Efficiency Loss in a Network Resource Allocation Game

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1 Introduction

In this report, we summarize the paper by Ramesh Johari and John Tsitsiklis [1]. This paper considers the problem of distributed resource allocation mechanisms for the Internet. The current Internet is used widely by a heterogenous population of users. Different users place different values on the perceived network performance. Moreover there are some fundamental constraints on the maximum rate each link can support. How can the bandwidth be efficiently allocated to different users in such a setting? To make the problem more concrete, consider the example shown in figure 1 taken from [2].

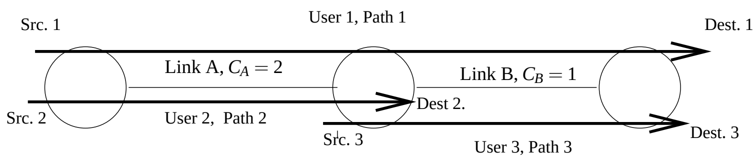


Figure 1: Two link Network

In the figure above, there are three users [1] labelled 1, 2 and 3 and two links A and B. The path belonging to user 1 uses both links A and B. Similarly user 2 uses link A while user 3 uses link B only. Also suppose the capacities of the two links are $C_A = 2$ and $C_B = 1$. How should the rates be allocated to the three users? One possible solution is to do a max-min fair allocation. This particular allocation has a desirable property that it gives the most poorly treated user (i.e. the user with the lowest rate) the maximum possible share without wasting network resources. The max min allocation in this example is : $d_1 = d_3 = 0.5$ and $d_2 = 1.5$.[2] However max-min fair allocation may not always be desirable. Suppose user 1 only requires 0.25 units of bandwidth and does not care about the rest. Then it is preferable to assign $d_1 = 0.25$, $d_2 = 1.75$ and $d_3 = 0.75$. In general one should consider the utility function of users before doing bandwidth allocation. Suppose each user $r \in \{1, 2, \dots, R\}$ has a utility function $U_r(\cdot)$. The network optimization problem is to allocate each user a data rate d_r to solve the following problem

$$\max_{\{d_1, d_2, \dots, d_R\}} \sum_{r=1}^{R} U_r(d_r)$$

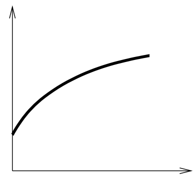
[1] Each source-destination pair refers to one "user".

[2] We divide link B equally between user 1 and 3. On link A, since user 1 uses only 0.5 units, user 2 gets 1.5 units.

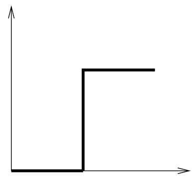
The set of feasible rate vectors $\{d_1, d_2, \dots, d_R\}$ must satisfy capacity constraints on each link.[3] We will impose the following constraint on the utility functions:

ASSUMPTION 1 For each r, $U_r(d_r)$ is a continuously differentiable, non-decreasing strictly concave function.

The above assumption has been used throughout the paper. An important implication of this assumption is that the demand of each user is elastic. The utility keeps increasing as the user gets higher rate. In particular utility functions shown in 2(b) are not permissible.



(a) Permissible Utility Function



(b) Not Permissible Utility Function

Figure 2: Permissible Utility Functions

To solve this optimization problem the network manager needs to know the utility function of each user. In practice it may not be possible. The idea behind *distributed resource allocation* is to introduce a pricing mechanism that solves the above optimization problem in a distributed manner. Each user performs a local computation based on its own utility function and submits a bid to the network. The network manager collects these bids and determines the price to charge the users. In the remainder of this report we will explain this distributed pricing mechanism in more detail. In Section 2 we will consider the case of a single link (All users wish to use a single link with total capacity of C). We will first describe the pricing mechanism that solves the global optimization problem. This particular mechanism assumes that the users are price taking (i.e. they do not anticipate the effect of their bids on the price). Then we will consider the case when the users are price anticipating. In this case there is a unique Nash equilibrium. Moreover at the Nash equilibrium, the aggregate utility is within 3/4 of the optimal value. Section 3 generalizes these single link results to the case of an arbitrary network.

2 Single Link Case

Suppose R users wish to communicate over a single link with capacity C. Each user is assigned a portion of this capacity say d_r. We wish to solve the following problem:

SYSTEM:

$$\text{maximize} \quad \sum_r U_r(d_r) \tag{1}$$

$$\text{subject to} \quad \sum_r d_r \leq C \tag{2}$$

$$d_r \geq 0, r = 1, 2, \dots R \tag{3}$$

[3] For the single link case, this constraint is $\sum_r d_r \leq C$. For the network case, this constraint will be formalized in section 3.

Under assumption 1, the above problem has a unique optimal solution. We will now describe a specific pricing mechanism that solves this problem in a distributed manner. Suppose user r gives a payment of w_r to the link manager; we assume that $w_r \geq 0$. Given the vector $\mathbf{w} = (w_1, w_2, \ldots w_r)$ of bids, the link manager chooses a rate allocation $\mathbf{d} = (d_1, \ldots, d_r)$ such that $d_r = w_r/\mu$, where

$$\mu = \frac{\sum_r w_r}{C} \tag{4}$$

In the subsequent sections we will consider two different ways in which the users will interact with this pricing mechanism. First we will consider the case when the users are price taking and establish the existence of a *competitive equilibrium*. Then we will consider the case when the users are price anticipating and establish the existence of a *Nash equilibrium* and study its properties.

2.1 Price taking Users and Competitive Equilibrium

In this section we consider a competitive equilibrium between the users and the link manager, which was first observed in [3]. A central assumption here is that the users are not price anticipating. More specifically, given a price $\mu > 0$, each user r, acts to maximize the following payoff function over $w_r \geq 0$:

$$P_r(w_r; \mu) = U_r\left(\frac{w_r}{\mu}\right) - w_r \tag{5}$$

The first term represents the utility to user r of receiving a rate allocation equal to w_r/μ. The second term is the payment w_r made to the manager. The users are price taking, in a sense that they take the price μ as a given quantity and do not anticipate its dependence on w_r. We say that a pair $(\mathbf{w}, \mu)$ is in competitive equilibrium if each user maximizes his/her payoff in (5) and the network sets the price according to (4). At the competitive equilibrium, we have the following conditions:

$$\begin{aligned} P_r(w_r; \mu) &\geq P_r(\bar{w}_r; \mu) \qquad \text{for } \bar{w}_r \geq 0,\ r = 1, \ldots, R && (6) \\ \mu &= \frac{\sum_r w_r}{C} && (7) \end{aligned}$$

We now present the main result for this pricing mechanism.

THEOREM 1 Under assumption 1, there exists a unique competitive equilibrium i.e. a unique pair $(\mathbf{w}, \mu)$ that satisfies (6)-(7). Furthermore the corresponding rate vector $\mathbf{d} = \mathbf{w}/\mu$ satisfies the SYSTEM problem (1)-(3).

Proof: (Outline) The system problem can be formulated as a lagrangian optimization problem:

$$\mathcal{L}(\mathbf{d}, \mu) = \sum_r U_r(d_r) - \mu \left(\sum_r d_r - C \right)$$

Differentiating with respect to d_r, we have:

$$\begin{aligned} U_r'(d_r) &= \mu \qquad \text{if } d_r > 0 \\ U_r'(d_r) &\leq \mu \qquad \text{if } d_r = 0 \end{aligned}$$

Substituting $d_r = w_r/\mu$, we have

$$U_r'(\frac{w_r}{\mu}) = \mu \quad \text{if } w_r > 0$$
$$U_r'(\frac{w_r}{\mu}) \leq \mu \quad \text{if } w_r = 0$$

The resulting w_r however precisely satisfies (6). Furthermore since $\sum_r d_r = C$, (7) is also satisfied. Similarly we can show that the solution to (6)-(7) satisfies the SYSTEM problem. $\square$

Note that in the above proof, we required that the users are not price anticipating. In particular the users do not attempt to exploit the dependence of μ on w_r. We now mention several remarks:

Remarks

- The above result that competitive equilibrium achieves global optimality is actually a special case of a more general problem of social welfare. This process of achieving competitive equilibrium is called *tatonnement* [5](section 5.4). Kelly's main contribution in [3] is to adapt this general result to the case of network with elastic demand.
- In this pricing mechanism users iteratively adjust their bids (w_r) based on the price (μ) they receive to maximize their payoff (5). However Theorem 1 does not consider the dynamics of this mechanism. It simply asserts that the process will converge to a unique equilibrium and the resulting allocation will maximize the SYSTEM problem.
- The choice of the payoff functions (5) appears unique upto a scaling constant. For example, with $P_r(w_r;\mu) = U_r(\frac{w_r}{\mu}) - 0.5w_r$ the result of Theorem 1 still holds. [4] However it is not clear if there are more general payoff functions which achieve global optimality.

2.2 Price anticipating users and Nash equilibrium.

Before we consider the case of price anticipating users, we will provide a simple example of the Nash equilibrium.

2.2.1 Nash Equilibrium - Background

Nash equilibrium is a celebrated result in the Game Theory literature. It refers to a set of strategies for a game with the property that no player can benefit by changing his strategy while the other players keep their strategies unchanged. It implicitly assumes that each user behaves selfishly to maximize his/her payoff. It is best understood by a simple example.

Prisoner's Dilemma Two suspects are arrested by police. The police have insufficient evidence for a conviction and having separated them, visit each of them and offer the same deal: If you confess and your accomplice remains silent, he gets 10 year sentence and you go free. If he confesses and you remain silent you get 10 year sentence. If both of you remain silent, each gets six months imprisonment but if both confess against each other, each gets a 6 year sentence. These rules can be summarized as:

[4] However now the bid and price at the equilibrium will scale accordingly. If (w_r, μ) satisfies (6) then $(2w_r, 2\mu)$ will maximize this modified payoff.

Strategy	You deny	You Confess
He denies	Both serve 6 months	He serves 10 yrs.; you are free
He Confesses	He goes free; you serve 10 yrs.	both serve 6 yrs

The question is that knowing the above rules but not knowing what your accomplice is going to do what should be your strategy?

Clearly if both could communicate, then both would agree to deny and serve six months. Since we do not know what the other suspect is going to do, we decide to act selfishly to reduce our sentence. Suppose the other suspect confesses. Then clearly it is in our best interest to confess as well, since that will reduce the sentence. On the other hand suppose he denies. It is still in our interest to confess. Thus no matter what the other suspect does, we will always confess. The second suspect, by a symmetric argument will also confess. Hence the Nash equilibrium occurs when both confess against each other and get a six year sentence.

The above example illustrates an important property of the Nash equilibrium that it need not be globally optimum. We will investigate this property further in the resource allocation problem. We will show that there exists a unique Nash equilibrium for the pricing mechanism considered in the previous section and it achieves within 3/4 of the global optimum. For a more elaborate treatment of Nash Equilibrium refer to a standard book on game theory such as [6].

2.2.2 Nash Equilibrium for the Pricing Scheme

Suppose now that the users consider the effect of their bids on the price μ. In particular given $\mathbf{w}_{-r} = (w_1, \ldots, w_{r-1}, w_{r+1}, \ldots w_R)$, each user will choose w_r to maximize:

$$Q_r(w_r; \mathbf{w}_{-r}) = \begin{cases} U_r\left(\frac{w_r}{\sum_s w_s} C\right) - w_r & \text{if } w_r > 0 \\ U_r(0) & \text{if } w_r = 0 \end{cases} \tag{8}$$

A Nash equilibrium exists for the above game if there exists a vector of bids $\mathbf{w}$ that satisfies the following:

$$Q_r(w_r; \mathbf{w}_{-r}) \geq Q_r(\bar{w}_r; \mathbf{w}_{-r}) \qquad \text{for all } \bar{w}_r \geq 0 \tag{9}$$

We now state the main result for Nash Equilibrium:

THEOREM 2 Under assumption 1, there exists a unique $\mathbf{w}$ that achieves the Nash equilibrium (9). Furthermore the corresponding rate assignments $d_r = \frac{w_r}{\sum_s w_s}$ maximize the SYSTEM problem (1)-(3) with the following modified utility function

$$\hat{U}_r(d_r) = \left(1 - \frac{d_r}{C}\right) U_r(d_r) + \left(\frac{d_r}{C}\right) \left(\frac{1}{d_r} \int_0^{d_r} U_r(z) dz\right) \tag{10}$$

Proof: (Outline) The existence of the Nash equilibrium follows from the fact that $Q_r(\cdot)$ is a concave function in w_r. The proof of its uniqueness follows by showing that it is equivalent to optimizing a SYSTEM function with modified utility functions. Since the SYSTEM has a unique solution, the uniqueness of Nash equilibrium follows.

Note that the optimal w_r that satisfies (9) also satisfies the following equation (obtained by differentiating $Q_r(\cdot)$.

$$U'\left(\frac{w_r}{\sum_s w_s} C\right)\left(1 - \frac{w_r}{\sum_s w_s}\right) = \frac{\sum_s w_s}{C} \text{ if } w_r > 0 \tag{11}$$

$$U'(0) \leq \frac{\sum_s w_s}{C} \tag{12}$$

Also note that the d_r satisfying the SYSTEM with utility functions $\hat{U}(\cdot)$ solves the following (obtained by differentiating $\hat{U}_r(\cdot)$ with ρ being the Lagrange multiplier):

$$U'(d_r)\left(1-\frac{d_r}{C}\right) = \rho \text{ if } d_r > 0 \quad (13)$$

$$U'(0) \leq \rho \quad (14)$$

It follows that (11)-(12) is equivalent to (13)-(14) by choosing $d_r = \frac{w_r}{\sum_s w_s}$ and $\rho = \frac{\sum_s w_s}{C}$ and this establishes the equivalence.

□

2.2.3 Price of Anarchy

It is well known that the Nash equilibrium does not achieve global optimum. In this section, we explicitly quantify the efficiency loss.

THEOREM 3 Suppose that the utility function U_r satisfy assumption 1 and that $U_r(0) \geq 0$ for all r. If $\mathbf{d}^G$ and $\mathbf{d}^S$ are the rate allocations for the Nash equilibrium and Competitive equilibrium respectively then

$$\sum_r U_r(d_r^G) \geq \frac{3}{4}\sum_r U_r(d_r^S)$$

Moreover, for every $\varepsilon > 0$ there exists an $R > 1$ and a choice of utility functions $U_r(\cdot)$ such that

$$\sum_{r=1}^{R} U_r(d_r^G) \leq \left(\frac{3}{4}+\varepsilon\right)\sum_{r=1}^{R} U_r(d_r^S)$$

Before we prove this result, we prove the following lemma:

LEMMA 1 Let $\mathbf{d} = (d_1,\ldots,d_R)$ satisfy, $\sum_r d_r \leq C$. Let $\mathbf{d}^S$ be the optimal solution to SYSTEM. Then the following inequality holds:

$$\frac{\sum_r U_r(d_r)}{\sum_r U_r(d_r^S)} \geq \frac{\sum_r U_r'(d_r)d_r}{(\max_r U_r'(d_r))C}$$

The equality occurs if $U_r(\cdot)$ are linear functions (i.e. linear utility functions are the worst case scenario).

Proof: (Outline) From concavity of $U_r(\cdot)$, we have that $U_r(d_r^S) \leq U_r(d_r) + U_r'(d_r)(d_r^S - d_r)$. Now the left hand side can be bounded as:

$$\begin{aligned}
\text{LHS} = \frac{\sum_r U_r(d_r)}{\sum_r U_r(d_r^S)} &\geq \frac{\sum_r (U_r(d_r) - U_r'(d_r)d_r) + \sum_r U_r'(d_r)d_r}{\sum_r (U_r(d_r) - U_r'(d_r)d_r) + \sum_r U_r'(d_r)d_r^S} \\
&\geq \frac{\sum_r (U_r(d_r) - U_r'(d_r)d_r) + \sum_r U_r'(d_r)d_r}{\sum_r (U_r(d_r) - U_r'(d_r)d_r) + \max_r U_r'(d_r)C} \qquad (\sum_r a_r x_r \leq \max_r a_r C \text{ if } \sum x_r \leq C) \\
&\geq \frac{\sum_r U_r'(d_r)d_r}{(\max_r U_r'(d_r))C} = \text{RHS} \qquad (\sum_r U_r(d_r) - U_r'(d_r)d_r \geq 0)
\end{aligned}$$

Furthermore the equality occurs if $\sum_r U_r'(d_r)d_r^S = \max_r U_r'(d_r)C$ and $U_r(d_r^S) = U_r(d_r) + U_r'(d_r)(d_r^S - d_r)$. This requires that $U_r(\cdot)$ be linear. □

Proof: (Theorem 3) The proof is constructive in that it gives the utility functions for which the loss is close to $1/4$ of the optimal value. From the above lemma we can only consider utility functions that are linear. Let $U_r(d_r) = \alpha_r d_r$. Also without loss in generality let $\alpha_1 = 1$ and $\alpha_r \leq 1$ for $r > 1$. Now from Lemma 1 it follows that to get the worst case efficiency, we must minimize $\sum_r \alpha_r d_r = d_1 + \sum_{r>1} \alpha_r d_r$ by treating $\{\alpha_r\}$ and $\{d_r\}$ as variables. The optimization problem can be stated as follows:

$$
\begin{aligned}
\text{minimize} \quad & d_1^G + \sum_{r=2}^{R} \alpha_r d_r^G && (15)\\
\text{subject to} \quad & \alpha_r(1-d_r^G) = 1 - d_1^G, \qquad \text{if } d_r^G > 0 && (16)\\
& \alpha_r(1-d_r^G) \leq 1 - d_1^G, \qquad \text{if } d_r^G = 0 && (17)\\
& \sum_r d_r^G = 1, \quad 0 \leq \alpha_r \leq 1 \quad d_r^G \geq 0 && (18)
\end{aligned}
$$

In the above optimization problem, we are minimizing the ratio of the aggregate utility at the Nash equilibrium to that of the global optimum. The second and third conditions above, ensure that $\mathbf{d}^G$ is indeed a Nash equilibrium. Note that since the Nash equilibrium is unique, the above optimization should have a unique minimizing $\mathbf{d}^G$. Without loss in generality we make the following simplifications:

- $d_r^G > 0$ for every user. (If not we can simply remove that user and the value of R decreases by 1). This leads to having $\alpha_r = (1-d_1^G)/(1-d_r^G)$.
- $d_2^G = d_3^G = \cdots = d_R^G$. This follows from the symmetry of the problem.

Hence the optimization problem reduces to

$$
\begin{aligned}
\text{minimize} \quad & d_1^G + (1-d_1^G)^2 \left(1 - \frac{1-d_1^G}{R-1}\right)^{-1} && (19)\\
\text{subject to} \quad & 1/R \leq d_1^G \leq 1 && (20)
\end{aligned}
$$

For the worst case scenario, we let $R \to \infty$, then we have to minimize the quadratic expression $d_1^G + (1-d_1^G)^2$, for $0 \leq d_1^G \leq 1$, which gives an efficiency loss of 3/4.

Finally, the worst case utility functions are of the form $U_1(d_1) = d_1$ and $U_r(d_r) \approx d_r/2$ for $r > 1$. The rate allocation at the Nash equilibrium is $d_1^G = \frac{1}{2}$ and $d_r^G = \frac{1}{(2(R-1))}$. By making R arbitrarily large we can achieve an efficiency loss arbitrarily close to 3/4. $\square$

2.2.4 Discussion

- While the proof above is constructive, it seems to lack intuition. We are searching over the entire space of utility functions and getting the worst case loss. Is there an intuition behind the solution? Note that the system problem with linear utility functions is trivial. We simply allocate all the bandwidth to the user with the greatest utility (user 1 in the proof above) and allocate nothing to everyone else. This solution cannot be a Nash equilibrium, since Nash equilibrium solution requires atleast two users to make positive bids. Now by making the utility function of all other users to be equal we are forcing all of them to be active at the Nash equilibrium. Thus we are trying to depart from the global optimum as much as possible with this choice.[5]

[5] The exact choice of $U_r(d_r) = d_r/2$ seems more difficult to explain.

- Theorem 3 holds for the specific pricing mechanism used by Kelly. Is it possible to do better by choosing a different pricing mechanism? In his thesis Johari [8] presents a set of conditions under which the efficiency loss cannot be better than 3/4. In particular, one cannot do better than 3/4 if the network manager does not price discriminate between users. More recently Sanghvi and Hajek [9] have proposed a pricing mechanism for a two user system that achieves within 7/8 of the optimum.

- While the result for single link may not be very practical for the wireline case, it serves as a basis for the network case discussed next.

- A wireless network such as the Multiple Access Channel can be reduced to the single link case and it follows that even for these channels there is atmost 1/4 loss in efficiency.

- The problem is solved for the constraint $\sum_r d_r < C$. It can be readily generalized to the case when there are weights associated with each rate: $\sum_r \beta_r d_r < C$. However the solution does not generalize when the constraint represents a convex set instead of a straight line.

3 General Network Case

We now consider the case where there are J links in the network with capacities $(C_1, C_2 \ldots C_J)$ with $C_j > 0$. Suppose there are P paths in this network and each path is associated with a unique user. We define two matrices A and H such that

$$A_{jp} = \begin{cases} 1 & \text{if link j is included in path p} \\ 0 & \text{otherwise} \end{cases}$$

$$H_{rp} = \begin{cases} 1 & \text{if path p belongs to user r} \\ 0 & \text{otherwise} \end{cases}$$

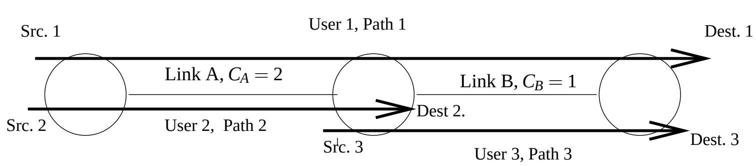


Figure 3: Two link Network

For the example in figure 3, we have $A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$ and H = I_3. The system problem is now given by

SYSTEM

$$\begin{aligned} \text{maximize} \quad & \sum_r U_r(d_r) \\ \text{subject to} \quad & Ay \leq C, Hy = d, y_p \geq 0, p \in P \end{aligned}$$

The set of feasible region is compact and hence if we have the utility functions satisfy assumption 1, then the optimal **d** vector is unique. The pricing mechanism for this network is an extension of the single link case: Let w_{jr} denote the bid of user r on link j. We denote $\mathbf{w}_r = (w_{jr}, j \in \mathcal{J})$ to be the set of bids that user r makes on the links. Let $\mathbf{w} = (\mathbf{w}_r, r \in \mathcal{R})$. Then the rate allocation on link j is determined by the price $\mu_j = \frac{\sum_r w_{jr}}{C}$. Thus portion of bandwidth user r gets on link j is given by:

$$x_{jr} = \begin{cases} \frac{w_{jr}}{\mu_j} & \text{if } w_{jr} > 0 \\ 0 & \text{otherwise} \end{cases} \tag{21}$$

Given the set of rates $(x_{jr}, r \in \mathcal{R})$ on the links, how does user r calculate the data rate d_r that he/she receives? If there is only one path assigned to a given user, the rate $d_r = \min_j x_{jr}$. However in general there may be multiple paths assigned to each user. Given the set of rates x_{jr}, each user solves the following *max flow* optimization problem:

$$\text{maximize} \quad \sum_{p \in r} y_p \tag{22}$$

$$\text{subject to} \quad \sum_{p \in r: j \in p} y_p \leq x_{jr} \tag{23}$$

$$y_p \geq 0, p \in r \tag{24}$$

The first equation above is the total data rate user r gets from his/her paths. The second constraint says that the aggregate of all paths of user r that pass through a link j cannot exceed x_{jr}. We denote the solution to the above optimization problem by $d_r(\mathbf{x_r}(\mathbf{w}))$

3.1 Nash Equilibrium

As in the single link case, we begin by defining

$$Q_r(\mathbf{w}_r; \mathbf{w}_{-r}) = U_r(d_r(\mathbf{x}(\mathbf{w}))) - \sum_j w_{jr} \tag{25}$$

We say that $\mathbf{w}$ is a Nash equilibrium if

$$Q_r(\mathbf{w}_r; \mathbf{w}_{-r}) \geq Q_r(\bar{\mathbf{w}}_r; \mathbf{w}_{-r}) \quad \text{for all } \bar{\mathbf{w}}_r \geq 0 \tag{26}$$

Unlike the single link case, it turns out that in the network case, a Nash equilibrium may not exist. This could happen because our pricing mechanism assumes that the allocation (21) is market clearing (i.e. the entire bandwidth is allocated). However there could some links that cannot be fully utilized because some other links are bottleneck in the network. In such case, there will be some surplus bandwidth remaining on the link. If a user is getting a surplus then he/she can reduce the bid to get less bandwidth. On the other hand one cannot bid 0, as from (21) this would give 0 bandwidth in return. This problem is fixed by defining an extended pricing mechanism that allow each user to receive non-zero bandwidth if no user has a positive bid on a given link. However, we will not go into the details of this particular extension as it is merely a technical detail. To follow the rest of this exposition assume that there are atleast two positive bids on each link.[6]

[6] There cannot be a Nash equilibrium with only one positive bid, since this user can reduce his/her bid but cannot make it 0.

3.2 Price of Anarchy

We now show that the Nash equilibrium loses atmost 1/4 of the global optimum in the network case as well. To simplify exposition and provide the main ideas in concise manner, we will consider the case when $U_r(d_r(\mathbf{x}_r))$ is differentiable for every x_{jr}. It is possible to establish similar result under milder constraint[7], however it requires significantly more mathematical machinery but not much more intuition. We will find it convenient to work in terms of the vector $\mathbf{x}_r$ instead of $\mathbf{w}_r$. We define

$$f_r(\mathbf{x}_r;\mathbf{w}_{-r}) = U_r(d_r(\mathbf{x}_r)) - \sum_j \frac{W_{jr}x_{jr}}{C_j - x_{jr}} \tag{27}$$

Here $W_{jr} = \sum_{s\neq r} w_{js}$.

CLAIM 1 A vector $\mathbf{w}$ is Nash equilibrium to the original system (25)-(26) iff $\mathbf{x}_r(\mathbf{w})$ satisfies the following for each r:

$$\mathbf{x}_r(\mathbf{w}) = \arg\max_{\mathbf{x}_r} f_r(\mathbf{x}_r;\mathbf{w}_{-r})$$

Proof: (Outline) The proof follows immediately from the rate allocation rule $x_{jr} = \frac{w_{jr}}{w_{jr}+W_{jr}}$ which is equivalent to $w_{jr} = \frac{W_{jr}x_{jr}}{C_j - x_{jr}}$. Maximizing $Q_r(\cdot)$ in (26) is equivalent to maximizing $f_r(\cdot)$. □

CLAIM 2 If $\mathbf{x}_r$ is a Nash equilibrium for the original problem then it also satisfies

$$\mathbf{x}_r = \arg\max_{\mathbf{x}_r}\left[\alpha_r^T\mathbf{x}_r - \sum_j \frac{W_{jr}x_{jr}}{C_j - x_{jr}}\right]$$

where $\alpha = \nabla U_r(d_r(\mathbf{x}_r))$.

Proof: (Outline) The proof follows immediately from the previous claim and noting that the optimal x_{jr} satisfies the same differential equations in the two claims. □

The above claim implies that the solution to a Nash equilibrium to the original problem is also a Nash equilibrium to a new system with linear utility functions $\hat{U}(\mathbf{x}_r) = \sum_j \alpha_{jr}x_{jr}$. It follows from this result that if the original system has a Nash equilibrium then each link in the network has a Nash equilibrium under modified utility functions $\alpha_{jr}x_{jr}$.

THEOREM 4 The efficiency loss for the network case is atmost 1/4.

Suppose $\mathbf{x}_r^G$ be a Nash equilibrium and $\mathbf{x}_r^S$ be the global optimum point. We derive the following set of inequalities as in the single link case.

$$\frac{\sum_r U_r(d_r(\mathbf{x}_r^G))}{\sum_r U_r(d_r(\mathbf{x}_r^S))} \geq \frac{\sum_r(U_r(d_r(\mathbf{x}_r^G)) - \alpha_r^T\mathbf{x}_r^G) + \sum_r \alpha_r^T\mathbf{x}_r^G}{\sum_r(U_r(d_r(\mathbf{x}_r^S)) - \alpha_r^T\mathbf{x}_r^G) + \sum_r \alpha_r^T\mathbf{x}_r^S} \tag{28}$$

$$\geq \frac{\sum_j\sum_r \alpha_{jr}x_{jr}}{\sum_j(\max_r \alpha_{jr})C_j} \tag{29}$$

If the overall system is at the Nash equilibrium each single link has a Nash equilibrium and we can invoke the single link result that $\sum_r \alpha_{jr}x_{jr} \geq \frac{3}{4}(\max_r \alpha_{jr})C$. Substituting this in 29, we get the desired result.

[7] $U_r(\cdot)$ need only be superdifferentiable in $\mathbf{x_r}$

4 Related Work

A substantial amount of work on application of Game theorey for network resource allocation problems has emerged in recent years. In particular subsequent work by Johari et. al. [7] have considered the case when the network links have elastic supply rather than hard capacity constraints. They show that under this assumption the loss in efficiency is $4\sqrt(2) - 5$. Johari has also shown that there is a class of pricing mechanisms for which the efficiency loss of $3/4$ is optimum. A necessary condition for these mechanism is that the network does not discriminate between users. A subsequent result by Sanghvi and Hajek [9] has shown that if we allow the network to discriminate between users than the worst case efficiency is $7/8$.

Another related work by Tim Roughgarden and Eva Tardos [10] considers a game for network routing to minimize the total latency. They show that if the intermediate nodes choose locally optimal routes than the total latency is no more than 4/3 of the optimal. It is not clear if there is any connection between this result and the present paper.

Finally there has been a growing interest in using ideas of game theory for resource allocation in wireless networks. For example efficient mechanisms for power control exist and have been shown to converge to a Nash equilibria [11].

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