Cooperative Strategies and Capacity Theorems for Relay Networks

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1 Outline

This report is based primarily on [KGG], which deals with relay networks — multi-terminal networks where a single pair of terminals wish to communicate and all other terminals function as relays, assisting communication between the pair. In the special case where there is only a single relay, forming a three-terminal network, we have a relay channel. Such channels were first studied by van der Meulen [vdM71] and finding their capacity has remained an open problem since. We therefore have little hope of finding the capacity of relay networks, but we can bound it through the design of particular strategies and assessing the rates that they achieve.

We commence by introducing a model for relay networks in Section 2. Then, in Section 3, we give an upper bound on the capacity of relay networks. The two strategies that we consider, decode-and-forward and compress-and-forward, are discussed in Sections 4 and 5, respectively, whilst a mixture of the two strategies is discussed in Section 6. We specialize to the wireless setting in Section 7.

2 Model

We consider networks consisting of T terminals: a source terminal (terminal 1), $T-2$ relays (terminals t with $t \in \mathcal{T} := \{2, 3, \ldots, T-1\}$), and a destination terminal (terminal T). The source has a message W that it wishes to

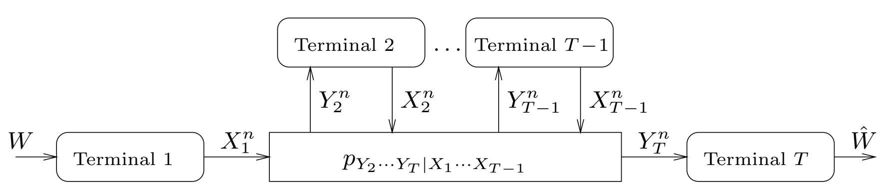

Figure 1: The relay network model.

communicate to the destination. At each time $i \in \{1, 2, \dots, n\}$, each terminal $t \in \{1, 2, \dots, T-1\}$ sends $X_{ti} \in \mathcal{X}_t$, and each terminal $t \in \{2, 3, \dots, T\}$ receives $Y_{ti} \in \mathcal{Y}_t$. The $\{X_{1i}\}$ are a function of W, and the X_{ti} are functions of terminal t's past outputs $Y_t^{i-1} = (Y_{t1}, Y_{t2}, \dots, Y_{t(i-1)})$. The network is memoryless and time-invariant, so

$$p_{Y_{2i}\cdots Y_{Ti}|X_1^i\cdots X_{T-1}^i Y_2^{i-1}\cdots Y_T^{i-1}}(y_{2i}, \dots, y_{Ti}|x_1^i, \dots, x_{T-1}^i, y_2^{i-1}, \dots, y_T^{i-1}) \\ = p_{Y_2\cdots, Y_T|X_1\cdots X_{T-1}}(y_{2i}, \dots, y_{Ti}|x_{1i}, \dots, x_{(T-1)i})$$

The destination makes its message estimate $\hat{W}$ from Y_T^n. See Figure 1 for an illustration of the model.

3 An upper bound on capacity

A capacity upper bound is obtained by straightforward application of the cut-set bound for multi-terminal networks [CT91, Section 14.10]. Let $X_{\mathcal{S}} := \{X_t\}_{t\in\mathcal{S}}$. Then we have

Proposition 1 *The T-terminal relay network capacity satisfies*

$$C \leq \max_{p_{X_1X_2\cdots X_{T-1}}} \min_{\mathcal{S}\subset\mathcal{T}} I(X_1X_{\mathcal{S}}; Y_{\mathcal{S}^c}Y_T|X_{\mathcal{S}^c})$$

where $\mathcal{S}^c$ is the complement of $\mathcal{S}$ in $\mathcal{T}$.

For the relay channel, for example, we have $T = 3$, so

$$C \leq \max_{p_{X_1X_2}} \min(I(X_1; Y_2Y_3|X_2), I(X_1X_2; Y_3)). \tag{1}$$

4 Decode-and-forward

In decode-and-forward schemes, the relays are able to fully decode the message and use their knowledge of the message to assist the source's transmission. Thus, in a wireless setting, decode-and-forward achieves the gains related to multi-antenna transmission. The particular decode-and-forward scheme we consider is that from [XK].

4.1 Single relay

For simplicity, we first consider the case of a single relay. We divide the message w into B blocks $w_1, w_2, \dots, w_B$ of nR bits each, where

$$R < \max_{p_{X_1X_2}} \min(I(X_1; Y_2|X_2), I(X_1X_2; Y_3)).$$

Note that the maximum allowable rate differs from the upper bound on capacity (1). The transmission is performed in $B+1$ blocks by using random codewords

	Block 1	Block 2	Block 3	Block 4
$X_1^n =$	$x_{11}^n(1, w_1)$	$x_{12}^n(w_1, w_2)$	$x_{13}^n(w_2, w_3)$	$x_{14}^n(w_3, 1)$
$X_2^n =$	$x_{21}^n(1)$	$x_{22}^n(w_1)$	$x_{23}^n(w_2)$	$x_{24}^n(w_3)$

Figure 2: A decode-and-forward strategy for the single-relay network.

of length n. Hence, the overall rate is $B \cdot nR/\{(B+1)n\} = R \cdot B/(B+1)$ bits per use, which can be made arbitrarily close to R by taking B arbitrarily large.
Code construction: Take a joint distribution $p_{X_1X_2}$. For block b, $b = 1, 2, \ldots, B+1$, generate 2^{nR} codewords $x_{2b}^n(v)$, $v = 1, 2, \ldots, 2^{nR}$, by choosing the symbols $\{x_{2bi}(v)\}$ independently using p_{X_2}. Then, for every $x_{2b}^n(v)$, generate 2^{nR} codewords $x_{1b}^n(v, w)$, $w = 1, 2, \ldots, 2^{nR}$, by choosing the $\{x_{1bi}(v, w)\}$ independently using $\{p_{X_1|X_2}(\cdot|x_{2bi}(v))\}$.
Source terminal: In block b, the source transmits $x_{1b}^n(w_{b-1}, w_b)$, where $w_0 = w_{B+1} = 1$.
Relay terminal: For the first block, the relay transmits $x_{21}^n(1)$. After transmission of block b, the relay tries to find a $\tilde{w}_b$ such that

$$(x_{1b}^n(\hat{w}_{b-1}^{(2)}, \tilde{w}_b), x_{2b}^n(\hat{w}_{b-1}^{(2)}), y_{2b}^n) \in A_\epsilon^{(n)}(p_{X_1X_2Y_2}), \tag{2}$$

where $\hat{w}_{b-1}^{(2)}$ is the relay terminal's estimate of w_{b-1}. If one or more such $\tilde{w}_b$ are found, then the relay chooses one of them and sets $\hat{w}_b^{(2)}$ equal to it; otherwise, it sets $\hat{w}_b^{(2)} = 1$. The relay transmits $x_{2(b+1)}^n(\hat{w}_b^{(2)})$ in block $b+1$.
Destination terminal: After transmission of block b, the destination tries to find a $\tilde{w}_{b-1}$ such that

$$(x_{1(b-1)}^n(\hat{w}_{b-2}^{(3)}, \tilde{w}_{b-1}), x_{2(b-1)}^n(\hat{w}_{b-2}^{(3)}), y_{3(b-1)}^n) \in A_\epsilon^{(n)}(p_{X_1X_2Y_3}) \tag{3}$$

and

$$(x_{2b}^n(\tilde{w}_{b-1}), y_{3b}^n) \in A_\epsilon^{(n)}(p_{X_2Y_3}), \tag{4}$$

where $\hat{w}_{b-2}^{(3)}$ is the destination terminal's estimate of w_{b-2}. If one or more such $\tilde{w}_{b-1}$ are found, then the destination chooses one of them and sets $\hat{w}_{b-1}^{(3)}$ equal to it; otherwise, it sets $\hat{w}_{b-1}^{(3)} = 1$.

The scheme is depicted in Figure 2 for $B = 3$. To see that it achieves reliable communication, note that the probability that there exists $\tilde{w}_b \neq w_b$ satisfying (2) can be made arbitrarily small by taking n arbitrarily large provided that its past message estimate $\hat{w}_{b-1}^{(2)}$ was correct and

$$R < I(X_1; Y_2|X_2), \tag{5}$$

and that the probability that there exists $\tilde{w}_{b-1} \neq w_{b-1}$ satisfying (3) and (4) can be made arbitrarily small by taking n arbitrarily large provided that its past message estimate $\hat{w}_{b-2}^{(3)}$ was correct and

$$R < I(X_1; Y_3|X_2) + I(X_2; Y_3) = I(X_1X_2; Y_3). \tag{6}$$

	Block 1	Block 2	Block 3
$X_1^n =$	$x_{11}^n(1,1,w_1)$	$x_{12}^n(1,w_1,w_2)$	$x_{13}^n(w_1,w_2,w_3)$
$X_2^n =$	$x_{21}^n(1,1)$	$x_{22}^n(1,w_1)$	$x_{23}^n(w_1,w_2)$
$X_3^n =$	$x_{31}^n(1)$	$x_{32}^n(1)$	$x_{33}^n(w_1)$

	Block 4	Block 5	Block 6
$X_1^n =$	$x_{14}^n(w_2,w_3,w_4)$	$x_{15}^n(w_3,w_4,1)$	$x_{16}^n(w_4,1,1)$
$X_2^n =$	$x_{24}^n(w_2,w_3)$	$x_{25}^n(w_3,w_4)$	$x_{26}^n(w_4,1)$
$X_3^n =$	$x_{34}^n(w_2)$	$x_{35}^n(w_3)$	$x_{36}^n(w_4)$

Figure 3: A decode-and-forward strategy for the two-relay network.

By assumption, there exists a distribution $p_{X_1X_2}$ such that conditions (5) and (6) are both satisfied.

4.2 Multiple relays

The generalization to multiple relays follows quite straightforwardly from the single relay case. For two relays, for example, we divide the message w into B blocks $w_1, w_2, \ldots, w_B$ of nR bits each, where

$$R < \max_{p_{X_1X_2X_3}} \min(I(X_1;Y_2|X_2X_3), I(X_1X_2;Y_3|X_3), I(X_1X_2X_3;Y_4)).$$

The transmission is performed in $B+2$ blocks by using random codewords of length n. Hence, the overall rate is $B \cdot nR/\{(B+2)n\} = R \cdot B/(B+2)$ bits per use, which can be made arbitrarily close to R by taking B arbitrarily large. The scheme for $B=4$ is depicted in Figure 3.

In this case, terminal 2 can reliably decode w_b after the transmission of block b if n is large, its past message estimates, $\hat{w}_{b-2}^{(2)}$ and $\hat{w}_{b-1}^{(2)}$, were correct, and

$$R < I(X_1;Y_2|X_2X_3).$$

Terminal 3 can reliably decode w_{b-1} after the transmission of block b if n is large, its past message estimates, $\hat{w}_{b-3}^{(3)}$ and $\hat{w}_{b-2}^{(3)}$ were correct, and

$$R < I(X_1X_2;Y_3|X_3).$$

Terminal 4 can reliably decode w_{b-2} after the transmission of block b if n is large, its past message estimates, $\hat{w}_{b-4}^{(4)}$ and $\hat{w}_{b-3}^{(4)}$ were correct, and

$$R < I(X_1X_2X_3;Y_4).$$

By assumption, there exists a distribution $p_{X_1X_2X_3}$ such that the above conditions are all satisfied, and reliable communication is achieved.

By generalizing to T-terminal relay networks, the following theorem can be proved. Let π be a permutation on $\mathcal{T}$, and define $\pi(1) := 1$, $\pi(T) := T$, and $\pi(i:j) := \{\pi(i), \pi(i+1), \ldots, \pi(j)\}$.

Theorem 1 *Decode-and-forward achieves any rate up to*

$$R_{DF} = \max_{p_{X_1 X_2 \cdots X_{T-1}}} \max_{\pi} \min_{1 \le t \le T-1} I(X_{\pi(1:t)}; Y_{\pi(t+1)} | X_{\pi(t+1:T-1)}). \tag{7}$$

5 Compress-and-forward

In compress-and-forward strategies, the relays do not decode the message and, rather, forward compressed versions of their observations. Thus, in a wireless setting, compress-and-forward achieves the gains related to multi-antenna reception.

5.1 Single relay

Again, we first consider the case of a single relay. The scheme we describe is essentially due to Cover and El Gamal [CEG79, Section VI]. We divide the message w into B blocks $w_1, w_2, \ldots, w_B$ of nR bits each, where

$$R < \max_{p_{X_1} p_{X_2} p_{\hat{Y}_2|X_2 Y_2}} I(X_1; \hat{Y}_2 Y_3 | X_2)$$

subject to the constraint

$$I(\hat{Y}_2; Y_2 | X_2 Y_3) \le I(X_2; Y_3).$$

The transmission is performed in $B+1$ blocks by using random codewords of length n. Hence, the overall rate is $B \cdot nR / \{(B+1)n\} = R \cdot B/(B+1)$ bits per use, which can be made arbitrarily close to R by taking B arbitrarily large.
Code construction: For block b, $b = 1, 2, \ldots, B+1$, generate 2^{nR} codewords $x_{1b}^n(w), w = 1, 2, \ldots, 2^{nR}$ by choosing the $\{x_{1bi}(w)\}$ independently using p_{X_1}. Similarly, generate 2^{nR_2} codewords $x_{2b}^n(v), w = 1, 2, \ldots, 2^{nR_2}$ by choosing the $\{x_{2bi}(v)\}$ independently using p_{X_2}. Define $p_{\hat{Y}_2|X_2}$ using p_{X_1}, $p_{\hat{Y}_2|X_2 Y_2}$, and $p_{Y_1 Y_2|X_1 X_2}$; then, for each $x_{2b}^n(v)$, generate a "quantization" code-book by generating $2^{n(R_2+R_2')}$ codewords $\hat{y}_{2b}^n(v,t,u)$, $t = 1, 2, \ldots, 2^{nR_2'}$, $u = 1, 2, \ldots, 2^{nR_2}$, by choosing the $\{\hat{y}_{2bi}(v,t,u)\}$ independently using $\{p_{\hat{Y}_2|X_2}(\cdot|x_{2bi}(v))\}$.
Source terminal: In block b, the source transmits $x_{1b}(w_b)$, where $w_{B+1} = 1$.
Relay terminal: For the first block, the relay transmits $x_{21}^n(1)$. After transmission of block b, the relay tries to find a $(\tilde{t}_b, \tilde{u}_b)$, such that

$$(\hat{y}_{2b}^n(v_b, \tilde{t}_b, \tilde{u}_b), x_{2b}^n(v_b), y_{2b}^n) \in A_\epsilon^{(n)}(p_{\hat{Y}_2 X_2 Y_2}).$$

If one or more such $(\tilde{t}_b, \tilde{u}_b)$ are found, then the relay chooses one of them and sets $v_{b+1} = \tilde{u}_b$; otherwise, it sets $v_{b+1} = 1$. The relay transmits $x_{2(b+1)}^n(v_{b+1})$ in block $b+1$.

	Block 1	Block 2	Block 3	Block 4
$X_1^n =$	$x_{11}^n(w_1)$	$x_{12}^n(w_2)$	$x_{13}^n(w_3)$	$x_{14}^n(w_4)$
$X_2^n =$	$x_{21}^n(1)$	$x_{22}^n(v_2)$	$x_{23}^n(v_3)$	$x_{24}^n(v_4)$
$\hat{Y}_2^n =$	$\hat{y}_{21}^n(1, t_1, v_2)$	$\hat{y}_{22}^n(1, t_2, v_3)$	$\hat{y}_{23}^n(v_2, t_3, v_4)$	

Figure 4: A compress-and-forward strategy for the single-relay network.

Destination terminal: After transmission of block b, the destination tries to find a $\tilde{v}_b$ such that

$$(x_{2b}^n(\tilde{v}_b), y_{3b}^n) \in A_\epsilon^{(n)}(p_{X_2Y_3}).$$

If one or more such $\tilde{v}_b$ are found, then the destination chooses one of them and sets $\hat{v}_b^{(3)}$ equal to it; otherwise, it sets $\hat{v}_b^{(3)} = 1$. Next, the destination tries to find a $\tilde{t}_{b-1}$ such that

$$(\hat{y}_{2(b-1)}^n(\hat{v}_{b-1}^{(3)}, \tilde{t}_{b-1}, \hat{v}_b^{(3)}), x_{2(b-1)}^n(\hat{v}_{b-1}^{(3)}), y_{3(b-1)}^n) \in A_\epsilon^{(n)}(p_{\hat{Y}_2X_2Y_3}),$$

where $\hat{v}_{b-1}^{(3)}$ is the destination terminal's estimate of v_{b-1}. If one or more such $\tilde{t}_{b-1}$ are found, then the destination chooses one of them and sets $\hat{t}_{b-1}^{(3)}$ equal to it; otherwise, it sets $\hat{t}_{b-1}^{(3)} = 1$. Finally, the destination tries to find a $\tilde{w}_{b-1}$ such that

$$(x_{1(b-1)})^n(\tilde{w}_{b-1}), \hat{y}_{2b}^n(\hat{v}_{b-1}^{(3)}, \hat{t}_{b-1}^{(3)}, \hat{v}_b^{(3)}), x_{2(b-1)}^n(\hat{v}_{b-1}^{(3)}), y_{3(b-1)}^n) \in A_\epsilon^{(n)}(p_{X_1\hat{Y}_2X_2Y_3}).$$

If one or more such $\tilde{w}_{b-1}$ are found, then the destination chooses one of them and sets $\hat{w}_{b-1}^{(3)}$ equal to it; otherwise, it sets $\hat{w}_{b-1}^{(3)} = 1$.

The scheme is depicted in Figure 4 for $B = 3$. The relay can reliably encode to (t_b, u_b) after the transmission of block b provided that n is large and

$$R_2 + R_2' > I(\hat{Y}_2; Y_2|X_2). \tag{8}$$

The destination can reliable decode (v_b, t_{b-1}, w_{b-1}) after the transmission of block b provided that n is large, its past estimate $\hat{v}_{b-1}^{(3)}$ was correct,

$$R_2 < I(X_2; Y_3), \tag{9}$$

$$R_2' < I(\hat{Y}_2; Y_3|X_2), \tag{10}$$

and

$$R < I(X_1; \hat{Y}_2Y_3|X_2). \tag{11}$$

By assumption, there exist distributions p_{X_1}, p_{X_2}, and $p_{\hat{Y}_2|X_2Y_2}$ such that condition (11) is satisfied. We can satisfy (10) by setting $R_2' := I(\hat{Y}_2; Y_3|X_2) - \epsilon$ for some $\epsilon > 0$. Then, condition (8) becomes

$$R_2 > I(\hat{Y}_2; Y_2|X_2) - I(\hat{Y}_2; Y_3|X_2) + \epsilon = I(\hat{Y}_2; Y_2|X_2Y_3) + \epsilon, \tag{12}$$

and we see that conditions (9) and (12) can be satisfied since we have imposed the constraint

$$I(\hat{Y}_2; Y_2|X_2Y_3) \leq I(X_2; Y_3).$$

5.2 Multiple relays

The compress-and-forward strategy just described does not generalize to multiple relays as straightforwardly as the decode-and-forward strategy of Section 4. The main complication that arises is that when the relays forward their observations to the destination, they do so simultaneously, interfering with each other's observations as well as causing interference at the destination. Kramer et al. [KGG] deal with this complication by allowing for terminals to partially decode each other's codewords, and they prove the following theorem.

Theorem 2 *Compress-and-forward achieves any rate up to*

$$R_{CF} = \max_{p_{X_1}\{p_{U_tX_t}p_{\hat{Y}_t|U_\mathcal{T}X_tY_t}\}_{t\in\mathcal{T}}} I(X_1; \hat{Y}_\mathcal{T} Y_T | U_\mathcal{T} X_\mathcal{T}) \tag{13}$$

where

$$\begin{aligned} I(\hat{Y}_\mathcal{S}; Y_\mathcal{S}|U_\mathcal{T}X_\mathcal{T}\hat{Y}_{\mathcal{S}^c}Y_T) &+ \sum_{t\in\mathcal{S}} I(\hat{Y}_t; X_{\mathcal{T}\setminus\{t\}}|U_\mathcal{T}X_t) \\ &\leq I(X_\mathcal{S}; Y_T|U_\mathcal{S}X_{\mathcal{S}^c}) + \sum_{m=1}^{M} I(U_{\mathcal{K}_m}; Y_{r(m)}|U_{\mathcal{K}_m^c}X_{r(m)}) \end{aligned} \tag{14}$$

for all $\mathcal{S} \subset \mathcal{T}$, all partitions $\{\mathcal{K}_m\}_{m=1}^M$ of $\mathcal{S}$, and all $r(m) \in \{2, 3, \ldots, T\}$ such that $r(m) \notin \mathcal{K}_m$. For $r(m) = T$, we set $X_T := 0$.

6 Mixed strategies

The decode-and-forward and compress-and-forward strategies can be combined so that the relays partially decode the source's message and use their partial decoding for co-operative transmission while compressing and forwarding the remainder. Such a scheme is described for the single-relay network in [CEG79, Section VI].

A more restrictive, but more analytically tractable, approach is taken by Kramer et al. [KGG] where each relay chooses either decode-and-forward or compress-and-forward. Thus, the relay indices are divided into two sets $\mathcal{T}_1 = \{2, 3, \ldots, T_1 + 1\}$ and $\mathcal{T}_2 = \{T_1 + 2, \ldots, T - 1\}$, and the relays in $\mathcal{T}_1$ use decode-and-forward while the relays in $\mathcal{T}_2$ use compress-and-forward. They prove the following theorem.

Theorem 3 *Choosing either decode-and-forward or compress-and-forward achieves* *any rate up to*

$$R_{DCF} = \max_{p_{X_1 X_{\mathcal{T}_1}} \{p_{U_t X_t} p_{\hat{Y}_t | U_{\mathcal{T}_2} X_t Y_t}\}_{t \in \mathcal{T}_2}} \min \Big(\min_{1 \le t \le T_1} I(X_{\pi(1:t)}; Y_{\pi(t+1)} | X_{\pi(t+1:T_1+1)}), \quad I(X_1 X_{\mathcal{T}_1}; \hat{Y}_{\mathcal{T}_2} Y_T | U_{\mathcal{T}_2} X_{\mathcal{T}_2}) \Big) \tag{15}$$

where π is a permutation on $\mathcal{T}_1$, we set $\pi(1) := 1$, and

$$I(\hat{Y}_{\mathcal{S}}, Y_{\mathcal{S}} | U_{\mathcal{T}_2} X_{\mathcal{T}_2} \hat{Y}_{\mathcal{S}^c} Y_T) + \sum_{t \in \mathcal{S}} I(\hat{Y}_t; X_{\mathcal{T}_2 \setminus \{t\}} | U_{\mathcal{T}_2} X_t) \le I(X_{\mathcal{S}}; Y_T | U_{\mathcal{S}} X_{\mathcal{S}^c}) + \sum_{m=1}^{M} I(U_{\mathcal{K}_m}; Y_{r(m)} | U_{\mathcal{K}_m^c} X_{r(m)})$$

for all $\mathcal{S} \subset \mathcal{T}_2$, all partitions $\{\mathcal{K}_m\}_{m=1}^M$ of $\mathcal{S}$, and all $r(m) \in \mathcal{T}_2 \cup \{T\}$ such that $r(m) \notin \mathcal{K}_m$. Here, $\mathcal{S}^c$ denotes the complement of $\mathcal{S}$ in $\mathcal{T}_2$. For $r(m) = T$, we set $X_T := 0$.

7 The wireless setting

In the wireless setting, we have

$$\underline{Y}_t = \sum_{s \ne t} \frac{A_{st}}{\sqrt{d_{st}^{\alpha}}} \underline{X}_s + \underline{Z}_t,$$

where d_{st} is the distance between terminals s and t, α is an attenuation exponent, $\underline{X}_s$ is a $n_s \times 1$ complex vector, A_{st} is a $n_t \times n_s$ matrix whose $A_{st}^{(i,j)}$ is a complex fading random variable, and $\underline{Z}_t$ is a $n_t \times 1$ noise vector whose entries are i.i.d. circularly-symmetric complex Gaussian random variables of unit variance. There are per-symbol power constraints $E[\underline{X}_s^{\dagger} \underline{X}_s] \le P_s$ for all s, where $\underline{X}_s^{\dagger}$ is the complex-conjugate transpose of $\underline{X}_s$.

Several different kinds of fading are considered in [KGG]. We shall only review the case of no fading, where $A_{st}^{(i,j)}$ is constant for all s, t, i, and j. The results with fading are of a somewhat similar nature.

7.1 Single relay

In this case, it is shown that the cut-set bound (1) is maximized by making (X_1, X_2) zero-mean jointly Gaussian, so the bound becomes

$$C \le \max_{0\le\rho\le 1} \min\left(\log\left(1 + P_1\left(\frac{1}{d_{12}^{\alpha}} + \frac{1}{d_{13}^{\alpha}}\right)(1-|\rho|^2)\right),\right.$$
$$\left.\log\left(1 + \frac{P_1}{d_{13}^{\alpha}} + \frac{P_2}{d_{23}^{\alpha}} + \frac{2\rho\sqrt{P_1P_2}}{d_{13}^{\alpha/2}d_{23}^{\alpha/2}}\right)\right),$$

where ρ is the correlation coefficient of X_1 and X_2.

The best decode-and-forward rate (7) is

$$R_{DF} = \max_{0\le\rho\le 1} \min\left(\log\left(1 + \frac{P_1}{d_{12}^{\alpha}}(1-|\rho|^2)\right),\right.$$
$$\left.\log\left(1 + \frac{P_1}{d_{13}^{\alpha}} + \frac{P_2}{d_{23}^{\alpha}} + \frac{2\rho\sqrt{P_1P_2}}{d_{13}^{\alpha/2}d_{23}^{\alpha/2}}\right)\right).$$

For the compress-and-forward strategy, we choose X_1 and X_2 to be Gaussian and $\hat{Y}_2 = Y_2 + \hat{Z}_2$, where $\hat{Z}_2$ is a Gaussian random variable with zero-mean and variance $\hat{N}_2$. The rate (13) is then

$$R_{CF} = \log\left(1 + \frac{P_1}{d_{12}^{\alpha}(1+\hat{N}_2)} + \frac{P_1}{d_{13}^{\alpha}}\right),$$

where the choice

$$\hat{N}_2 = \frac{P_1(1/d_{12}^{\alpha}) + 1/d_{13}^{\alpha}) + 1}{P_2/d_{23}^{\alpha}}$$

satisfies (14) with equality.

Suppose $d_{13} = 1$. As the relay moves towards the source ($d_{12} \to 0$), the quantities C and R_{DF} converge towards

$$\log\left(1 + P_1 + P_2 + 2\sqrt{P_1P_2}\right),$$

whilst R_{CF} converges towards

$$R_{CF} = \log(1 + P_1 + P_2),$$

so R_{DF} tends towards capacity. Similarly, as the relay moves towards the destination ($d_{23} \to 0$), the quantities C and R_{CF} converge towards

$$\log(1 + 2P_1),$$

whilst R_{DF} converges towards

$$\log(1 + P_1),$$

so R_{CF} tends towards capacity. These observations are consistent with the interpretation that decode-and-forward achieves the gains related to multi-antenna transmission while compress-and-forward achieves the gains related to multi-antenna reception.

7.2 Multiple relays

Suppose we have two relays and that d_{14}, the distance between the source and destination terminals, is 1. If the relays are within a distance d of the source, then the decode-and-forward rate (7) becomes the capacity as $d \to 0$, which is

$$R_{DF} = \log\left(1 + \left[\sqrt{P_1} + \sqrt{P_2} + \sqrt{P_3}\right]^2\right).$$

Similarly, if the relays are within a distance d of the destination, then the compress-and-forward rate (13) becomes the capacity as $d \to 0$, which is

$$R_{CF} = \log(1 + 3P_1).$$

Finally, if one of the relays moves towards the source and the other towards the destination (for example, $d_{12} \to 0$ and $d_{34} \to 0$), then the mixed-strategy rate (15) tends towards the capacity, which is

$$R_{DCF} = \log\left(1 + 2\left[P_1 + P_2 + 2\sqrt{P_1 P_2}\right]\right).$$

These results relating to the case of two relays generalize to T terminals; hence, if we have T terminals forming two closely spaced clusters, then choosing either decode-and-forward or compress-and-forward at each relay approaches capacity.

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