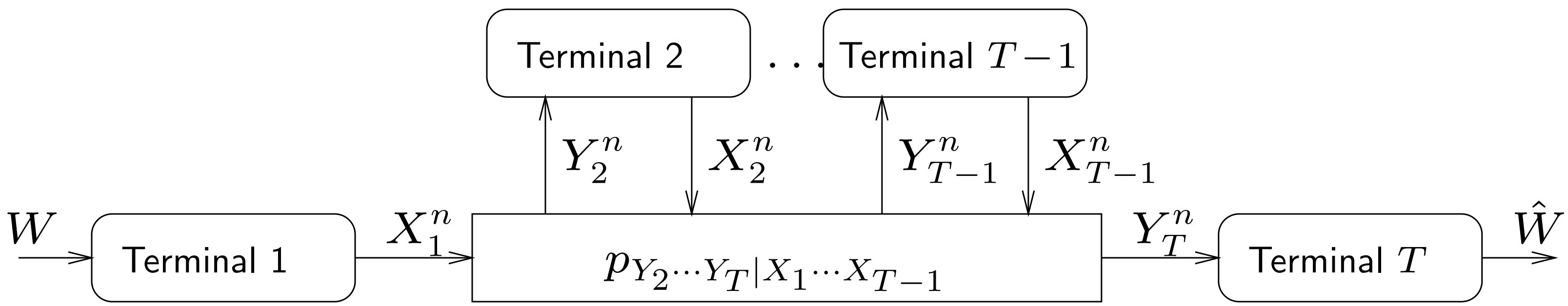
- We have little hope of finding the capacity of relay networks.

- So we focus on achievable rates $\longrightarrow$ design particular schemes and assess the rates they achieve.

- Schemes:
  - decode-and-forward (related to multi-antenna transmission);
  - compress-and-forward (related to multi-antenna reception);
  - mixtures of the two.

# Model

- $T$ terminals: source is terminal 1, destination is terminal $T$.
- Network is memoryless and time-invariant.



# Upper bound on capacity

- Can get upper bound from cut-set bound for multi-terminal networks:

$$C \leq \max_{p_{X_1 X_2 \cdots X_{T-1}}} \min_{\mathcal{S} \subset \mathcal{T}} I(X_1 X_{\mathcal{S}}; Y_{\mathcal{S}^c} Y_T | X_{\mathcal{S}^c})$$

- For relay channel:

$$C \leq \max_{p_{X_1 X_2}} \min(I(X_1; Y_2 Y_3 | X_2), I(X_1 X_2; Y_3)).$$

# Decode-and-forward

- Relays fully decode message and use knowledge of the message to assist the source.
- In a wireless setting: achieves the gains related to multi-antenna transmission.
- Scheme we discuss is due to Xie and Kumar.

# Decode-and-forward: Single relay

- Divide message $w$ into $B$ blocks $w_1, w_2, \ldots, w_B$ of $nR$ bits each, where

$$R < \max_{p_{X_1 X_2}} \min(I(X_1; Y_2 | X_2), I(X_1 X_2; Y_3)).$$

- Transmission is performed in $B+1$ blocks using random codewords of length $n$.

- Rate is

$$\frac{B \cdot nR}{(B+1)n} = R\frac{B}{B+1} \text{ bits per use.}$$

Arbitrarily close to $R$ for $B$ arbitrarily large.

# Decode-and-forward: Single relay

- Code construction:
  - Take joint distribution $p_{X_1 X_2}$.
  - For block $b$:
    - Generate $2^{nR}$ codewords $x_{2b}^n(v)$, choosing symbols $\{x_{2bi}(v)\}$ independently using $p_{X_2}$.
    - Generate $2^{nR}$ codewords $x_{1b}^n(v, w)$, choosing symbols $\{x_{1bi}(v, w)\}$ independently using $\{p_{X_1|X_2}(\cdot|x_{2bi}(v))\}$.

# Decode-and-forward: Single relay

- $B = 3$:

| | Block 1 | Block 2 | Block 3 | Block 4 |
|---|---|---|---|---|
| $X_1^n =$ | $x_{11}^n(1, w_1)$ | $x_{12}^n(w_1, w_2)$ | $x_{13}^n(w_2, w_3)$ | $x_{14}^n(w_3, 1)$ |
| $X_2^n =$ | $x_{21}^n(1)$ | $x_{22}^n(w_1)$ | $x_{23}^n(w_2)$ | $x_{24}^n(w_3)$ |

- After transmission of block $b$,
  - relay decodes $w_b$;
  - destination decodes $w_{b-1}$.

# Decode-and-forward: Single relay

- Relay decodes $w_b$ reliably if $n$ is large, $\hat{w}_{b-1}^{(2)} = w_{b-1}$, and

$$R < I(X_1; Y_2 | X_2).$$

- Destination decodes $w_b$ reliably if $n$ is large, $\hat{w}_{b-1}^{(3)} = w_{b-1}$, and

$$R < I(X_1; Y_3 | X_2) + I(X_2; Y_3) = I(X_1 X_2; Y_3).$$

- There exists a distribution $p_{X_1 X_2}$ that satisfies both conditions by assumption.

# Decode-and-forward: Multiple relays

- Consider two relays.
- Divide message $w$ into $B$ blocks $w_1, w_2, \ldots, w_B$ of $nR$ bits each, where

$$R < \max_{p_{X_1X_2X_3}} \min(I(X_1;Y_2|X_2X_3), I(X_1X_2;Y_3|X_3), I(X_1X_2X_3;Y_4)).$$

- Transmission is performed in $B+2$ blocks using random codewords of length $n$.
- Rate is

$$\frac{B \cdot nR}{(B+2)n} = R\frac{B}{B+2} \text{ bits per use.}$$

# Decode-and-forward: Multiple relays

- $B = 4$:

| | Block 1 | Block 2 | Block 3 |
|---|---|---|---|
| $X_1^n =$ | $x_{11}^n(1, 1, w_1)$ | $x_{12}^n(1, w_1, w_2)$ | $x_{13}^n(w_1, w_2, w_3)$ |
| $X_2^n =$ | $x_{21}^n(1, 1)$ | $x_{22}^n(1, w_1)$ | $x_{23}^n(w_1, w_2)$ |
| $X_3^n =$ | $x_{31}^n(1)$ | $x_{32}^n(1)$ | $x_{33}^n(w_1)$ |

- After transmission of block $b$, terminal 2 decodes $w_b$, terminal 3 decodes $w_{b-1}$, destination decodes $w_{b-2}$.

# Decode-and-forward: Multiple relays

- $B = 4$:

| | Block 4 | Block 5 | Block 6 |
|---|---|---|---|
| $X_1^n =$ | $x_{14}^n(w_2, w_3, w_4)$ | $x_{15}^n(w_3, w_4, 1)$ | $x_{16}^n(w_4, 1, 1)$ |
| $X_2^n =$ | $x_{24}^n(w_2, w_3)$ | $x_{25}^n(w_3, w_4)$ | $x_{26}^n(w_4, 1)$ |
| $X_3^n =$ | $x_{34}^n(w_2)$ | $x_{35}^n(w_3)$ | $x_{36}^n(w_4)$ |

- After transmission of block $b$, terminal 2 decodes $w_b$, terminal 3 decodes $w_{b-1}$, destination decodes $w_{b-2}$.

# Decode-and-forward: Multiple relays

- Terminal 2 decodes $w_b$ reliably if $n$ is large, $\hat{w}_{b-2}^{(2)} = w_{b-2}$ and $\hat{w}_{b-1}^{(2)} = w_{b-1}$, and

$$R < I(X_1; Y_2 | X_2 X_3).$$

- Terminal 3 decodes $w_b$ reliably if $n$ is large, $\hat{w}_{b-2}^{(3)} = w_{b-2}$ and $\hat{w}_{b-1}^{(3)} = w_{b-1}$, and

$$R < I(X_1 X_2; Y_3 | X_3).$$

- Destination decodes $w_b$ reliably if $n$ is large, $\hat{w}_{b-2}^{(4)} = w_{b-2}$ and $\hat{w}_{b-1}^{(4)} = w_{b-1}$, and

$$R < I(X_1 X_2 X_3; Y_4).$$

- There exists a distribution $p_{X_1 X_2 X_3}$ that satisfies all three conditions by assumption.

# Decode-and-forward: Multiple relays

- Straightforward to generalize scheme to $T$-terminal relay networks.

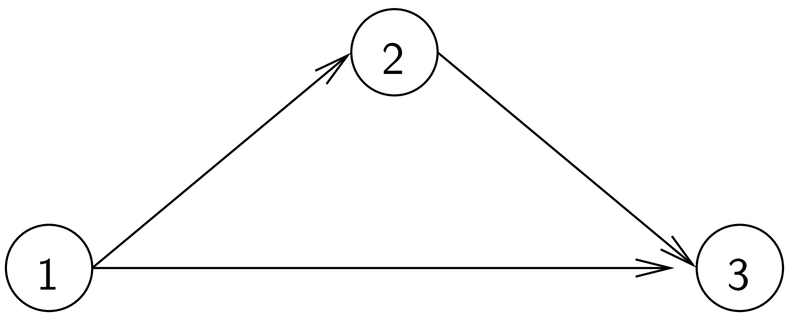
- **Theorem 1.** *Decode-and-forward achieves any rate up to*

$$R_{DF} = \max_{p_{X_1 X_2 \cdots X_{T-1}}} \max_{\pi} \min_{1 \leq t \leq T-1} I(X_{\pi(1:t)}; Y_{\pi(t+1)} | X_{\pi(t+1:T-1)}).$$

- $\pi$ is a permutation on $T$ with $\pi(1) := 1$ and $\pi(T) := T$.

# Decode-and-forward: Sub-optimality

- Requiring the relays to decode can be a severe constraint.
- Consider




Links are independent with unit capacity.

- Capacity is clearly 2 bits per use, but decode-and-forward only achieves 1 bit per use.

# Compress-and-forward

- Relays do not decode message and, rather, forward compressed versions of their observations.

- In a wireless setting: achieves the gains related to multi-antenna reception.

- Scheme we discuss is due to Cover and El Gamal (1979) for the single relay network and Kramer et al. for the multiple relay network.

# Compress-and-forward: Single relay

- Divide message $w$ into $B$ blocks $w_1, w_2, \ldots, w_B$ of $nR$ bits each, where

$$R < \max_{p_{X_1} p_{X_2} p_{\hat{Y}_2|X_2 Y_2}} I(X_1; \hat{Y}_2 Y_3 | X_2)$$

  subject to the constraint

$$I(\hat{Y}_2; Y_2 | X_2 Y_3) \leq I(X_2; Y_3).$$

- Transmission is performed in $B+1$ blocks using random codewords of length $n$.

- Rate is again $R \cdot B/(B+1)$. Arbitrarily close to $R$ for $B$ arbitrarily large.

# Compress-and-forward: Single relay

- Code construction:
  - Take distributions $p_{X_1}$, $p_{X_2}$ and $p_{\hat{Y}_2|X_2Y_2}$.
  - For block $b$:
    - Generate $2^{nR}$ codewords $x_{1b}^n(w)$, choosing symbols $\{x_{1bi}(w)\}$ independently using $p_{X_1}$.
    - Generate $2^{nR}$ codewords $x_{2b}^n(v)$, choosing symbols $\{x_{2bi}(v)\}$ independently using $p_{X_2}$.
    - ("Quantization" codebook:) Generate $2^{n(R_2'+R_2)}$ codewords $\hat{y}_{2b}^n(v,t,u)$, choosing symbols $\{\hat{y}_{2bi}(v,t,u)\}$ independently using $\{p_{\hat{Y}_2|X_2}(\cdot|x_{2bi}(v))\}$.

# Compress-and-forward: Single relay

- $B = 3$:

| | Block 1 | Block 2 | Block 3 | Block 4 |
|---|---|---|---|---|
| $X_1^n =$ | $x_{11}^n(w_1)$ | $x_{12}^n(w_2)$ | $x_{13}^n(w_3)$ | $x_{14}^n(w_4)$ |
| $X_2^n =$ | $x_{21}^n(1)$ | $x_{22}^n(v_2)$ | $x_{23}^n(v_3)$ | $x_{24}^n(v_4)$ |
| $\hat{Y}_2^n =$ | $\hat{y}_{21}^n(1, t_1, v_2)$ | $\hat{y}_{22}^n(1, t_2, v_3)$ | $\hat{y}_{23}^n(v_2, t_3, v_4)$ | |

- After transmission of block $b$,
  - relay encodes to $(t_b, v_{b+1})$,
  - destination decodes $v_b$, then $t_{b-1}$, then $w_{b-1}$.

# Compress-and-forward: Single relay

- Relay encodes to $(t_b, v_{b+1})$ reliably if $n$ is large and

$$R_2 + R_2' > I(\hat{Y}_2; Y_2|X_2).$$

- Destination decodes $(v_b, t_{b-1}, w_{b-1})$ reliably if $n$ is large, $\hat{v}_{b-1} = v_{b-1}$,

$$R_2 < I(X_2; Y_3), \qquad R_2' < I(\hat{Y}_2; Y_3|X_2), \qquad R < I(X_1; \hat{Y}_2 Y_3|X_2).$$

- Can find $R_2$ and $R_2'$ to satisfy these conditions given that assumption on $R$ is satisfied.

# Compress-and-forward: Multiple relays

- Compress-and-forward does not generalize to multiple relays as straightforwardly as decode-and-forward.

- Main complication: Relays forward their observations simultaneously $\longrightarrow$ interference at other relays and at destination.

- Kramer et al. deal with complication by allowing for partial decoding at relays of each other's codewords.

# Compress-and-forward: Multiple relays

- **Theorem 2.** *Compress-and-forward achieves any rate up to*

$$R_{CF} = \max_{p_{X_1}\{p_{U_t X_t} p_{\hat{Y}_t|U_{\mathcal{T}} X_t Y_t}\}_{t\in\mathcal{T}}} I(X_1; \hat{Y}_{\mathcal{T}} Y_T | U_{\mathcal{T}} X_{\mathcal{T}})$$

*where*

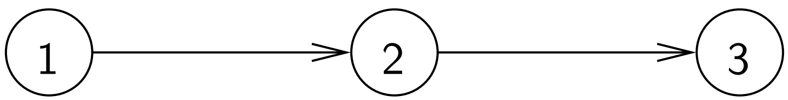
$$I(\hat{Y}_{\mathcal{S}}; Y_{\mathcal{S}} | U_{\mathcal{T}} X_{\mathcal{T}} \hat{Y}_{\mathcal{S}^c} Y_T) + \sum_{t\in\mathcal{S}} I(\hat{Y}_t; X_{\mathcal{T}\setminus\{t\}} | U_{\mathcal{T}} X_t)$$
$$\leq I(X_{\mathcal{S}}; Y_T | U_{\mathcal{S}} X_{\mathcal{S}^c}) + \sum_{m=1}^{M} I(U_{\mathcal{K}_m}; Y_{r(m)} | U_{\mathcal{K}_m^c} X_{r(m)})$$

*for all* $\mathcal{S} \subset \mathcal{T}$, *all partitions* $\{\mathcal{K}_m\}_{m=1}^{M}$ *of* $\mathcal{S}$, *and all* $r(m) \in \{2, 3, \ldots, T\}$ *such that* $r(m) \notin \mathcal{K}_m$. *For* $r(m) = T$, *we set* $X_T := 0$.

# Compress-and-forward: Sub-optimality

- Not decoding at relays introduces sub-optimality.

- Consider




Links are independent with unit capacity.

- Capacity is clearly 1 bit per use, but compress-and-forward cannot achieve it in general.

# Mixed strategies

- What about mixing decode-and-forward and compress-and-forward?
- Relays can partially decode message and
  - use partial decoding for co-operative transmission,
  - compress and forward the remainder.
- Such a scheme described for the single-relay network by Cover and El Gamal (1979).
- Kramer et al. consider a more restrictive mixed strategy: each relay chooses either decode-and-forward or compress-and-forward $\longrightarrow$ achievable rate $R_{DCF}$.

# Wireless setting

- We have

$$\underline{Y}_t = \sum_{s \neq t} \frac{A_{st}}{\sqrt{d_{st}^{\alpha}}} \underline{X}_s + \underline{Z}_t,$$

  where

  - $d_{st}$: distance between terminals $s$ and $t$,
  - $\alpha$: attenuation exponent,
  - $\underline{X}_s$: $n_s \times 1$ complex vector,
  - $A_{st}$: $n_t \times n_s$ complex fading matrix, and
  - $\underline{Z}_t$: $n_t \times 1$ noise vector with i.i.d. circularly-symmetric complex Gaussian entries of unit variance.

- Assume no fading: $A_{st}^{(i,j)}$ constant for all $s$, $t$, $i$, and $j$.

# Wireless setting: Single relay

- As relay moves towards source,
  - decode-and-forward achieves capacity,
  - compress-and-forward does not.
- As relay moves towards destination,
  - compress-and-forward achieves capacity,
  - decode-and-forward does not.
- Consistent with multi-antenna interpretation.

# Wireless setting: Single relay

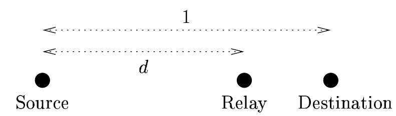



Fig. 6. A single relay on a line.

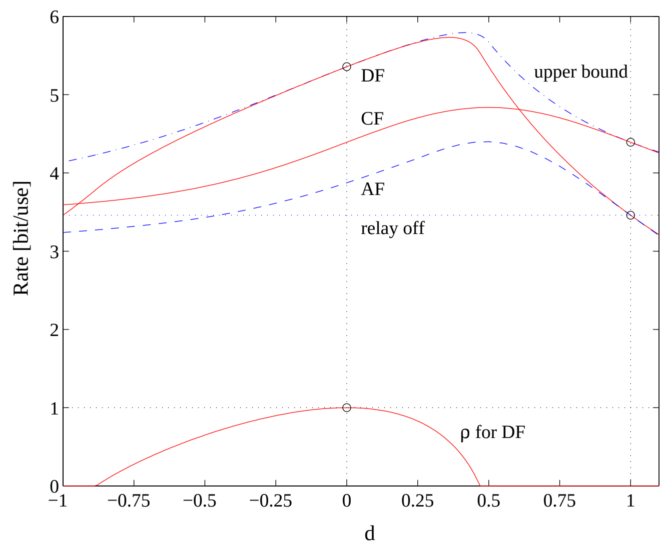



Fig. 7. Rates for a single-relay network with $P_1 = P_2 = 10$ and $\alpha = 2$.

# Wireless setting: Multiple relays

- Observation generalizes to multiple relays.
- If the $T$ terminals form two closely-spaced clusters, then capacity approached by choosing
  - decode-and-forward at terminals close to source, and
  - compress-and-forward at terminals close to destination.

# Summary

- Decode-and-forward:
  - Relays fully decode message and use knowledge of the message to assist the source.
  - Achieves the gains related to multi-antenna transmission.

- Compress-and-forward:
  - Relays do not decode message and, rather, forward compressed versions of their observations.
  - Achieves the gains related to multi-antenna reception.