Cooperative Strategies and Capacity Theorems for Relay Networks

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What are relay networks?

- Relay network:
  - multi-terminal network;
  - single pair of terminals wish to communicate (source and destination);
  - all other terminals assist (relays).

- Special case of single relay → relay channel.

- Relay channel:
  - first studied by van der Meulen (1971);
  - capacity of the relay channel is an open problem.
What problem is addressed?

- We have little hope of finding the capacity of relay networks.

- So we focus on achievable rates → design particular schemes and assess the rates they achieve.

- Schemes:
  - decode-and-forward (related to multi-antenna transmission);
  - compress-and-forward (related to multi-antenna reception);
  - mixtures of the two.
Model

- $T$ terminals: source is terminal 1, destination is terminal $T$.

- Network is memoryless and time-invariant.
Upper bound on capacity

- Can get upper bound from cut-set bound for multi-terminal networks:

\[
C \leq \max_{p_{X_1 X_2 \ldots X_{T-1}}} \min_{S \subseteq T} I(X_1 X_S; Y_S Y_T | X_{S^c})
\]

- For relay channel:

\[
C' \leq \max_{p_{X_1 X_2}} \min(I(X_1; Y_2 Y_3 | X_2), I(X_1 X_2; Y_3)).
\]
Decode-and-forward

- Relays fully decode message and use knowledge of the message to assist the source.

- In a wireless setting: achieves the gains related to multi-antenna transmission.

- Scheme we discuss is due to Xie and Kumar.
Decode-and-forward: Single relay

- Divide message $w$ into $B$ blocks $w_1, w_2, \ldots, w_B$ of $nR$ bits each, where

$$R < \max_{p_{X_1X_2}} \min(I(X_1; Y_2|X_2), I(X_1X_2; Y_3)).$$

- Transmission is performed in $B + 1$ blocks using random codewords of length $n$.

- Rate is

$$\frac{B \cdot nR}{(B + 1)n} = R \frac{B}{B + 1} \text{ bits per use.}$$

Arbitrarily close to $R$ for $B$ arbitrarily large.
Decode-and-forward: Single relay

- Code construction:
  - Take joint distribution \( p_{X_1X_2} \).
  - For block \( b \):
    * Generate \( 2^{nR} \) codewords \( x_{2b}^n(v) \), choosing symbols \( \{x_{2bi}(v)\} \) independently using \( p_{X_2} \).
    * Generate \( 2^{nR} \) codewords \( x_{1b}^n(v, w) \), choosing symbols \( \{x_{1bi}(v, w)\} \) independently using \( p_{X_1|X_2}(\cdot|x_{2bi}(v)) \).
Decode-and-forward: Single relay

- \( B = 3 \):

\[
\begin{align*}
X^n_1 &= \begin{cases} 
  x^n_{11}(1, w_1) \\
  x^n_{21}(1) 
\end{cases} & \text{Block 1} \\
X^n_2 &= \begin{cases} 
  x^n_{12}(w_1, w_2) \\
  x^n_{22}(w_1) 
\end{cases} & \text{Block 2} \\
X^n_3 &= \begin{cases} 
  x^n_{13}(w_2, w_3) \\
  x^n_{23}(w_2) 
\end{cases} & \text{Block 3} \\
X^n_4 &= \begin{cases} 
  x^n_{14}(w_3, 1) \\
  x^n_{24}(w_3) 
\end{cases} & \text{Block 4}
\end{align*}
\]

- After transmission of block \( b \),
  - relay decodes \( w_b \);
  - destination decodes \( w_{b-1} \).
Decode-and-forward: Single relay

- Relay decodes $w_b$ reliably if $n$ is large, $\hat{w}_{b-1}^{(2)} = w_{b-1}$, and
  \[ R < I(X_1; Y_2|X_2). \]

- Destination decodes $w_b$ reliably if $n$ is large, $\hat{w}_{b-1}^{(3)} = w_{b-1}$, and
  \[ R < I(X_1; Y_3|X_2) + I(X_2; Y_3) = I(X_1X_2; Y_3). \]

- There exists a distribution $p_{X_1X_2}$ that satisfies both conditions by assumption.
Decode-and-forward: Multiple relays

- Consider two relays.

- Divide message $w$ into $B$ blocks $w_1, w_2, \ldots, w_B$ of $nR$ bits each, where

  $$ R < \max_{p_{X_1X_2X_3}} \min(I(X_1; Y_2|X_2X_3), I(X_1X_2; Y_3|X_3), I(X_1X_2X_3; Y_4)). $$

- Transmission is performed in $B + 2$ blocks using random codewords of length $n$.

- Rate is

  $$ \frac{B \cdot nR}{(B + 2)n} = R \frac{B}{B + 2} \text{ bits per use}. $$
### Decode-and-forward: Multiple relays

- \( B = 4 \):

<table>
<thead>
<tr>
<th>Block 1</th>
<th>Block 2</th>
<th>Block 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( X_1^n = )</td>
<td>( x_{11}^n(1, 1, w_1) )</td>
<td>( x_{12}^n(1, w_1, w_2) )</td>
</tr>
<tr>
<td>( X_2^n = )</td>
<td>( x_{21}^n(1, 1) )</td>
<td>( x_{22}^n(1, w_1) )</td>
</tr>
<tr>
<td>( X_3^n = )</td>
<td>( x_{31}^n(1) )</td>
<td>( x_{32}^n(1) )</td>
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</table>

- After transmission of block \( b \), terminal 2 decodes \( w_b \), terminal 3 decodes \( w_{b-1} \), destination decodes \( w_{b-2} \).
Decode-and-forward: Multiple relays

- $B = 4$:

<table>
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<tr>
<th>Block 4</th>
<th>Block 5</th>
<th>Block 6</th>
</tr>
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<td>$X_1^n = x_{14}^n(w_2, w_3, w_4)$</td>
<td>$x_{15}^n(w_3, w_4, 1)$</td>
<td>$x_{16}^n(w_4, 1, 1)$</td>
</tr>
<tr>
<td>$X_2^n = x_{24}^n(w_2, w_3)$</td>
<td>$x_{25}^n(w_3, w_4)$</td>
<td>$x_{26}^n(w_4, 1)$</td>
</tr>
<tr>
<td>$X_3^n = x_{34}^n(w_2)$</td>
<td>$x_{35}^n(w_3)$</td>
<td>$x_{36}^n(w_4)$</td>
</tr>
</tbody>
</table>

- After transmission of block $b$, terminal 2 decodes $w_b$, terminal 3 decodes $w_{b-1}$, destination decodes $w_{b-2}$. 
Decode-and-forward: Multiple relays

- Terminal 2 decodes $w_b$ reliably if $n$ is large, $\hat{w}_{b-2}^{(2)} = w_{b-2}$ and $\hat{w}_{b-1}^{(2)} = w_{b-1}$, and
  $$R < I(X_1; Y_2|X_2X_3).$$

- Terminal 3 decodes $w_b$ reliably if $n$ is large, $\hat{w}_{b-2}^{(3)} = w_{b-2}$ and $\hat{w}_{b-1}^{(3)} = w_{b-1}$, and
  $$R < I(X_1X_2; Y_3|X_3).$$

- Destination decodes $w_b$ reliably if $n$ is large, $\hat{w}_{b-2}^{(4)} = w_{b-2}$ and $\hat{w}_{b-1}^{(4)} = w_{b-1}$, and
  $$R < I(X_1X_2X_3; Y_4).$$

- There exists a distribution $p_{X_1X_2X_3}$ that satisfies all three conditions by assumption.
Decode-and-forward: Multiple relays

• Straightforward to generalize scheme to $T$-terminal relay networks.

• **Theorem 1.** Decode-and-forward achieves any rate up to

\[
R_{DF} = \max_{p_{X_1X_2\ldots X_{T-1}}} \max_{\pi} \min_{1 \leq t \leq T-1} I(X_{\pi(1:t)}; Y_{\pi(t+1)} | X_{\pi(t+1:T-1)}).
\]

• $\pi$ is a permutation on $T$ with $\pi(1) := 1$ and $\pi(T) := T$. 
Decode-and-forward: Sub-optimality

- Requiring the relays to decode can be a severe constraint.

- Consider

```
1 --2 --3
```

Links are independent with unit capacity.

- Capacity is clearly 2 bits per use, but decode-and-forward only achieves 1 bit per use.
Compress-and-forward

- Relays do not decode message and, rather, forward compressed versions of their observations.

- In a wireless setting: achieves the gains related to multi-antenna reception.

- Scheme we discuss is due to Cover and El Gamal (1979) for the single relay network and Kramer et al. for the multiple relay network.
Compress-and-forward: Single relay

- Divide message $w$ into $B$ blocks $w_1, w_2, \ldots, w_B$ of $nR$ bits each, where

  \[ R < \max_{pX_1pX_2p\hat{Y}_2|X_2Y_2} I(X_1; \hat{Y}_2Y_3|X_2) \]

  subject to the constraint

  \[ I(\hat{Y}_2; Y_2|X_2Y_3) \leq I(X_2; Y_3). \]

- Transmission is performed in $B + 1$ blocks using random codewords of length $n$.

- Rate is again $R \cdot B/(B + 1)$. Arbitrarily close to $R$ for $B$ arbitrarily large.
Compress-and-forward: Single relay

- Code construction:
  - Take distributions $p_{X_1}, p_{X_2}$ and $p_{\hat{Y}_2|X_2Y_2}$.
  - For block $b$:
    * Generate $2^{nR}$ codewords $x_{1b}^n(w)$, choosing symbols $\{x_{1bi}(w)\}$ independently using $p_{X_1}$.
    * Generate $2^{nR}$ codewords $x_{2b}^n(v)$, choosing symbols $\{x_{2bi}(v)\}$ independently using $p_{X_2}$.
    * ("Quantization" codebook:) Generate $2^{n(R'_2+R_2)}$ codewords $\hat{y}_{2b}^n(v, t, u)$, choosing symbols $\{\hat{y}_{2bi}(v, t, u)\}$ independently using $\{p_{\hat{Y}_2|X_2}(\cdot|x_{2bi}(v))\}$. 
Compress-and-forward: Single relay

- $B = 3$:

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<td>$x_{24}^n(v_4)$</td>
</tr>
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<td>$\hat{Y}<em>2^n = \hat{y}</em>{21}^n(1, t_1, v_2)$</td>
<td>$\hat{y}_{22}^n(1, t_2, v_3)$</td>
<td>$\hat{y}_{23}^n(v_2, t_3, v_4)$</td>
<td></td>
</tr>
</tbody>
</table>

- After transmission of block $b$,
  - relay encodes to $(t_b, v_{b+1})$,
  - destination decodes $v_b$, then $t_{b-1}$, then $w_{b-1}$. 
Compress-and-forward: Single relay

- Relay encodes to \((t_b, v_{b+1})\) reliably if \(n\) is large and

\[ R_2 + R'_2 > I(\hat{Y}_2; Y_2|X_2). \]

- Destination decodes \((v_b, t_{b-1}, w_{b-1})\) reliably if \(n\) is large, \(\hat{v}_{b-1} = v_{b-1},\)

\[ R_2 < I(X_2; Y_3), \quad R'_2 < I(\hat{Y}_2; Y_3|X_2), \quad R < I(X_1; \hat{Y}_2 Y_3|X_2). \]

- Can find \(R_2\) and \(R'_2\) to satisfy these conditions given that assumption on \(R\) is satisfied.
Compress-and-forward: Multiple relays

- Compress-and-forward does not generalize to multiple relays as straightforwardly as decode-and-forward.

- Main complication: Relays forward their observations simultaneously → interference at other relays and at destination.

- Kramer et al. deal with complication by allowing for partial decoding at relays of each other’s codewords.
Compress-and-forward: Multiple relays

- **Theorem 2.** Compress-and-forward achieves any rate up to

\[
R_{CF} = \max_{p_{X_1} \{p_{U_t X_t} p_{\hat{Y}_t | U_T X_t Y_t} \} t \in T} I(X_1; \hat{Y}_T Y_T | U_T X_T)
\]

where

\[
I(\hat{Y}_S; Y_S | U_T X_T \hat{Y}_S^c Y_T) + \sum_{t \in S} I(\hat{Y}_t; X_T \{t\} | U_T X_T)
\]

\[
\leq I(X_S; Y_T | U_S X_S^c) + \sum_{m=1}^{M} I(U_{K_m}; Y_{r(m)} | U_{K_m^c} X_{r(m)})
\]

for all \( S \subset T \), all partitions \( \{K_m\}_{m=1}^M \) of \( S \), and all \( r(m) \in \{2, 3, \ldots, T\} \) such that \( r(m) \notin K_m \). For \( r(m) = T \), we set \( X_T := 0 \).
Compress-and-forward: Sub-optimality

- Not decoding at relays introduces sub-optimality.

- Consider

```
1 ----> 2 ----> 3
```

Links are independent with unit capacity.

- Capacity is clearly 1 bit per use, but compress-and-forward cannot achieve it in general.
Mixed strategies

- What about mixing decode-and-forward and compress-and-forward?
- Relays can partially decode message and
  - use partial decoding for co-operative transmission,
  - compress and forward the remainder.
- Such a scheme described for the single-relay network by Cover and El Gamal (1979).
- Kramer et al. consider a more restrictive mixed strategy: each relay chooses either decode-and-forward or compress-and-forward → achievable rate $R_{DCF}$. 
Wireless setting

- We have

\[ Y_t = \sum_{s \neq t} \frac{A_{st}}{\sqrt{d_{st}^\alpha}} X_s + Z_t, \]

where

- \( d_{st} \): distance between terminals \( s \) and \( t \),
- \( \alpha \): attenuation exponent,
- \( X_s \): \( n_s \times 1 \) complex vector,
- \( A_{st} \): \( n_t \times n_s \) complex fading matrix,
- \( Z_t \): \( n_t \times 1 \) noise vector with i.i.d. circularly-symmetric complex Gaussian entries of unit variance.

- Assume no fading: \( A_{st}^{(i,j)} \) constant for all \( s, t, i, \) and \( j \).
Wireless setting: Single relay

- As relay moves towards source,
  - decode-and-forward achieves capacity,
  - compress-and-forward does not.

- As relay moves towards destination,
  - compress-and-forward achieves capacity,
  - decode-and-forward does not.

- Consistent with multi-antenna interpretation.
Wireless setting: Single relay

Fig. 6. A single relay on a line.

Fig. 7. Rates for a single-relay network with $P_1 = P_2 = 10$ and $\alpha = 2$. 
Wireless setting: Multiple relays

- Observation generalizes to multiple relays.

- If the $T$ terminals form two closely-spaced clusters, then capacity approached by choosing
  - decode-and-forward at terminals close to source, and
  - compress-and-forward at terminals close to destination.
Summary

- **Decode-and-forward:**
  - Relays fully decode message and use knowledge of the message to assist the source.
  - Achieves the gains related to multi-antenna transmission.

- **Compress-and-forward:**
  - Relays do not decode message and, rather, forward compressed versions of their observations.
  - Achieves the gains related to multi-antenna reception.