Efficiency Loss in a Network Resource Allocation Game

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Abstract

We explore the properties of a congestion game where users of a congested resource anticipate the effect of their actions on the price of the resource. When users are sharing a single resource, we establish that the aggregate utility received by the users is at least 3/4 of the maximum possible aggregate utility. We also consider extensions to a network context, where users submit individual payments for each link in the network which they may wish to use. In this network model, we again show that the selfish behavior of the users leads to an aggregate utility which is no worse than 3/4 the maximum possible aggregate utility. We also show that the same analysis extends to a wide class of resource allocation systems where end users simultaneously require multiple scarce resources. These results form part of a growing literature on the "price of anarchy," i.e., the extent to which selfish behavior affects system efficiency.

1 Introduction

The current Internet is used by a widely heterogeneous population of users; not only are different types of traffic sharing the same network, but different end users place different values on their perceived network performance. As a result, characterizing "good" use of the network is difficult: how should resources be shared between a file transfer and a peer-to-peer connection? Partly in response to this heterogeneity, a variety of models for *congestion pricing* in the future Internet have emerged. These models propose a traditional economic solution to the problem of heterogeneous demand: they treat the collection of network resources as a market, and price their use accordingly.

The last decade has witnessed a dramatic rise in research suggesting the use of market mechanisms to manage congestion in networks; see, e.g., [1] for an early overview of some of the issues involved, and [2, 3] for more recent discussion. The proposals have varied widely in approach and simplicity, including applications of auction theory [4] as well as fixed rate pricing mechanisms [5].

In this paper, we will consider a framework with a single network manager, who wishes to allocate network capacity efficiently among a collection of users, each endowed with a utility function depending on their allocated rate. In [6], a market is proposed where each user submits a "bid," or willingness-to-pay per unit time, to the network; the network accepts these submitted bids and determines the price of each network link. A user is then allocated rate in proportion to his bid, and inversely proportional to the price of links he wishes to use. Under certain assumptions, it is shown in [6] that such a scheme maximizes aggregate utility.

In the special case where the network consists of only a single link, a given user is allocated a fraction of the link equal to his bid divided by the sum of all users' bids. This "proportional" allocation mechanism has been considered in a variety of other contexts as well. Hajek and Gopalakrishnan have considered such a mechanism in the context of Internet autonomous system competition [7]. They suggest that smaller Internet providers might bid for resources from larger Internet providers upstream using the proportional allocation mechanism. In the economics literature, such a mechanism is referred to as a "raffle"; it has been analyzed in the context of charitable fundraising [8]. In the computer science community, this mechanism is known as the "proportional share" mechanism, where it has been investigated for time-sharing of resources [9].

In this paper, we wish to understand the extent to which the analysis proposed in [6] accurately models the interactions of network users. Specifically, a fundamental assumption in the model of [6] is that each user acts as a price taker; that is, users do not anticipate the effect of their actions on the prices of the links. In contrast, we relax this assumption, and ask whether price anticipating behavior significantly worsens the performance of the network. If we assume that users can anticipate the effects of their actions, then the model becomes a game; we will show that the Nash equilibria of this game lead to allocations at which total utility is no worse than 3/4 the aggregate system utility.

The fact that Nash equilibria of a game may not achieve full efficiency has been well known in the economics literature [10]. Recent research efforts have focused on quantifying this loss for specific game environments; the resulting degree of efficiency loss is known as the "price of anarchy" [11]. Most of the results on price of anarchy have focused on routing [12], traffic networks [13, 14], and network design [15, 16], as well as a special class of submodular games including facility location games [17]. Stated in the language of this literature, the central result of our paper is that the price of anarchy of the network pricing mechanism studied is an efficiency loss of no more than 25%. The investigation of the price of anarchy provides a foundation for design of engineering systems with robustness against selfish behavior; in particular, our results suggest that selfish behavior of individual network users need not degrade network performance arbitrarily, provided the network pricing mechanism is carefully chosen.

The remainder of the paper is organized as follows. In Section 2 we give background on the model formulation. We recapitulate the key results of [6], and precisely define the notion of price taking. We prove the main theorem of [6] for a single link: if users are price taking, then aggregate utility is maximized. We then consider a game where users are price anticipating. We give a proof of a result due to Hajek and Gopalakrishnan establishing existence and uniqueness of a Nash equilibrium, by showing that at a Nash equilibrium, it is as if aggregate utility is maximized but with *modified* utility functions [7]. In Section 3, we consider the loss of efficiency at the Nash equilibrium of the single link game. Theorem 3 is key result of this paper: when users are price anticipating, the price of anarchy is a 25% efficiency loss.

In Section 4, we extend the earlier analysis to networks. We consider a game where each user requests service from multiple links by submitting a bid to each link. Users have multiple routes available to them for sending traffic, so that this is a model including alternative routing. Links then allocate rates using the same scheme as in the single link model, and each user sends the maximum flow possible, given the vector of rates allocated from links in the network. Although this definition of the game is natural, we demonstrate that Nash equilibria may not exist, due to a discontinuity in the payoff functions of individual players. (This problem also arises in the single link setting, but is irrelevant there as long as more than two players share the link.) To address the discontinuity, we extend the strategy space by allowing each user to request a nonzero rate without submitting a positive bid to a link, if the total payment made by other users at that link is zero; this extension is sufficient to guarantee existence of a Nash equilibrium. Furthermore, if a Nash equilibrium exists in the original game, it corresponds naturally to a Nash equilibrium of the extended game. Finally, we show that in this network setting, the total utility achieved at any Nash equilibrium of the game is no less than 3/4 of the maximum possible aggregate utility. This extends the price of anarchy result from the single link case to the network setting.

In Section 5, we consider a more general resource allocation game. We suppose that users bid for multiple resources, as in Section 4; but we no longer define utility as a function of the maximum flow that a user can send. Rather, we allow the user's utility to be *any* concave function of the vector of resources allocated. Such a game can also be interpreted more generally; for example, each resource may be a raw material, and each end user may be a manufacturing facility that takes these raw materials as input. We show that such a game can be analyzed using the same methods as Section 4, and in particular prove once again that the efficiency loss is no worse than 25% relative to the system optimal operating point. We conclude in Section 6.

2 Background

Suppose R users share a communication link of capacity $C > 0$. Let d_r denote the rate allocated to user r. We assume that user r receives a *utility* equal to $U_r(d_r)$ if the allocated rate is d_r; we assume that utility is measured in monetary units. We also assume the utility function $U_r(d_r)$ is concave, strictly increasing, and continuously differentiable, with domain $d_r \geq 0$; concavity corresponds to the assumption of *elastic* traffic, as defined by Shenker [18]. Given complete knowledge and centralized control of the system, a natural problem for the network manager to try to solve is the following optimization problem [6]:

SYSTEM:

$$\text{maximize} \quad \sum_r U_r(d_r) \tag{1}$$

$$\text{subject to} \quad \sum_r d_r \leq C; \tag{2}$$

$$d_r \geq 0, \quad r = 1, \dots, R. \tag{3}$$

Since the objective function is continuous and the feasible region is compact, an optimal solution $\boldsymbol{d} = (d_1, \dots, d_R)$ exists; since the feasible region is convex, if the functions U_r are strictly concave, then the optimal solution is unique.

In general, the utility functions are not available to the link manager. As a result, we consider the following pricing scheme for rate allocation. Each user r gives a payment (also called a *bid*) of w_r to the link manager; we assume $w_r \geq 0$. Given the vector $\boldsymbol{w} = (w_1, \dots, w_r)$, the link manager chooses a rate allocation $\boldsymbol{d} = (d_1, \dots, d_r)$. We assume the manager treats all users alike—in other words, the network manager does not *price discriminate*. Each user is charged the same price $\mu > 0$, leading to $d_r = w_r/\mu$. We further assume the manager always seeks to allocate the entire link capacity C; in this case, following the analysis of [6], we expect the price μ to satisfy:

$$\sum_r \frac{w_r}{\mu} = C.$$

The preceding equality can only be satisfied if $\sum_r w_r > 0$, in which case we have:

$$\mu = \frac{\sum_r w_r}{C}. \tag{4}$$

In other words, if the manager chooses to allocate the entire available rate at the link, and does not price discriminate between users, then for every nonzero $\boldsymbol{w}$ there is a *unique* price $\mu > 0$ which must be chosen by the network, given by the previous equation.

In the remainder of the section, we consider two different models for how users might interact with this price mechanism. In Section 2.1, we consider a model where users do not anticipate the effect of their bids on the price, and establish existence of a competitive equilibrium (a result due to Kelly [6]). Furthermore, this competitive equilibrium leads to an allocation which solves *SYSTEM*. In Section 2.2, we change the model and assume users are price anticipating, and establish existence and uniqueness of a Nash equilibrium (a result due to Hajek and Gopalakrishnan [7]). Section 3 then considers the loss of efficiency at this Nash equilibrium, relative to the optimal solution to *SYSTEM*.

2.1 Price Taking Users and Competitive Equilibrium

In this section, we consider a *competitive equilibrium* between the users and the link manager [19], following the development of Kelly [6]. A central assumption in the definition of competitive equilibrium is that each user does not anticipate the effect of their payment w_r on the price μ, i.e., each user acts as a *price taker*. In this case, given a price $\mu > 0$, user r acts to maximize the following payoff function over $w_r \geq 0$:

$$P_r(w_r; \mu) = U_r\left(\frac{w_r}{\mu}\right) - w_r. \tag{5}$$

The first term represents the utility to user r of receiving a rate allocation equal to w_r/μ; the second term is the payment w_r made to the manager. Observe that since utility is measured in monetary units, the payoff is *quasilinear* in money, a typical assumption in modeling market mechanisms [19].

We now say a pair $(\boldsymbol{w}, \mu)$ with $\boldsymbol{w} \geq 0$ and $\mu > 0$ is a *competitive equilibrium* if users maximize their payoff as defined in (5), and the network "clears the market" by setting the price μ according to (4):

$$P_r(w_r; \mu) \;\geq\; P_r(\overline{w}_r; \mu) \quad \text{for } \overline{w}_r \geq 0, \;\; r = 1, \ldots, R; \tag{6}$$

$$\mu \;=\; \frac{\sum_r w_r}{C}. \tag{7}$$

Kelly shows in [6] that when users are price takers, there exists a competitive equilibrium, and the resulting allocation solves *SYSTEM*. This is formalized in the following theorem, adapted from [6]; we also present a proof for completeness.

Theorem 1 (Kelly, [6]) *Assume that for each r, the utility function U_r is concave, strictly increasing, and continuously differentiable. Then there exists a competitive equilibrium, i.e., a vector $\boldsymbol{w} = (w_1, \ldots, w_R) \geq 0$ and a scalar $\mu > 0$ satisfying* (6)-(7).

In this case, the scalar μ is uniquely determined, and the vector $\boldsymbol{d} = \boldsymbol{w}/\mu$ is a solution to SYSTEM. If the functions U_r are strictly concave, then $\boldsymbol{w}$ is uniquely determined as well.

Proof. The key idea in the proof is to use Lagrangian techniques to establish that optimality conditions for (6)-(7) are identical to the optimality conditions for the problem *SYSTEM*, under the identification $\boldsymbol{d} = \boldsymbol{w}/\mu$.

Step 1: Given $\mu > 0$, $\boldsymbol{w}$ satisfies (6) *if and only if:*

$$
\begin{aligned}
U_r'\left(\frac{w_r}{\mu}\right) &= \mu, \quad \text{if } w_r > 0; &(8)\\
U_r'(0) &\leq \mu, \quad \text{if } w_r = 0. &(9)
\end{aligned}
$$

Indeed, since U_r is concave, P_r is concave as well; and thus (8)-(9) are necessary and sufficient optimality conditions for (6).

Step 2: There exists a vector $\boldsymbol{d} \geq 0$ and a unique scalar $\mu > 0$ such that:

$$
\begin{aligned}
U_r'(d_r) &= \mu, \quad \text{if } d_r > 0; &(10)\\
U_r'(0) &\leq \mu, \quad \text{if } d_r = 0; &(11)\\
\sum_r d_r &= C. &(12)
\end{aligned}
$$

The vector $\boldsymbol{d}$ is then a solution to SYSTEM. If the functions U_r are strictly concave, then $\boldsymbol{d}$ is unique as well. As discussed above, at least one optimal solution to *SYSTEM* exists since the feasible region is compact and the objective function is continuous. We form the Lagrangian for the problem *SYSTEM*:

$$
\mathcal{L}(\boldsymbol{d}, \mu) = \sum_r U_r(d_r) - \mu \left(\sum_r d_r - C \right)
$$

Here the second term is a penalty for the link capacity constraint. The Slater constraint qualification ([20], Section 5.3) holds for the problem *SYSTEM* at the point $\boldsymbol{d} = 0$, since then $0 = \sum_r d_r < C$; this guarantees the existence of a *Lagrange multiplier* μ. In other words, since the objective function is concave and the feasible region is convex, a feasible vector $\boldsymbol{d}$ is optimal if and only if there exists $\mu \geq 0$ such that the conditions (10)-(12) hold. Since there exists at least one optimal solution $\boldsymbol{d}$ to *SYSTEM*, there exists at least one pair $(\boldsymbol{d}, \mu)$ satisfying (10)-(12).

Since $C > 0$, at least one d_r is positive, so $\mu > 0$ (since U_r is strictly increasing). We now claim that μ is uniquely determined. Suppose not; then there exist $(\boldsymbol{d}, \mu)$, $(\overline{\boldsymbol{d}}, \overline{\mu})$ that satisfy (10)-(12), where (without loss of generality) $\mu < \overline{\mu}$. For any r such that $\overline{d}_r > 0$, we will have

$U_r'(d_r) \leq \mu < \overline{\mu} = U_r'(\overline{d}_r)$, which implies that $d_r > \overline{d}_r > 0$. Summing over all r, we obtain $\sum_r d_r > \sum_r \overline{d}_r$, which contradicts the feasibility condition $\sum_r d_r = C = \sum_r \overline{d}_r$. Thus μ is unique.

Step 3: If the pair $(\boldsymbol{d}, \mu)$ satisfies (10)-(12)*, and we let $\boldsymbol{w} = \mu \boldsymbol{d}$, then the pair $(\boldsymbol{w}, \mu)$ satisfies* (6)-(7). By Step 2, $\mu > 0$; thus, under the identification $\boldsymbol{w} = \mu \boldsymbol{d}$, (12) becomes equivalent to (7). Furthermore, (10)-(11) become equivalent to (8)-(9); by Step 1, this guarantees that (6) holds.

Step 4: If $\boldsymbol{w}$ and $\mu > 0$ satisfy (6)-(7)*, and we let $\boldsymbol{d} = \boldsymbol{w}/\mu$, then the pair $(\boldsymbol{d}, \mu)$ satisfies* (10)-(12). We simply reverse the argument of Step 3. Under the identification $\boldsymbol{d} = \boldsymbol{w}/\mu$, (8)-(9) become equivalent to (10)-(11); and (7) becomes equivalent to (12).

Step 5: Completing the proof. By Steps 2 and 3, there exists a vector $\boldsymbol{w}$ and a scalar $\mu > 0$ satisfying (6)-(7); by Step 4, μ is uniquely determined, and the vector $\boldsymbol{d} = \boldsymbol{w}/\mu$ is a solution to *SYSTEM*. Finally, if the utility functions U_r are strictly concave, then by Steps 2 and 4, $\boldsymbol{w}$ is uniquely determined as well (since the transformation from $(\boldsymbol{w}, \mu)$ to $(\boldsymbol{d}, \mu)$ is one-to-one). □

2.2 Price Anticipating Users and Nash Equilibrium

We now consider an alternative model where the users of a single link are price anticipating, rather than price takers. The key difference is that while the payoff function P_r takes the price μ as a fixed parameter in (5), price anticipating users will realize that μ is set according to (4), and adjust their payoff accordingly; this makes the model a game between the R players.

We use the notation $\boldsymbol{w}_{-r}$ to denote the vector of all bids by users other than r; i.e., $\boldsymbol{w}_{-r} = (w_1, w_2, \ldots, w_{r-1}, w_{r+1}, \ldots, w_R)$. Given $\boldsymbol{w}_{-r}$, each user r chooses w_r to maximize:

$$Q_r(w_r; \boldsymbol{w}_{-r}) = \begin{cases} U_r\left(\dfrac{w_r}{\sum_s w_s} C\right) - w_r, & \text{if } w_r > 0; \\ U_r(0), & \text{if } w_r = 0. \end{cases} \tag{13}$$

over nonnegative w_r. The second condition is required so that the rate allocation to user r is zero when $w_r = 0$, even if all other users choose $\boldsymbol{w}_{-r}$ so that $\sum_{s \neq r} w_s = 0$. The payoff function Q_r is similar to the payoff function P_r, except that the user anticipates that the network will set the price μ according to (4). A *Nash equilibrium* of the game defined by $(Q_1, \ldots, Q_R)$ is a vector $\boldsymbol{w} \geq 0$ such that for all r:

$$Q_r(w_r; \boldsymbol{w}_{-r}) \geq Q_r(\overline{w}_r; \boldsymbol{w}_{-r}), \quad \text{for all } \overline{w}_r \geq 0. \tag{14}$$

Note that the payoff function in (13) may be discontinuous at $w_r = 0$, if $\sum_{s \neq r} w_s = 0$. This discontinuity may preclude existence of a Nash equilibrium, as the following example shows.

Example 1 Suppose there is a single user with strictly increasing utility function U. In this case, the user is not playing a game against anyone else, so any positive payment results in the entire link being allocated to the single user. The payoff to the user is thus:

$$Q(w) = \begin{cases} U(C) - w, & \text{if } w > 0; \\ U(0), & \text{if } w = 0. \end{cases}$$

Since U has been assumed to be strictly increasing, we know $U(C) > U(0)$. Thus, at a bid of $w = 0$, a profitable deviation for the user is any bid $\overline{w}$ such that $0 < \overline{w} < U(C) - U(0)$. On the other hand, at any bid $w > 0$, a profitable deviation for the user is any bid $\overline{w}$ such that $0 < \overline{w} < w$. Thus no optimal choice of bid exists for the user, which implies that no Nash equilibrium exists.$\Box$

We will find the previous discontinuity plays a larger role in the network context, where an extended strategy space is required to ensure existence of a Nash equilibrium. In the single link setting, Hajek and Gopalakrishnan have shown that there exists a unique Nash equilibrium when multiple users share the link, by showing that at a Nash equilibrium it is *as if* the users are solving another optimization problem of the same form as the problem *SYSTEM*, but with "modified" utility functions. This is formalized in the following theorem, adapted from [7]; we also present a proof for completeness.

Theorem 2 (Hajek and Gopalakrishnan, [7]) *Assume that $R > 1$, and that for each r, the utility function U_r is concave, strictly increasing, and continuously differentiable. Then there exists a unique Nash equilibrium $\boldsymbol{w} \geq 0$ of the game defined by $(Q_1, \ldots, Q_R)$, and it satisfies $\sum_r w_r > 0$.*

In this case, the vector $\boldsymbol{d}$ defined by:

$$d_r = \frac{w_r}{\sum_s w_s} C, \quad r = 1, \ldots, R, \tag{15}$$

is the unique solution to the following optimization problem:

GAME:

$$\text{maximize} \quad \sum_r \hat{U}_r(d_r) \tag{16}$$

$$\text{subject to} \quad \sum_r d_r \leq C; \tag{17}$$

$$d_r \geq 0, \quad r = 1, \ldots, R, \tag{18}$$

where

$$\hat{U}_r(d_r) = \left(1 - \frac{d_r}{C}\right) U_r(d_r) + \left(\frac{d_r}{C}\right) \left(\frac{1}{d_r} \int_0^{d_r} U_r(z)\, dz\right). \tag{19}$$

Proof. The proof proceeds in a number of steps. We first show that at a Nash equilibrium, at least two components of $\boldsymbol{w}$ must be positive. This suffices to show that the payoff function Q_r is strictly concave and continuously differentiable for each user r. We then establish necessary and sufficient conditions for $\boldsymbol{w}$ to be a Nash equilibrium; these conditions look similar to the optimality conditions (8)-(9) in the proof of Theorem 1, but for "modified" utility functions defined according to (19). Mirroring the proof of Theorem 1, we then show the correspondence between these conditions and the optimality conditions for the problem *GAME*. This correspondence establishes existence and uniqueness of a Nash equilibrium.

Step 1: If $\boldsymbol{w}$ is a Nash equilibrium, then at least two coordinates of $\boldsymbol{w}$ are positive. Fix a user r, and suppose $w_s = 0$ for every $s \neq r$. If $w_r > 0$, user r can maintain the same rate allocation and reduce his payment by reducing w_r slightly; and since U_r is strictly increasing, if $w_r = 0$, then user r can profitably deviate by infinitesimally increasing his bid w_r and capturing the entire link capacity C. Thus at a Nash equilibrium, $w_s > 0$ for some $s \neq r$. Since this holds for every user r, at least two coordinates of $\boldsymbol{w}$ must be positive.

Step 2: If the vector $\boldsymbol{w} \geq 0$ has at least two positive components, then the function $Q_r(\overline{w}_r; \boldsymbol{w}_{-r})$ is strictly concave and continuously differentiable in $\overline{w}_r$, for $\overline{w}_r \geq 0$. This follows from (13), because when $\sum_{s \neq r} w_s > 0$, the expression $\overline{w}_r/(\overline{w}_r + \sum_{s\neq r} w_s)$ is a strictly increasing function of $\overline{w}_r$ (for $\overline{w}_r \geq 0$); in addition, $U_r(\cdot)$ is a strictly increasing concave, and differentiable function by assumption.

Step 3: The vector $\boldsymbol{w}$ is a Nash equilibrium if and only if at least two components of $\boldsymbol{w}$ are positive, and for each r, the following conditions hold:

$$U_r'\left(\frac{w_r}{\sum_s w_s} C\right)\left(1 - \frac{w_r}{\sum_s w_s}\right) = \frac{\sum_s w_s}{C}, \quad \text{if } w_r > 0; \tag{20}$$

$$U_r'(0) \leq \frac{\sum_s w_s}{C}, \quad \text{if } w_r = 0. \tag{21}$$

Let $\boldsymbol{w}$ be a Nash equilibrium. By Steps 1 and 2, $\boldsymbol{w}$ has at least two positive components and $Q_r(\overline{w}_r; \boldsymbol{w}_{-r})$ is strictly concave and continuously differentiable in $\overline{w}_r \geq 0$. Thus w_r must be the unique maximizer of $Q_r(\overline{w}_r; \boldsymbol{w}_{-r})$ over $\overline{w}_r \geq 0$, and satisfy the following first order optimality conditions:

$$\frac{\partial Q_r}{\partial w_r}(w_r; \boldsymbol{w}_{-r}) \begin{cases} = 0, & \text{if } w_r > 0; \\ \leq 0, & \text{if } w_r = 0. \end{cases}$$

After multiplying through by $\sum_s w_s/C$, these are precisely the conditions (20)-(21).

Conversely, suppose that $\boldsymbol{w}$ has at least two strictly positive components, and satisfies (20)-(21). Then we may simply reverse the argument: by Step 2, $Q_r(\overline{w}_r; \boldsymbol{w}_{-r})$ is strictly concave and continuously differentiable in $\overline{w}_r \geq 0$, and in this case the conditions (20)-(21) imply that w_r maximizes $Q_r(\overline{w}_r; \boldsymbol{w}_{-r})$ over $\overline{w}_r \geq 0$. Thus $\boldsymbol{w}$ is a Nash equilibrium.

If we let $\mu = \sum_r w_r/C$, note that the conditions (20)-(21) have the same form as the optimality conditions (8)-(9), but for a different utility function given by $\hat{U}_r$. We now use this relationship to complete the proof in a manner similar to the proof of Theorem 1.

Step 4: The function $\hat{U}_r$ defined in (19) *is strictly concave and strictly increasing over $0 \leq d_r \leq C$.* The proof follows by differentiating, which yields $\hat{U}_r'(d_r) = U_r'(d_r)(1 - d_r/C)$. Since U_r is concave and strictly increasing, we know that $U_r'(d_r) > 0$, and that U_r' is nonincreasing; we conclude that $\hat{U}_r'(d_r)$ is nonnegative and strictly decreasing in d_r over the region $0 \leq d_r \leq C$, as required.

Step 5: There exists a unique vector $\mathbf{d}$ *and scalar* ρ *such that:*

$$U_r'(d_r)\left(1-\frac{d_r}{C}\right)=\rho, \quad \text{if } d_r>0; \tag{22}$$

$$U_r'(0)\leq\rho, \quad \text{if } d_r=0; \tag{23}$$

$$\sum_r d_r = C. \tag{24}$$

The vector $\mathbf{d}$ *is then the unique solution to GAME.* By Step 4, since $\hat{U}_r$ is continuous and strictly concave over the convex, compact feasible region for each r, we know that *GAME* has a unique solution. This solution $\boldsymbol{d}$ is uniquely identified by the stationarity conditions (22)-(23), together with the constraint that $\sum_r d_r \leq C$. Since $\hat{U}_r$ is strictly increasing for each r, the constraint (24) must hold as well. That ρ is unique then follows because at least one d_r must be strictly positive at the unique solution to *GAME*.

Step 6: If $(\boldsymbol{d},\rho)$ *satisfy* (22)-(24)*, then the vector* $\boldsymbol{w}=\rho\boldsymbol{d}$ *is a Nash equilibrium.* We first check that at least two components of $\boldsymbol{d}$ are positive, and that $\rho>0$. We know from (24) that at least one component of $\boldsymbol{d}$ is strictly positive. Suppose now that $d_r>0$, and $d_s=0$ for $s\neq r$. Then we must have $d_r=C$. But then by (22), we have $\rho=0$; on the other hand, since U_s is strictly increasing and concave, we have $U_s'(0)>0$ for all s, so (23) cannot hold for $s\neq r$. Thus at least two components of $\boldsymbol{d}$ are positive. In this case, it is seen from (22) that $\rho>0$ as well.

By Step 3, to check that $\boldsymbol{w}=\rho\boldsymbol{d}$ is a Nash equilibrium, we must only check the stationarity conditions (20)-(21). We simply note that under the identification $\boldsymbol{w}=\rho\boldsymbol{d}$, using (24) we have that:

$$\rho=\frac{\sum_r w_r}{C}; \text{ and } d_r=\frac{w_r}{\sum_s w_s}C.$$

Substitution of these expressions into (22)-(23) leads immediately to (20)-(21). Thus $\boldsymbol{w}$ is a Nash equilibrium.

Step 7: If $\mathbf{w}$ *is a Nash equilibrium, then the vector* $\mathbf{d}$ *defined by* (15) *and scalar* ρ *defined by* $\rho=(\sum_r w_r)/C$ *are the unique solution to* (22)-(24). We simply reverse the argument of Step 6. By Step 3, $\boldsymbol{w}$ satisfies (20)-(21). Under the identifications of (15) and $\rho=\sum_r w_r/C$, we find that $\boldsymbol{d}$ and ρ satisfy (22)-(24). By Step 5, such a pair $(\boldsymbol{d},\rho)$ is unique.

Step 8: There exists a unique Nash equilibrium $\mathbf{w}$*, and the vector* $\mathbf{d}$ *defined by* (15) *is the unique solution of GAME.* This conclusion is now straightforward. Existence follows by Steps 5 and 6, and uniqueness follows by Step 7 (since the transformation from $\boldsymbol{w}$ to $(\boldsymbol{d},\rho)$ is one-to-one). Finally, that $\boldsymbol{d}$ solves *GAME* follows by Steps 5 and 7. □

Theorem 2 shows that the unique Nash equilibrium of the single link game is characterized by the optimization problem *GAME*. Other games have also profited from such relationships—notably traffic routing games, in which Nash equilibria can be found as solutions to a global optimization problem. Roughgarden and Tardos use this fact to their advantage in computing the price of anarchy for such games [13]; Schulz and Stier-Moses also use this relationship to consider routing games in capacitated networks [14].

Theorem 2 is also closely related to *potential games* [21], where best responses of players are characterized by changes in a global potential function. In such games, the global minima of the potential function correspond to Nash equilibria, as we observed for the problem *GAME*. However, we note that despite this correspondence the objective function of the problem *GAME* is not a potential function.

Finally, we note that for the congestion game presented here, several authors have derived results similar to Theorem 2. Gibbens and Kelly [22] considered the special case where all the functions U_r are linear, and demonstrated existence and uniqueness of the Nash equilibrium in this setting. The first result for general utility functions was given by La and Anantharam [23], who showed that if the users' strategies are restricted to a compact set $[W_{\min}, W_{\max}]$, where $0 < W_{\min} < W_{\max} < \infty$, then there exists a unique Nash equilibrium. Maheswaran and Basar consider a model where a fixed value of $\epsilon > 0$ is added to the price of the link [24]; the allocation to user r is thus $d_r = w_r/(\sum_s w_s + \epsilon)$, which avoids the possible discontinuity of Q_r when $w_r = 0$. The authors demonstrate existence and uniqueness of the Nash equilibrium in this setting. It is possible to use the model of [24] to show existence (but not uniqueness) of the Nash equilibrium of the congestion game defined by $(Q_1, \ldots, Q_R)$, by taking a limit as $\epsilon \to 0$; indeed, such a limit forms the basis of our proof of existence of Nash equilibria in the network context (see Theorem 6).

3 Price of Anarchy of the Single Link Game

We let $\boldsymbol{d}^S$ denote an optimal solution to *SYSTEM*, and let $\boldsymbol{d}^G$ denote the unique optimal solution to *GAME*. We now investigate the *price of anarchy* of this system [11]; that is, how much utility is lost because the users attempt to "game" the system? To answer this question, we must compare the utility $\sum_r U_r(d_r^G)$ obtained when the users fully evaluate the effect of their actions on the price, and the utility $\sum_r U_r(d_r^S)$ obtained by choosing the point which maximizes aggregate utility. (We know, of course, that $\sum_r U_r(d_r^G) \leq \sum_r U_r(d_r^S)$, by definition of $\boldsymbol{d}^S$.)

An easy lower bound on $\sum_r \hat{U}_r(d_r^G)$ may be constructed by using the modified utility functions $\hat{U}_r$ defined in (19). Notice that $\hat{U}_r(d_r)$ may be viewed as the "expectation" of U_r with respect to a probability distribution which places a mass of $1 - d_r/C$ on the rate d_r (the first term of (19)), and uniformly distributes the remaining mass of d_r/C on the interval $[0, d_r]$ (the second term of (19)). From this interpretation and the fact that U_r is strictly increasing, it follows that $\hat{U}(d_r) \leq U_r(d_r)$ if $0 \leq d_r \leq C$. Furthermore, if we assume that $U_r(0) \geq 0$, then using concavity of U_r, it is straightforward to establish that $\hat{U}(d_r) \geq U_r(d_r)/2$ for all d_r such that $0 \leq d_r \leq C$. Recalling that $\boldsymbol{d}^G$ solves *GAME*, and assuming that $U_r(0) \geq 0$ for all r, we can bound $\sum_r U_r(d_r^G)$ as follows:

$$\frac{1}{2}\sum_r U_r(d_r^S) \leq \sum_r \hat{U}_r(d_r^S) \leq \sum_r \hat{U}_r(d_r^G) \leq \sum_r U_r(d_r^G).$$

The preceding argument shows that the price of anarchy is no more than a 50% efficiency loss when users are price anticipating. However, this bound is not tight. As we show in the following theorem, the efficiency loss is exactly 25% in the worst case.

Theorem 3 *Assume that for each r, the utility function U_r is concave, strictly increasing, and continuously differentiable. Suppose also that $U_r(0) \geq 0$ for all r. If $\boldsymbol{d}^S$ is any solution to SYSTEM,* *and $\boldsymbol{d}^G$ is the unique solution to GAME, then:*

$$\sum_r U_r(d_r^G) \geq \frac{3}{4} \sum_r U_r(d_r^S).$$

Furthermore, this bound is tight: for every $\epsilon > 0$, there exists a choice of R, and a choice of (linear) utility functions U_r, $r = 1, \dots, R$, such that:

$$\sum_r U_r(d_r^G) \leq \left(\frac{3}{4} + \epsilon\right) \left(\sum_r U_r(d_r^S)\right).$$

In other words, for this system the price of anarchy is a 25% efficiency loss.

Proof. We first show that because of the assumption that U_r is concave and strictly increasing for each r, the worst case occurs with linear utility functions. This is summarized in the following lemma.

Lemma 4 *Suppose that $U_r(0) \geq 0$ for all r. Let $\overline{\boldsymbol{d}} = (\overline{d}_1, \dots, \overline{d}_r)$ satisfy $\sum_r \overline{d}_r \leq C$, and let $\boldsymbol{d}^S$ be any solution to SYSTEM. Then the following inequality holds:*

$$\frac{\sum_r U_r(\overline{d}_r)}{\sum_r U_r(d_r^S)} \geq \frac{\sum_r U_r'(\overline{d}_r)\overline{d}_r}{\left(\max_r U_r'(\overline{d}_r)\right) C}. \tag{25}$$

Proof of Lemma. Using concavity, we have $U_r(d_r^S) \leq U_r(\overline{d}_r) + U_r'(\overline{d}_r)(d_r^S - \overline{d}_r)$. Thus:

$$\frac{\sum_r U_r(\overline{d}_r)}{\sum_r U_r(d_r^S)} \geq \frac{\sum_r (U_r(\overline{d}_r) - U_r'(\overline{d}_r)\overline{d}_r) + \sum_r U_r'(\overline{d}_r)\overline{d}_r}{\sum_r (U_r(\overline{d}_r) - U_r'(\overline{d}_r)\overline{d}_r) + \sum_r U_r'(\overline{d}_r) d_r^S}.$$

Furthermore, since $\sum_r d_r^S = C$, we have the following trivial inequality:

$$\sum_r U_r'(\overline{d}_r) d_r^S \leq \left(\max_r U_r'(\overline{d}_r)\right) C.$$

This yields:

$$\frac{\sum_r U_r(\overline{d}_r)}{\sum_r U_r(d_r^S)} \geq \frac{\sum_r (U_r(\overline{d}_r) - U_r'(\overline{d}_r)\overline{d}_r) + \sum_r U_r'(\overline{d}_r)\overline{d}_r}{\sum_r (U_r(\overline{d}_r) - U_r'(\overline{d}_r)\overline{d}_r) + \left(\max_r U_r'(\overline{d}_r)\right) C}.$$

Now notice that because we have assumed $U_r(0) \geq 0$, we again have by concavity that $U_r'(\overline{d}_r)\overline{d}_r \leq U_r(\overline{d}_r)$. Thus the expression $\sum_r (U_r(\overline{d}_r) - U_r'(\overline{d}_r)\overline{d}_r)$ is nonnegative, so we conclude that:

$$\frac{\sum_r U_r(\overline{d}_r)}{\sum_r U_r(d_r^S)} \geq \frac{\sum_r U_r'(\overline{d}_r)\overline{d}_r}{\left(\max_r U_r'(\overline{d}_r)\right) C},$$

since the right hand side of the expression above is less than or equal to 1. $\square$

Let $\boldsymbol{d}^G$ be the unique Nash equilibrium of the game with utility functions $U_1, \dots, U_R$. We define a new collection of linear utility functions by:

$$\overline{U}_r(d_r) = U_r'(d_r^G) d_r.$$

Notice that the stationarity conditions (22)-(24) only involve the first derivatives of the utility functions U_r, $r = 1, \ldots, R$, at $\boldsymbol{d}^G$; thus, the unique Nash equilibrium of the game with utility functions $\overline{U}_1, \ldots, \overline{U}_R$ is given by $\boldsymbol{d}^G$ as well. Formally, $\boldsymbol{d}^G$ satisfies the stationarity conditions (22)-(24) for the family of utility functions $\overline{U}_1, \ldots, \overline{U}_R$. Furthermore, the system optimal aggregate utility for this family of utility functions is given by $\left(\max_r U_r'(d_r^G)\right) C$. Applying Lemma 4 with $\overline{\boldsymbol{d}} = \boldsymbol{d}^G$, we thus see that the worst case price of anarchy occurs in the case of linear utility functions. We now proceed to calculate this price of anarchy.

Assume for the remainder of the proof, therefore, that U_r is linear, with $U_r(d_r) = \alpha_r d_r$, where $\alpha_r > 0$. Let $\boldsymbol{d}^G$ be the Nash equilibrium of the game with these utility functions. From the discussion in the preceding paragraph, the ratio of aggregate utility at the Nash equilibrium to aggregate utility at the social optimum is given by:

$$\frac{\sum_r \alpha_r d_r^G}{(\max_r \alpha_r)\, C}.$$

By scaling and relabeling if necessary, we assume without loss of generality that $\max_r \alpha_r = \alpha_1 = 1$, and $C = 1$. To identify the worst case situation, we would like to find $\alpha_2, \ldots, \alpha_R$ such that $d_1^G + \sum_{r=2}^R \alpha_r d_r^G$, the total utility associated with the Nash equilibrium, is as small as possible; this results in the following optimization problem (with unknowns $d_1^G, \ldots, d_R^G, \alpha_2, \ldots, \alpha_R$):

$$\begin{aligned}
\text{minimize} \quad & d_1^G + \sum_{r=2}^R \alpha_r d_r^G && && (26)\\
\text{subject to} \quad & \alpha_r(1 - d_r^G) = 1 - d_1^G, && \text{if } d_r^G > 0; && (27)\\
& \alpha_r \leq 1 - d_1^G, && \text{if } d_r^G = 0; && (28)\\
& \sum_r d_r^G = 1; && && (29)\\
& 0 \leq \alpha_r \leq 1, && r = 2, \ldots, R; && (30)\\
& d_r^G \geq 0, && r = 1, \ldots, R. && (31)
\end{aligned}$$

This optimization problem *chooses* linear utility functions with slopes less than or equal to 1 for players $2, \ldots, R$. The constraints in the problem require that given linear utility functions $U_r(d_r) = \alpha_r d_r$ for $r = 1, \ldots, R$, the allocation $\boldsymbol{d}^G$ must in fact be the unique Nash equilibrium allocation of the resulting game. As a result, the optimal objective function value is precisely the lowest possible aggregate utility achieved, among all such games. In addition, since $C = 1$, and the largest α_r is $\alpha_1 = 1$, the system optimal aggregate utility is exactly 1; thus, the optimal objective function value of this problem also directly gives the price of anarchy.

Suppose now $(\boldsymbol{\alpha}, \boldsymbol{d})$ is an optimal solution to (26)-(31) in which $n < R$ users, say users $r = R - n + 1, \ldots, R$, have $d_r^G = 0$. Then the first $R - n$ coordinates of $\boldsymbol{\alpha}$ and $\boldsymbol{d}$ must be an optimal solution to the problem (26)-(31), with $R - n$ users. Therefore, in finding the worst case game, it suffices to assume that $d_r^G > 0$ for all $r = 2, \ldots, R$, and then consider the optimal objective function value for $R = 2, 3, \ldots$. This allows us to consider only the constraint:

$$\alpha_r(1 - d_r^G) = 1 - d_1^G. \tag{32}$$

This constraint then implies that $\alpha_r = (1 - d_1^G)/(1 - d_r^G)$. We will solve the resulting "reduced" optimization problem by decomposing it into two stages. First, we fix a choice of d_1^G and optimize over d_r^G, $r = 2, \dots, R$; then, we choose the optimal value of d_1^G.

Given these observations, we fix d_1^G, and consider the following, simpler optimization problem:

$$\begin{aligned} \text{minimize} \quad & d_1^G + \sum_{r=2}^{R} \frac{d_r^G(1 - d_1^G)}{1 - d_r^G} \\ \text{subject to} \quad & \sum_{r=2}^{R} d_r^G = 1 - d_1^G; \\ & 0 \le d_r^G \le d_1^G, \quad r = 2, \dots, R. \end{aligned}$$

Notice that we have substituted for α_r in the objective function. The constraint $\alpha_r \le 1$ becomes equivalent to $d_r^G \le d_1^G$ under the identification (32).

This simpler problem is only well defined if $d_1^G \ge 1/R$; otherwise the feasible region is empty—in other words, there exist no Nash equilibria with $d_1^G < 1/R$. If we assume that $d_1^G \ge 1/R$, then the feasible region is convex, compact, and nonempty, and the objective function is strictly convex in each of the variables d_r^G, $r = 2, \dots, R$. This is sufficient to ensure that there exists a unique optimal solution as a function of d_1^G; further, by symmetry, this optimal solution must be:

$$d_r^G = \frac{1 - d_1^G}{R - 1},$$

for $r = 2, \dots, R$.

We now optimize over d_1^G. After substituting, we have the following optimization problem:

$$\begin{aligned} \text{minimize} \quad & d_1^G + (1 - d_1^G)^2 \left(1 - \frac{1 - d_1^G}{R - 1}\right)^{-1} \\ \text{subject to} \quad & \frac{1}{R} \le d_1^G \le 1. \end{aligned}$$

The objective function for the preceding optimization problem is decreasing in R for every value of γ; in the limit where $R \to \infty$, the worst case price of anarchy is given by the solution to:

$$\begin{aligned} \text{minimize} \quad & d_1^G + (1 - d_1^G)^2 \\ \text{subject to} \quad & 0 \le d_1^G \le 1. \end{aligned}$$

It is simple to establish that the solution to this problem occurs at $d_1^G = 1/2$, which yields a worst case aggregate utility of $3/4$, as required. This bound is tight in the limit where the number of users increases to infinity; using this fact, we obtain the tightness claimed in the theorem. $\square$

The preceding theorem shows that in the worst case, aggregate utility falls by no more than 25% when users are able to anticipate the effects of their actions on the price of the link. Furthermore, this bound is essentially tight. In fact, it follows from the proof that the worst case consists of a link of capacity 1, where user 1 has utility $U_1(d_1) = d_1$, and all other users have utility $U_r(d_r) \approx d_r/2$ (when R is large). As R goes to infinity, at the Nash equilibrium of this game user 1 receives a rate

$d_1^G = 1/2$, while the remaining users uniformly split the rate $1 - d_1^G = 1/2$ among themselves, yielding an aggregate utility of $3/4$.

We note that a similar bound was observed by Roughgarden and Tardos for traffic routing games with affine link latency functions [13]. They found that the ratio of worst case Nash equilibrium cost to optimal cost was $4/3$. However, it remains an open question whether a relationship can be drawn between the two games; in particular, we note that while Theorem 3 holds even if the utility functions are nonlinear, Roughgarden and Tardos have shown that the price of anarchy in traffic routing may be arbitrarily high if link latency functions are nonlinear.

4 General Networks

In this section we will consider an extension of the single link model to general networks. We consider a network consisting of J links, numbered $1, \ldots, J$. Link j has a *capacity* given by $C_j > 0$; we let $\boldsymbol{C} = (C_1, \ldots, C_J)$ denote the vector of capacities. As before, a set of users numbered $1, \ldots, R$ shares this network of links. We assume there exists a set of paths through the network, numbered $1, \ldots, P$. By an abuse of notation, we will use J, R, and P to also denote the sets of links, users, and paths, respectively. Each path $p \in P$ uses a subset of the set of links J; if link j is used by path p, we will denote this by writing $j \in p$. Each user $r \in R$ has a collection of paths available through the network; if path p serves user r, we will denote this by writing $p \in r$. We will assume without loss of generality that paths are uniquely identified with users, so that for each path p there exists a unique user r such that $p \in r$. (There is no loss of generality because if two users share the same path, that is captured in our model by creating two paths which use exactly the same subset of links.) For notational convenience, we note that the links required by individual paths are captured by the *path-link incidence matrix* $\boldsymbol{A}$, defined by:

$$A_{jp} = \begin{cases} 1, & \text{if } j \in p; \\ 0, & \text{if } j \notin p. \end{cases}$$

Furthermore, we can capture the relationship between paths and users by the *path-user incidence matrix* $\boldsymbol{H}$, defined by:

$$H_{rp} = \begin{cases} 1, & \text{if } p \in r; \\ 0, & \text{if } p \notin r. \end{cases}$$

Note that by our assumption on paths, for each path p we have $H_{rp} = 1$ for exactly one user r.

Let $y_p \geq 0$ denote the rate allocated to path p, and let $d_r = \sum_{p \in r} y_p \geq 0$ denote the rate allocated to user r; using the matrix $\boldsymbol{H}$, we may write the relation between $\boldsymbol{d} = (d_r, r \in R)$ and $\boldsymbol{y} = (y_p, p \in P)$ as $\boldsymbol{H}\boldsymbol{y} = \boldsymbol{d}$. Any feasible rate allocation $\boldsymbol{y}$ must satisfy the following constraint:

$$\sum_{p: j \in p} y_p \leq C_j, \quad j \in J.$$

Using the matrix $\boldsymbol{A}$, we may write this constraint as $\boldsymbol{A}\boldsymbol{y} \leq \boldsymbol{C}$.

We continue to assume that user r receives a utility $U_r(d_r)$ from an amount of rate d_r, where the utility function U_r is concave, nondecreasing, and continuous, with domain $d_r \geq 0$. (Observe that we no longer require that U_r be strictly increasing or differentiable, as in the previous development.) The natural generalization of the problem *SYSTEM* to a network context is given by the following optimization problem:

SYSTEM:

$$\begin{aligned} \text{maximize} \quad & \sum_r U_r(d_r) && (33) \\ \text{subject to} \quad & \boldsymbol{A}\boldsymbol{y} \leq \boldsymbol{C}; && (34) \\ & \boldsymbol{H}\boldsymbol{y} = \boldsymbol{d}; && (35) \\ & y_p \geq 0, \;\; p \in P. && (36) \end{aligned}$$

Since the objective function is continuous and the feasible region is compact, an optimal solution $\boldsymbol{y}$ exists; since the feasible region is also convex, if the functions U_r are strictly concave, then the optimal vector $\boldsymbol{d} = \boldsymbol{H}\boldsymbol{y}$ is uniquely defined (though $\boldsymbol{y}$ need not be unique). As in the previous section, we will use the solution to *SYSTEM* as a benchmark for the outcome of the network congestion game.

We now define the resource allocation mechanism for this network setting. The natural extension of the single link model is defined as follows. Each user r submits a *bid* w_{jr} for each link j; this defines a strategy for user r given by $\boldsymbol{w}_r = (w_{jr}, j \in J)$, and a composite strategy vector given by $\boldsymbol{w} = (\boldsymbol{w}_1, \ldots, \boldsymbol{w}_R)$. We then assume that each link takes these bids as input, and uses the pricing scheme developed in the previous section. Formally, each link sets a price $\mu_j(\boldsymbol{w})$, given by:

$$\mu_j(\boldsymbol{w}) = \frac{\sum_r w_{jr}}{C_j}. \tag{37}$$

As before, we assume the rate allocated to a user is proportional to his payment. We denote by $x_{jr}(\boldsymbol{w})$ the rate allocated to user r by link j; we thus have:

$$x_{jr}(\boldsymbol{w}) = \begin{cases} \dfrac{w_{jr}}{\mu_j(\boldsymbol{w})}, & \text{if } w_{jr} > 0; \\ 0, & \text{otherwise.} \end{cases} \tag{38}$$

We define the vector $\boldsymbol{x}_r(\boldsymbol{w})$ by:

$$\boldsymbol{x}_r(\boldsymbol{w}) = (x_{jr}(\boldsymbol{w}), j \in J).$$

Now given any vector $\overline{\boldsymbol{x}}_r = (\overline{x}_{jr}, j \in J)$, we define $d_r(\overline{\boldsymbol{x}}_r)$ to be the optimal value of the following optimization problem:

$$\begin{aligned} \text{maximize} \quad & \sum_{p \in r} y_p && (39) \\ \text{subject to} \quad & \sum_{p \in r : j \in p} y_p \leq \overline{x}_{jr}, \;\; j \in J; && (40) \\ & y_p \geq 0, \;\; p \in r. && (41) \end{aligned}$$

Given the strategy vector $\boldsymbol{w}$, we then define the rate allocated to user r as $d_r(\boldsymbol{x}_r(\boldsymbol{w}))$. To gain some intuition for this allocation mechanism, notice that when the vector of bids is $\boldsymbol{w}$, user r is allocated

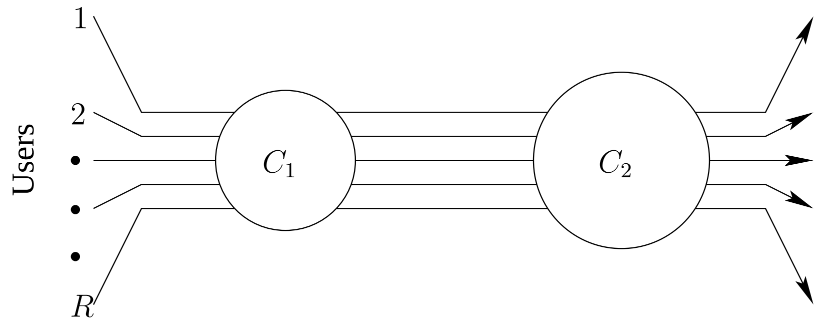

Figure 1: (Example 2) Link 1 has capacity C_1, and link 2 has capacity C_2, where $C_1 < C_2$. Each one of R users requires service from both links.

a rate $x_{jr}(\boldsymbol{w})$ at each link j. Since the utility to user r is nondecreasing in the total amount of rate allocated, user r's utility is maximized if he solves the preceding optimization problem, which is a *max-flow* problem constrained by the rate x_{jr} available at each link j. In other words, user r is allocated the maximum possible rate $d_r(\boldsymbol{x}_r(\boldsymbol{w}))$, given that each link j has granted him rate $x_{jr}(\boldsymbol{w})$.

Define the notation $\boldsymbol{w}_{-r} = (\boldsymbol{w}_1, \ldots, \boldsymbol{w}_{r-1}, \boldsymbol{w}_{r+1}, \ldots, \boldsymbol{w}_R)$. Based on the definition of $d_r(\boldsymbol{x}_r(\boldsymbol{w}))$ above, the payoff to user r is given by:

$$Q_r(\boldsymbol{w}_r; \boldsymbol{w}_{-r}) = U_r\big(d_r(\boldsymbol{x}_r(\boldsymbol{w}))\big) - \sum_j w_{jr}. \tag{42}$$

A *Nash equilibrium* of the game defined by $(Q_1, \ldots, Q_R)$ is a vector $\boldsymbol{w} \geq 0$ such that for all r:

$$Q_r(\boldsymbol{w}_r; \boldsymbol{w}_{-r}) \geq Q_r(\overline{\boldsymbol{w}}_r; \boldsymbol{w}_{-r}), \quad \text{for all } \overline{\boldsymbol{w}}_r \geq 0. \tag{43}$$

Although this pricing scheme is very natural, the fact that the payoff Q_r may be discontinuous can prevent existence of a Nash equilibrium, as we first observed in Example 1. Although we were able to prove a Nash equilibrium exists with $R > 1$ users for the single link case, the following example shows that Nash equilibria may not exist in the network context even if $R > 1$.

Example 2 Consider an example consisting of two links, labeled $j = 1$, and $j = 2$. The first link has capacity C_1, and the second link has capacity $C_2 > C_1$, as depicted in Figure 1. The system consists of R users, where we assume that each user r has a strictly increasing, concave, continuous utility function U_r. For this example, we will assume each user r is identified with a single path consisting of both links 1 and 2. This simplifies the analysis, since the solution to the problem (39)-(41) is then given by:

$$d_r(\boldsymbol{x}_r(\boldsymbol{w})) = \min\{x_{1r}(\boldsymbol{w}), x_{2r}(\boldsymbol{w})\}.$$

We will show that no Nash equilibrium exists for this network. Suppose, to the contrary, that $\boldsymbol{w}$ is a Nash equilibrium. We first show that $\sum_r w_{jr} > 0$, for $j = 1, 2$. If not, then all users are allocated zero rate. First suppose that $\sum_r w_{jr} = 0$ for both $j = 1, 2$. Then any user r can profitably deviate by infinitesimally increasing w_{1r} and w_{2r}, say by $\Delta > 0$; this deviation will give user r

rate $\min\{C_1, C_2\} = C_1$, and increase the total payment by 2Δ. For small enough Δ, this will strictly improve the payoff of player r; thus no Nash equilibrium exists where $\sum_r w_{jr} = 0$ for both $j = 1, 2$. A similar argument follows if $\sum_r w_{1r} = 0$, but $\sum_r w_{2r} > 0$: in this case, for any user r such that $w_{2r} > 0$, a profitable deviation exists where w_{2r} is reduced to zero; this leaves user r's rate allocation unchanged at zero, while reducing his total payment to the network. Symmetrically, the same argument may be used when $\sum_r w_{1r} > 0$, and $\sum_r w_{2r} = 0$. As a result, we conclude that if $\boldsymbol{w}$ is a Nash equilibrium, we must have $\sum_r w_{jr} > 0$ for both $j = 1, 2$.

Now note that (trivially) we have the relations:

$$\sum_r \frac{w_{1r}}{\sum_s w_{1s}} C_1 = C_1; \quad \text{and} \quad \sum_r \frac{w_{2r}}{\sum_s w_{2s}} C_2 = C_2.$$

Since $C_1 < C_2$, there must exist at least one user r for whom $(w_{1r}C_1)/(\sum_s w_{1s}) < (w_{2r}C_2)/(\sum_s w_{2s})$. Recall that user r is allocated a total rate equal to:

$$\min\left\{\frac{w_{1r}}{\sum_s w_{1s}} C_1, \frac{w_{2r}}{\sum_s w_{2s}} C_2\right\}.$$

As a result, user r can profitably deviate by reducing w_{2r}, since this reduces his payment, without altering his rate allocation. Thus no such $\boldsymbol{w}$ can be a Nash equilibrium. $\square$

As will be seen in the following development, the issue in the previous example is that link 2 is not a bottleneck in the network (since $C_1 < C_2$, link 2 will never be fully utilized). As a result, as long as the total payment $\sum_s w_{2s}$ to link 2 is strictly positive, there will always be some user r who is overpaying—i.e., this user could profitably deviate by reducing w_{2r}. Thus the only equilibrium outcome is one where the total payment to link 2 becomes zero; but in this case, because of the discontinuity in the payoff function defined in (42) (or, more precisely, the discontinuity in (38)), all users are allocated zero rate.

We will see in the following section that a resolution to this problem can be found if users are allowed to request and be allocated a nonzero rate from links for which the total payment is zero. We show that Nash equilibria are always guaranteed to exist for this "extended" game; furthermore, we show that any Nash equilibrium for the game defined by $(Q_1, \ldots, Q_R)$ corresponds in a natural way to a Nash equilibrium of the extended game. Finally, in Subsection 4.2, we show that the aggregate utility at any Nash equilibrium of the extended game is no less than 3/4 times the *SYSTEM* optimal aggregate utility, matching the result achieved for the single link game.

4.1 An Extended Game

In this section, we consider an extended game, where users not only submit bids, but also rate requests. We consider an allocation mechanism under which the rate requests are only taken into account by a link when the total payment to that link is zero. This behavior ensures that when a link is not congested (as in Example 2), or is not in sufficient demand (as in Example 1), users may still be allocated a nonzero rate on that link. In particular, this modification addresses the degeneracies which arise due to the discontinuity of Q_r in the original definition of the network game. We will show that Nash equilibria always exist for this extended game. (We note that extended strategy spaces have also proven fruitful for other games with payoff discontinuities; see, e.g., [25].)

Formally, we suppose that the strategy of user r includes a *rate request* $\phi_{jr} \geq 0$ at each link j; that is, the strategy of user r is a vector $\boldsymbol{\sigma}_r = (\boldsymbol{\phi}_r, \boldsymbol{w}_r)$, where $\boldsymbol{\phi}_r = (\phi_{jr}, j \in J)$, and $\boldsymbol{w}_r = (w_{jr}, j \in J)$, as before. We will write $\boldsymbol{\sigma} = (\boldsymbol{\sigma}_1, \ldots, \boldsymbol{\sigma}_R)$ to denote the composite strategy vector of all players; and we will write $\boldsymbol{\sigma}_{-r} = (\boldsymbol{\sigma}_1, \ldots, \boldsymbol{\sigma}_{r-1}, \boldsymbol{\sigma}_{r+1}, \ldots, \boldsymbol{\sigma}_R)$ to denote all components of $\boldsymbol{\sigma}$ other than $\boldsymbol{\sigma}_r$. We now suppose that each link j provides a rate $x_{jr}(\boldsymbol{\sigma})$ to user r, which is determined as follows:

1. If $\sum_s w_{js} > 0$, then:
$$x_{jr}(\boldsymbol{\sigma}) = \frac{w_{jr}}{\sum_s w_{js}} C_j. \tag{44}$$

2. If $\sum_s w_{js} = 0$, but $\sum_s \phi_{js} \leq C_j$, then:
$$x_{jr}(\boldsymbol{\sigma}) = \phi_{jr}. \tag{45}$$

3. If $\sum_s w_{js} = 0$ and $\sum_s \phi_{js} > C_j$, then:
$$x_{jr}(\boldsymbol{\sigma}) = 0. \tag{46}$$

In the first instance, when link j receives a positive payment from the users, rate is allocated in proportion to the bids. The second two cases apply only when the total payment to link j is zero; in this event, if the total requested rate is less than the capacity C_j, then the requests are granted. However, if the total requested rate exceeds capacity, no rate is allocated. We note here that the precise definition in case 3 above is not essential; any mechanism which splits the capacity C_j according to a preset deterministic rule will lead to the same results below. For example, if requests exceed capacity, a link may choose to allocate the same rate to all users who share the link; or the link may choose to allocate all the entire capacity to some predetermined "preferred" user.

As before, we define:
$$\boldsymbol{x}_r(\boldsymbol{\sigma}) = (x_{jr}(\boldsymbol{\sigma}), j \in J).$$
The rate of user r is then $d_r(\boldsymbol{x}_r(\boldsymbol{\sigma}))$ (where d_r is defined as the optimal value to the optimization problem (39)-(41)). The payoff T_r to user r is given by:
$$T_r(\boldsymbol{\sigma}_r; \boldsymbol{\sigma}_{-r}) = U_r\big(d_r(\boldsymbol{x}_r(\boldsymbol{\sigma}))\big) - \sum_j w_{jr}. \tag{47}$$

(Note that this is an abuse of notation in the definition of $\boldsymbol{x}_r$ and x_{jr}, since we previously had defined them as functions of $\boldsymbol{w}$. However, the definition in use will be clear from the argument of the function.)

A *Nash equilibrium* of the game defined by $(T_1, \ldots, T_R)$ is a vector $\boldsymbol{\sigma} \geq 0$ such that for all r:
$$T_r(\boldsymbol{\sigma}_r; \boldsymbol{\sigma}_{-r}) \geq T_r(\overline{\boldsymbol{\sigma}}_r; \boldsymbol{\sigma}_{-r}), \quad \text{for all } \overline{\boldsymbol{\sigma}}_r \geq 0. \tag{48}$$

We start with a theorem which states that the game defined in this subsection is indeed an extension of the original network game, defined by $(Q_1, \ldots, Q_R)$.

Theorem 5 *Assume that for each r, the utility function U_r is concave, nondecreasing, and continuous. Suppose that $\boldsymbol{w}$ is a strategy vector for the game defined by $(Q_1, \ldots, Q_R)$. For each user r, define:*

$$\phi_{jr} = \begin{cases} \dfrac{w_{jr}}{\sum_s w_{js}} C_j, & \textit{if } w_{jr} > 0; \\ 0, & \textit{otherwise}. \end{cases}$$

For each user r, let $\boldsymbol{\sigma}_r = (\boldsymbol{\phi}_r, \boldsymbol{w}_r)$. Then user r receives the same payoff in either game:

$$T_r(\boldsymbol{\sigma}_r; \boldsymbol{\sigma}_{-r}) = Q_r(\boldsymbol{w}_r; \boldsymbol{w}_{-r}).$$

Furthermore, if $\boldsymbol{w}$ is a Nash equilibrium of the game defined by $(Q_1, \ldots, Q_R)$, then $\boldsymbol{\sigma}$ is a Nash equilibrium of the game defined by $(T_1, \ldots, T_R)$.

Proof. We will refer to the game defined by $(Q_1, \ldots, Q_R)$ as the "original game," and the game defined by $(T_1, \ldots, T_R)$ as the "extended game." We first note that given the definition of ϕ_{jr} above, we have the identity $x_{jr}(\boldsymbol{\sigma}) = x_{jr}(\boldsymbol{w})$ for each link j; that is, the allocation from link j to user r in the extended game is identical to the allocation made by link j in the original game. Furthermore, the total payment made by user r remains unchanged in the extended game. Thus the payoff to user r is the same in both games, under the mapping from $\boldsymbol{w}$ to $\boldsymbol{\sigma}$ defined in the statement of the theorem.

Now suppose that $\boldsymbol{w}$ is a Nash equilibrium of the original game, and define $\boldsymbol{\sigma}$ as in the statement of the theorem. For each link j and each user r, define $W_{jr} = \sum_{s \neq r} w_{js}$. Suppose there exists a strategy vector $\overline{\boldsymbol{\sigma}}_r = (\overline{\boldsymbol{\phi}}_r, \overline{w}_r)$ such that:

$$U_r\big(d_r(\boldsymbol{x}_r(\overline{\boldsymbol{\sigma}}_r, \boldsymbol{\sigma}_{-r}))\big) - \sum_j \overline{w}_{jr} > U_r\big(d_r(\boldsymbol{x}_r(\boldsymbol{\sigma}))\big) - \sum_j w_{jr}.$$

Fix $\epsilon > 0$. For each j, we define $\hat{w}_{jr} = \overline{w}_{jr}$ if $W_{jr} > 0$, and $\hat{w}_{jr} = \epsilon$ if $W_{jr} = 0$. Then:

$$x_{jr}(\hat{\boldsymbol{w}}_r, \boldsymbol{w}_{-r}) \geq x_{jr}(\overline{\boldsymbol{\sigma}}_r, \boldsymbol{\sigma}_{-r}).$$

The preceding inequality follows because from each link $j \in r$ with $W_{jr} = 0$, user r is allocated the entire capacity C_j in return for the payment of $\epsilon > 0$. From this we may conclude that:

$$d_r(\boldsymbol{x}_r(\hat{\boldsymbol{w}}_r, \boldsymbol{w}_{-r})) \geq d_r(\boldsymbol{x}_r(\overline{\boldsymbol{\sigma}}_r, \boldsymbol{\sigma}_{-r})).$$

Now as $\epsilon \to 0$, we have $\sum_j \hat{w}_{jr} \to \sum_j \overline{w}_{jr}$. Thus for sufficiently small $\epsilon > 0$, we will have:

$$\begin{aligned} U_r\big(d_r(\boldsymbol{x}_r(\hat{\boldsymbol{w}}_r, \boldsymbol{w}_{-r}))\big) - \sum_j \hat{w}_{jr} &\geq U_r\big(d_r(\boldsymbol{x}_r(\overline{\boldsymbol{\sigma}}_r, \boldsymbol{\sigma}_{-r}))\big) - \sum_j \hat{w}_{jr} \\ &> U_r\big(d_r(\boldsymbol{x}_r(\boldsymbol{\sigma}))\big) - \sum_j w_{jr} \\ &= U_r\big(d_r(\boldsymbol{x}_r(\boldsymbol{w}))\big) - \sum_j w_{jr}. \end{aligned}$$

Thus the vector $\hat{\boldsymbol{w}}_r = (\hat{w}_{jr}, j \in r)$ is a profitable deviation for user r in the original game, a contradiction. Therefore no profitable deviation exists for user r in the extended game. We conclude

$\boldsymbol{\sigma}$ is a Nash equilibrium for the extended game, as required. $\square$

The preceding theorem shows that any Nash equilibrium of the original game corresponds naturally to a Nash equilibrium of the extended game. To construct a partial converse to this result, suppose that we are given a Nash equilibrium $\boldsymbol{\sigma} = (\boldsymbol{\phi}, \boldsymbol{w})$ of the extended game, but that $\sum_r w_{jr} > 0$ for all links j. We first note that for each link j, at least two distinct users submit positive bids. If not, then there is some link j where a single user r submits a positive bid—but this user can leave his rate allocation unchanged and reduce his payment by lowering the bid submitted to link j. Thus we conclude that for each link j and each user r, the payment by all other users $\sum_{s\neq r} w_{js}$ is positive. This ensures the rate requests $\boldsymbol{\phi}_r$ do not have any effect on the rate allocation made to user r, so that the payoffs are determined entirely by the bid vectors $\boldsymbol{w}_r$, for $r \in R$. This is sufficient to conclude that $\boldsymbol{w}$ must actually be a Nash equilibrium for the original game. To summarize, we have shown that whenever all link prices are positive at a Nash equilibrium in the extended game, then in fact we have a Nash equilibrium for the original game as well.

We now turn our attention to showing that a Nash equilibrium always exists for the extended game.

Theorem 6 *Assume that for each r, the utility function U_r is concave, nondecreasing, and continuous. Then a Nash equilibrium exists for the game defined by $(T_1, \ldots, T_R)$.*

Proof. Our technique is to consider a perturbed version of the *original* game, where a "virtual" user submits a bid of $\epsilon > 0$ to each link j in the network. Formally, this means that at a bid vector $\boldsymbol{w}$, user r is allocated a rate $x^{\epsilon}_{jr}(\boldsymbol{w})$ at link j, given by:

$$x^{\epsilon}_{jr}(\boldsymbol{w}) = \frac{w_{jr}}{\epsilon + \sum_s w_{js}} C_j.$$

We define the vector $\boldsymbol{x}^{\epsilon}_r(\boldsymbol{w}) = (x^{\epsilon}_{jr}(\boldsymbol{w}), j \in J)$, and the rate attained by user r is then $d_r(\boldsymbol{x}^{\epsilon}_r(\boldsymbol{w}))$, where d_r is the optimal value to the optimization problem (39)-(41).

The modified allocation defined by $\boldsymbol{x}^{\epsilon}_r$ was also considered by Maheswaran and Basar in the context of a single link [24]; we will use this allocation mechanism to prove existence for our game by taking a limit as $\epsilon \to 0$. Our approach will be to first apply standard fixed point techniques to establish existence of a Nash equilibrium $\boldsymbol{w}^{\epsilon}$ for this perturbed game, with an associated allocation to each user given by $\boldsymbol{x}^{\epsilon}_r(\boldsymbol{w}^{\epsilon})$. We will then show that $\boldsymbol{w}^{\epsilon}$ and $\boldsymbol{x}^{\epsilon}_r(\boldsymbol{w}^{\epsilon})$ (for each r) lie in compact sets, respectively. If we then choose $\boldsymbol{w}$ and $\boldsymbol{\phi} = (\boldsymbol{\phi}_r, r \in R)$ as limit points when $\epsilon \to 0$, we will find that $(\boldsymbol{w}, \boldsymbol{\phi})$ is a Nash equilibrium of the extended game.

Step 1: A Nash equilibrium $\boldsymbol{w}^{\epsilon}$ exists in the perturbed game. We first observe that since $\epsilon > 0$, $x^{\epsilon}_{jr}(\boldsymbol{w})$ is a continuous, strictly concave, and strictly increasing function of $w_{jr} \geq 0$ (in particular, there is no longer any discontinuity in the rate allocation at $w_{jr} = 0$). Furthermore, since d_r is defined as the maximal objective value of a linear program, $d_r(\boldsymbol{x}_r)$ is concave and continuous as a function of $\boldsymbol{x}_r$ ([26], Section 5.2); and if $x_{jr} \geq \overline{x}_{jr}$ for all j, then clearly $d_r(\boldsymbol{x}_r) \geq d_r(\overline{\boldsymbol{x}}_r)$, i.e., d_r is *nondecreasing* (this follows immediately from the problem (39)-(41)).

We will now combine these facts to show that user r's payoff in this perturbed game is concave as a function of $\boldsymbol{w}_r$, and continuous as a function of the composite strategy $\boldsymbol{w}$. The payoff in the perturbed game, denoted Q_r^ϵ, is given by:

$$Q_r^\epsilon(\boldsymbol{w}_r;\boldsymbol{w}_{-r}) = U_r\big(d_r(\boldsymbol{x}_r^\epsilon(\boldsymbol{w}))\big) - \sum_{j\in r} w_{jr}.$$

Continuity of Q_r^ϵ as a function of $\boldsymbol{w}$ follows immediately from continuity of x_{jr}^ϵ, d_r, and U_r. To show that Q_r^ϵ is concave as a function of $\boldsymbol{w}_r$, it suffices to show that $U_r(d_r(\boldsymbol{x}_r^\epsilon(\boldsymbol{w}_r,\boldsymbol{w}_{-r})))$ is a concave function of $\boldsymbol{w}_r$. Since for each j the function x_{jr}^ϵ is concave in w_{jr}, and does not depend on w_{kr} for $k \neq j$, we conclude that each component of $\boldsymbol{x}_r^\epsilon(\boldsymbol{w}_r,\boldsymbol{w}_{-r})$ is a concave function of $\boldsymbol{w}_r$. If we fix the bids of the other players as $\boldsymbol{w}_{-r}$, then since d_r is nondecreasing and concave in its argument, we have for any two bid vectors $\boldsymbol{w}_r, \overline{\boldsymbol{w}}_r$, and δ such that $0 \le \delta \le 1$:

$$\begin{aligned} d_r(\boldsymbol{x}_r^\epsilon(\delta\boldsymbol{w}_r + (1-\delta)\overline{\boldsymbol{w}}_r,\boldsymbol{w}_{-r})) &\ge d_r(\delta\boldsymbol{x}_r^\epsilon(\boldsymbol{w}_r,\boldsymbol{w}_{-r}) + (1-\delta)\boldsymbol{x}_r^\epsilon(\overline{\boldsymbol{w}}_r,\boldsymbol{w}_{-r}))\\ &\ge \delta d_r(\boldsymbol{x}_r^\epsilon(\boldsymbol{w}_r,\boldsymbol{w}_{-r})) + (1-\delta)d_r(\boldsymbol{x}_r^\epsilon(\overline{\boldsymbol{w}}_r,\boldsymbol{w}_{-r})).\end{aligned}$$

We now apply the fact that U_r is nondecreasing and concave to conclude that:

$$\begin{aligned} U_r\big(d_r(\boldsymbol{x}_r^\epsilon(\delta\boldsymbol{w}_r + (1-\delta)\overline{\boldsymbol{w}}_r,\boldsymbol{w}_{-r}))\big) &\ge U_r\big(\delta d_r(\boldsymbol{x}_r^\epsilon(\boldsymbol{w}_r,\boldsymbol{w}_{-r})) + (1-\delta)d_r(\boldsymbol{x}_r^\epsilon(\overline{\boldsymbol{w}}_r,\boldsymbol{w}_{-r}))\big)\\ &\ge \delta U_r\big(d_r(\boldsymbol{x}_r^\epsilon(\boldsymbol{w}_r,\boldsymbol{w}_{-r}))\big) + (1-\delta)U_r\big(d_r(\boldsymbol{x}_r^\epsilon(\overline{\boldsymbol{w}}_r,\boldsymbol{w}_{-r}))\big).\end{aligned}$$

Thus user r's payoff function $Q_r^\epsilon(\boldsymbol{w}_r;\boldsymbol{w}_{-r})$ is concave in $\boldsymbol{w}_r$.

Finally, we observe that in searching for a Nash equilibrium of the perturbed game defined by $(Q_1^\epsilon,\ldots,Q_R^\epsilon)$, we can restrict the strategy space of each user to a compact, convex subset of $\mathbb{R}^J$. To see this, fix a user r, and choose $B_r > U_r(\sum_j C_j) - U_r(0)$. When user r sets $\boldsymbol{w}_r = 0$, his payoff is $U_r(0)$. On the other hand, the maximum rate user r can be allocated from the network is bounded above by $\sum_j C_j$; and thus, if user r chooses any strategy $\boldsymbol{w}_r$ such that $\sum_j w_{jr} > B_r$, then regardless of the strategies $\boldsymbol{w}_{-r}$ of all other players, we have:

$$U_r(d_r(\boldsymbol{x}_r^\epsilon(\boldsymbol{w}_r,\boldsymbol{w}_{-r})) - \sum_j w_{jr} \le U_r(\sum_j C_j) - B_r < U_r(0).$$

Thus, if we define the compact set $S_r = \{\boldsymbol{w}_r : \sum_j w_{jr} \le B_r\}$, we observe that user r would never choose a strategy vector that lies outside S_r; this allows us to restrict the strategy space of user r to the set S_r.

The game defined by $(Q_1^\epsilon,\ldots,Q_R^\epsilon)$ together with the strategy spaces $(S_1,\ldots,S_R)$ is then a *concave R-person game*: each payoff function is continuous in the composite strategy vector $\boldsymbol{w}$; Q_r^ϵ is concave in $\boldsymbol{w}_r$; and the strategy space of each user r is a compact, convex, nonempty subset of $\mathbb{R}^J$. Applying Rosen's existence theorem [27] (proven using Kakutani's fixed point theorem), we conclude that a Nash equilibrium $\boldsymbol{w}^\epsilon$ exists for this game.

Step 2: There exists a limit point $\boldsymbol{\sigma} = (\boldsymbol{\phi},\boldsymbol{w})$ of the Nash equilibria of the perturbed games. For each user r, define $\phi_{jr}^\epsilon = x_{jr}^\epsilon(\boldsymbol{w}^\epsilon)$. Let $\boldsymbol{\phi}_r^\epsilon = (\phi_{jr}^\epsilon, j \in J)$, and $\boldsymbol{\phi}^\epsilon = (\boldsymbol{\phi}_r^\epsilon, r \in R)$. We now note that for all $\epsilon > 0$, the pair $(\boldsymbol{\phi}^\epsilon,\boldsymbol{w}^\epsilon)$ lies in a compact subset of Euclidean space. To see this, note that $\boldsymbol{w}^\epsilon$ lies in the compact set $S_1 \times \cdots \times S_R$, and that $0 \le \phi_{jr}^\epsilon \le C_j$ for all j and r. Thus, there exists a sequence $\epsilon_k \to 0$ such that the sequence $(\boldsymbol{\phi}^{\epsilon_k},\boldsymbol{w}^{\epsilon_k})$ converges to some $\boldsymbol{\sigma} = (\boldsymbol{\phi},\boldsymbol{w})$, where $\boldsymbol{w} \in S_1 \times \cdots \times S_R$ and $0 \le \phi_{jr} \le C_j$.

We expect that at the limit point $\boldsymbol{\sigma}$, the rates allocated to each user are the limits of the rates allocated in the perturbed games. Formally, we show that we have:

$$x_{jr}(\boldsymbol{\sigma}) = \lim_{k\to\infty} x_{jr}^{\epsilon_k}(\boldsymbol{w}^{\epsilon_k}). \tag{49}$$

Fix a link j, and suppose that $\sum_r w_{jr} = 0$. By definition, $\phi_{jr} = \lim_{k\to\infty} x_{jr}^{\epsilon_k}(\boldsymbol{w}^{\epsilon_k})$ for each r. We thus only need to show that $x_{jr}(\boldsymbol{\sigma}) = \phi_{jr}$ for each r, which follows from the rate allocation mechanism since:

$$\sum_r \phi_{jr} = \lim_{k\to\infty} \sum_r x_{jr}^{\epsilon_k}(\boldsymbol{w}^{\epsilon_k}) \leq C_j.$$

On the other hand, suppose that $\sum_r w_{jr} > 0$. In this case we have $x_{jr}(\boldsymbol{\sigma}) = (w_{jr}C_j)/(\sum_s w_{js}) = \lim_{k\to\infty} x_{jr}^{\epsilon_k}(\boldsymbol{w}^{\epsilon_k})$ for each r, as required.

Step 3: The vector $\boldsymbol{\sigma}$ is a Nash equilibrium of the extended game. Suppose $\boldsymbol{\sigma}$ is not a Nash equilibrium of the extended game; then there exists some user r, and a strategy vector $\overline{\boldsymbol{\sigma}}_r = (\overline{\boldsymbol{\phi}}_r, \overline{\boldsymbol{w}}_r)$, such that $T_r(\overline{\boldsymbol{\sigma}}_r; \boldsymbol{\sigma}_{-r}) > T_r(\boldsymbol{\sigma})$. Our goal will be to show that in this case, for sufficiently small $\epsilon > 0$, a profitable deviation exists for user r from the strategy vector $\boldsymbol{w}_r^\epsilon$ (i.e., from the chosen Nash equilibrium for the game defined by $Q_1^\epsilon, \ldots, Q_R^\epsilon$).

For fixed $\epsilon > 0$, we now construct a new strategy vector $\overline{\boldsymbol{w}}_r^\epsilon$ for user r. First fix a link j such that $W_{jr} > 0$; we then define $\overline{w}_{jr}^\epsilon > 0$ by:

$$\overline{w}_{jr}^\epsilon = \frac{W_{jr}^\epsilon + \epsilon}{W_{jr}} \overline{w}_{jr}.$$

Observe that with this definition, as $k \to \infty$, we have $\overline{w}_{jr}^{\epsilon_k} \to \overline{w}_{jr}$. We also have:

$$\frac{\overline{w}_{jr}}{\overline{w}_{jr} + W_{jr}} C_j = \frac{\overline{w}_{jr}^\epsilon}{\overline{w}_{jr}^\epsilon + W_{jr}^\epsilon + \epsilon} C_j.$$

This implies that $x_{jr}(\overline{\boldsymbol{\sigma}}_r, \boldsymbol{\sigma}_{-r}) = x_{jr}^\epsilon(\overline{\boldsymbol{w}}_r^\epsilon, \boldsymbol{w}_{-r}^\epsilon)$, regardless of how we define the remaining components of the vector $\overline{\boldsymbol{w}}_r^\epsilon$.

To complete this definition, suppose now that we fix a link j such that $W_{jr} = 0$. In this case we define $\overline{w}_{jr}^\epsilon = \sqrt{W_{jr}^\epsilon + \epsilon}$. (The specific form is not important here; for the proof we only require that when $W_{jr} = 0$, we have $\overline{w}_{jr}^\epsilon/(W_{jr}^\epsilon + \epsilon) \to \infty$ as $\epsilon \to 0$.) Then we have $\overline{w}_{jr}^{\epsilon_k} \to 0$ as $k \to \infty$. Furthermore:

$$x_{jr}^\epsilon(\overline{\boldsymbol{w}}_r^\epsilon, \boldsymbol{w}_{-r}^\epsilon) = \frac{\sqrt{W_{jr}^\epsilon + \epsilon}}{\sqrt{W_{jr}^\epsilon + \epsilon} + W_{jr}^\epsilon + \epsilon} C_j.$$

Since $W_{jr}^{\epsilon_k} + \epsilon_k \to 0$ as $k \to \infty$, we conclude that $x_{jr}^{\epsilon_k}(\overline{\boldsymbol{w}}_r^{\epsilon_k}, \boldsymbol{w}_{-r}^{\epsilon_k}) \to C_j$ as $k \to \infty$.

Define $\hat{w}_{jr}$ and $\hat{x}_{jr}$ as the limit of $\overline{w}_{jr}^{\epsilon_k}$ and $x_{jr}^{\epsilon_k}(\overline{\boldsymbol{w}}_r^{\epsilon_k}, \boldsymbol{w}_{-r}^{\epsilon_k})$, respectively. From the preceding discussion, as $k \to \infty$ we have the following relations:

$$\hat{w}_{jr} = \lim_{k\to\infty} \overline{w}_{jr}^{\epsilon^k} = \begin{cases} \overline{w}_{jr}, & \text{if } W_{jr} > 0; \\ 0, & \text{if } W_{jr} = 0. \end{cases} \tag{50}$$

$$\hat{x}_{jr} = \lim_{k\to\infty} x_{jr}^{\epsilon_k}(\overline{\boldsymbol{w}}_r^{\epsilon_k}, \boldsymbol{w}_{-r}^{\epsilon_k}) = \begin{cases} x_{jr}(\overline{\boldsymbol{w}}_r, \boldsymbol{w}_{-r}), & \text{if } W_{jr} > 0; \\ C_j, & \text{if } W_{jr} = 0. \end{cases} \tag{51}$$

(52)

From (50) we conclude $\hat{w}_{jr} \leq \overline{w}_{jr}$; and from (51) we conclude that $\hat{x}_{jr} \geq x_{jr}(\overline{\boldsymbol{\sigma}}_r, \boldsymbol{\sigma}_{-r})$. But then since the functions d_r and U_r are nondecreasing, we conclude that:

$$\begin{aligned} U_r\big(d_r(\hat{\boldsymbol{x}}_r)\big) - \sum_j \hat{w}_{jr} &\geq U_r\big(d_r(\boldsymbol{x}_r(\overline{\boldsymbol{\sigma}}_r, \boldsymbol{\sigma}_{-r}))\big) - \sum_j \overline{w}_{jr} \\ &> U_r\big(d_r(\boldsymbol{x}_r(\boldsymbol{\sigma}))\big) - \sum_j w_{jr}. \end{aligned}$$

The last inequality follows since $\overline{\boldsymbol{w}}_r$ is a profitable deviation for user r.

But now recall that the composite function $U_r(d_r(\cdot))$ is continuous in its argument; as a result, from the limits in (49), (50), and (51), we conclude that for sufficiently large k we will have:

$$U_r\big(d_r(\boldsymbol{x}_r^{\epsilon_k}(\overline{\boldsymbol{w}}_r^{\epsilon_k}, \boldsymbol{w}_{-r}^{\epsilon_k}))\big) - \sum_j \overline{w}_{jr}^{\epsilon_k} > U_r\big(d_r(\boldsymbol{x}_r^{\epsilon_k}(\boldsymbol{w}^{\epsilon_k}))\big) - \sum_j w_{jr}^{\epsilon_k}.$$

But this contradicts the fact that $\boldsymbol{w}^{\epsilon_k}$ is a Nash equilibrium for the game defined by $(Q_1^{\epsilon}, \ldots, Q_R^{\epsilon})$, since we have found a profitable deviation for user r. As a result, no profitable deviation $\overline{\boldsymbol{\sigma}}_r$ can exist for user r in the extended game with respect to the strategy vector $\boldsymbol{\sigma}$; thus we conclude that $\boldsymbol{\sigma}$ is a Nash equilibrium for the extended game, as required. $\square$

The previous theorem demonstrates that the "extended" strategy space eliminates the possibility of the nonexistence of a Nash equilibrium. Indeed, with the extended strategy space, both Examples 1 and 2 will possess at least one Nash equilibrium. In Example 1, the Nash equilibrium is for the single user to submit a bid of $w = 0$, and to request a rate $\phi = C$. In Example, 2, the Nash equilibrium is constructed as follows. First, all users play a single link game for link 1; suppose this results in the Nash equilibrium bid vector $(w_{11}, \ldots, w_{1R})$, with rate allocation to user r given by $x_{1r} = (w_{1r}C_1)/(\sum_s w_{1s})$. We may choose ϕ_{1r} arbitrarily, since it plays no role in the resulting allocation. Suppose each user then submits a bid of $w_{2r} = 0$ to link 2, but requests rate $\phi_{2r} = x_{1r}$ from link 2; since $\sum_r x_{1r} = C_1 < C_2$, these requests will be granted. It is straightforward to check that the strategy vector $(\boldsymbol{\phi}, \boldsymbol{w})$ is a Nash equilibrium for the extended game. We observe that at this Nash equilibrium, the total payment to link 2 is zero, reflecting the fact that link 2 is not a bottleneck.

We conclude by noting that while Theorem 6 establishes existence of a Nash equilibrium in the network case, we have not shown that such a Nash equilibrium is unique. In the special case where $C_j = C$ for all j (all capacities are equal), and each user is identified with exactly one path through the network (fixed routing), it is possible to use an argument analogous to the proof of Theorem 2 to show that a Nash equilibrium is unique; in particular, the Nash equilibrium conditions become equivalent to the optimality conditions for a network form of the problem *GAME*. In general, however, such a technique does not apply, and uniqueness of the Nash equilibrium remains an open question.

4.2 Price of Anarchy

Let $\boldsymbol{\sigma}$ be a Nash equilibrium of the extended game, i.e., the game defined by $(T_1, \ldots, T_R)$, and let $\boldsymbol{d}^G = (d_r(\boldsymbol{x}_r(\boldsymbol{\sigma})), r \in R)$ be the allocation at this Nash equilibrium. Let $\boldsymbol{d}^S$ denote any optimal solution to the network *SYSTEM* problem. The following theorem demonstrates that the utility lost at any Nash equilibrium is no worse than 25% of the maximum possible aggregate utility, matching the result derived in the single link model. We also note that this result does not require $R > 1$, or U_r to be strictly increasing and continuously differentiable; it is therefore a stronger version of Theorem 3 for the single link case.

Theorem 7 *Assume that for each r, the utility function U_r is concave, nondecreasing, and continuous. Assume also that $U_r(0) \geq 0$ for all users r. If $\boldsymbol{d}^G$ is any Nash equilibrium allocation for the extended network game, and $\boldsymbol{d}^S$ is any SYSTEM optimal allocation, then:*

$$\sum_r U_r(d_r^G) \geq \frac{3}{4} \sum_r U_r(d_r^S).$$

Proof. For the single user case ($R = 1$), at any Nash equilibrium the single user makes no payment to the link, and is granted any feasible capacity request. Thus any Nash equilibrium allocation yields a rate to user 1 given by $d_1(\boldsymbol{C})$, where $\boldsymbol{C}$ is the vector of link capacities. This allocation is an optimal solution to *SYSTEM*, so the theorem is trivially true. We assume without loss of generality, therefore, that $R > 1$ for the remainder of the proof.

As in the proof of Theorem 3, we also assume without loss of generality that $U_r(0) = 0$ for all users r. Our basic approach in this proof is to describe the entire problem in terms of the vector $\boldsymbol{x}_r = (x_{jr}, j \in J)$ of the rate allocation to user r from the network. We begin by redefining the problem *SYSTEM* as follows:

$$\text{maximize} \quad \sum_r U_r(d_r(\overline{\boldsymbol{x}}_r)) \tag{53}$$

$$\text{subject to} \quad \sum_r \overline{x}_{jr} \leq C_j, \quad j \in J; \tag{54}$$

$$\overline{x}_{jr} \geq 0, \quad j \in J,\ r \in R. \tag{55}$$

(The notation $\overline{\boldsymbol{x}}_r$ is used here to distinguish from the function $\boldsymbol{x}_r(\boldsymbol{\sigma})$.) In this problem, the network only chooses how to allocate rate at each link to the users. The users then solve a max-flow problem to determine the maximum rate at which they can send (this is captured by the function $d_r(\cdot)$). This problem is equivalent to the problem *SYSTEM* as defined in (33)-(36), because of the definition of $d_r(\cdot)$ in (39)-(41). We label an optimal solution to this problem by $(\boldsymbol{x}_r^S, r \in R)$.

Next, we prove a lemma which states that a Nash equilibrium may be characterized in terms of users optimally choosing rate allocations $(\overline{\boldsymbol{x}}_r, r \in R)$. As before, given a bid vector $\boldsymbol{w}$, for each link j and each user r we let $W_{jr} = \sum_{s \neq r} w_{js}$. In addition, we define the set $\mathcal{C} \subset \mathbb{R}^J$ by $\mathcal{C} = \{\overline{\boldsymbol{x}} = (x_j, j \in J) : 0 \leq x_j \leq C_j\}$. For $\overline{\boldsymbol{x}}_r \in \mathcal{C}$, we define a function $f_r(\overline{\boldsymbol{x}}_r; \boldsymbol{\sigma}_{-r})$ as follows:

$$f_r(\overline{\boldsymbol{x}}_r; \boldsymbol{\sigma}_{-r}) = \begin{cases} -\infty, & \text{if } \overline{x}_{jr} = C_j \text{ for some } j \text{ with } W_{jr} > 0; \\ U_r(d_r(\overline{\boldsymbol{x}}_r)) - \displaystyle\sum_{j: W_{jr} > 0} \frac{W_{jr}\overline{x}_{jr}}{C_j - \overline{x}_{jr}}, & \text{otherwise}. \end{cases} \tag{56}$$

Lemma 8 *A vector $\boldsymbol{\sigma} = (\boldsymbol{\phi}, \boldsymbol{w})$ is a Nash equilibrium for the extended game if and only if the following two conditions hold:*

1. *For each link j and each user r, if $W_{jr} = 0$ then $w_{jr} = 0$.*

2. *For each user r:*

$$\boldsymbol{x}_r(\boldsymbol{\sigma}) \in \arg\max_{\overline{\boldsymbol{x}}_r \in \mathcal{C}} f_r(\overline{\boldsymbol{x}}_r; \boldsymbol{\sigma}_{-r}). \tag{57}$$

Proof of Lemma. Suppose first that $\boldsymbol{\sigma}$ is a Nash equilibrium. Then consider a link j and user r such that $W_{jr} = 0$. If $w_{jr} > 0$, then user r can achieve exactly the same rate allocation, but lower his total payment, by choosing a bid $\overline{w}_{jr}$ to link j such that $0 < \overline{w}_{jr} < w_{jr}$. This is a profitable deviation, contradicting the assumption that $\boldsymbol{\sigma}$ is a Nash equilibrium. So Condition 1 must hold.

Next, suppose there exists a vector $\overline{\boldsymbol{x}}_r \in \mathcal{C}$ such that:

$$f_r(\overline{\boldsymbol{x}}_r; \boldsymbol{\sigma}_{-r}) > f_r(\boldsymbol{x}_r(\boldsymbol{\sigma}); \boldsymbol{\sigma}_{-r}). \tag{58}$$

First, notice that if $W_{jr} > 0$, then the rate allocation rule:

$$x_{jr}(\boldsymbol{\sigma}) = \frac{w_{jr}}{w_{jr} + W_{jr}} C_j$$

implies that:

$$w_{jr} = \frac{W_{jr} x_{jr}(\boldsymbol{\sigma})}{C_j - x_{jr}(\boldsymbol{\sigma})}. \tag{59}$$

Since we have already shown $w_{jr} = 0$ if $W_{jr} = 0$, we have:

$$f_r(\boldsymbol{x}_r(\boldsymbol{\sigma}); \boldsymbol{\sigma}_{-r}) = U_r(d_r(\boldsymbol{x}_r(\boldsymbol{\sigma}))) - \sum_{j: W_{jr} > 0} \frac{W_{jr} x_{jr}(\boldsymbol{\sigma})}{C_j - x_{jr}(\boldsymbol{\sigma})} = T_r(\boldsymbol{\sigma}_r; \boldsymbol{\sigma}_{-r}).$$

On the other hand, consider the following bid vector for user r. If $W_{jr} > 0$, we define:

$$\overline{w}_{jr} = \frac{W_{jr} \overline{x}_{jr}}{C_j - \overline{x}_{jr}}.$$

If $W_{jr} = 0$, then we define $\overline{w}_{jr} = \epsilon > 0$. We may define $\overline{\phi}_{jr}$ arbitrarily for each link j; it will play no role in user r's payoff.

With the strategy $\overline{\boldsymbol{\sigma}}_r = (\overline{\boldsymbol{\phi}}_r, \overline{\boldsymbol{w}}_r)$, user r will be allocated a rate $x_{jr}(\overline{\boldsymbol{\sigma}}_r, \boldsymbol{\sigma}_{-r})$ given by:

$$x_{jr}(\overline{\boldsymbol{\sigma}}_r, \boldsymbol{\sigma}_{-r}) = \begin{cases} \overline{x}_{jr}, & \text{if } W_{jr} > 0; \\ C_j, & \text{if } W_{jr} = 0. \end{cases}$$

In particular, we conclude that $x_{jr}(\overline{\boldsymbol{\sigma}}_r, \boldsymbol{\sigma}_{-r}) \geq \overline{x}_{jr}$ for all links j, so that:

$$U_r(d_r(\boldsymbol{x}_r(\overline{\boldsymbol{\sigma}}_r, \boldsymbol{\sigma}_{-r}))) \geq U_r(d_r(\overline{\boldsymbol{x}}_r)).$$

The payoff to user r at the strategy vector $\overline{\boldsymbol{\sigma}}_r$ is:

$$\begin{aligned} T_r(\overline{\boldsymbol{\sigma}}_r; \boldsymbol{\sigma}_{-r}) &= U_r(d_r(\boldsymbol{x}_r(\overline{\boldsymbol{\sigma}}_r, \boldsymbol{\sigma}_{-r}))) - \sum_{j: W_{jr} > 0} \overline{w}_{jr} - \sum_{j: W_{jr} = 0} \epsilon \\ &\geq f_r(\overline{\boldsymbol{x}}_r; \boldsymbol{\sigma}_{-r}) - \sum_{j: W_{jr} = 0} \epsilon. \end{aligned}$$

As a result, for small enough $\epsilon > 0$ we conclude from (58) that $T_r(\overline{\boldsymbol{\sigma}}_r; \boldsymbol{\sigma}_{-r}) > T_r(\boldsymbol{\sigma}_r; \boldsymbol{\sigma}_{-r})$, contradicting the assumption that $\boldsymbol{\sigma}$ was a Nash equilibrium. So Condition 2 must hold as well.

Conversely, suppose that Conditions 1 and 2 of the lemma hold, but that $\boldsymbol{\sigma}$ is not a Nash equilibrium. Fix a user r, and let $\overline{\boldsymbol{\sigma}}_r$ be a profitable deviation for user r. Define $\overline{x}_{jr} = x_{jr}(\overline{\boldsymbol{\sigma}}_r, \boldsymbol{\sigma}_{-r})$ for each link j. Also, observe that if $W_{jr} > 0$, then the relation (59) holds, so we have:

$$\begin{aligned} T_r(\overline{\boldsymbol{\sigma}}_r; \boldsymbol{\sigma}_{-r}) &= U_r(d_r(\overline{\boldsymbol{x}}_r)) - \sum_{j: W_{jr} > 0} \frac{W_{jr}\overline{x}_{jr}}{C_j - \overline{x}_{jr}} - \sum_{j: W_{jr} = 0} \overline{w}_{jr} \\ &\le U_r(d_r(\overline{\boldsymbol{x}}_r)) - \sum_{j: W_{jr} > 0} \frac{W_{jr}\overline{x}_{jr}}{C_j - \overline{x}_{jr}} \\ &= f_r(\overline{\boldsymbol{x}}_r; \boldsymbol{\sigma}_{-r}). \end{aligned}$$

On the other hand, from Condition 1 together with (59), we also have:

$$T_r(\boldsymbol{\sigma}_r; \boldsymbol{\sigma}_{-r}) = U_r(d_r(\boldsymbol{x}_r(\boldsymbol{\sigma}))) - \sum_{j \in r: W_{jr} > 0} \frac{W_{jr} x_{jr}(\boldsymbol{\sigma})}{C_j - x_{jr}(\boldsymbol{\sigma})} = f_r(\boldsymbol{x}_r(\boldsymbol{\sigma}); \boldsymbol{\sigma}_{-r}).$$

Since $\overline{\boldsymbol{\sigma}}_r$ is a profitable deviation for user r, we have $T_r(\overline{\boldsymbol{\sigma}}_r; \boldsymbol{\sigma}_{-r}) > T_r(\boldsymbol{\sigma}_r; \boldsymbol{\sigma}_{-r})$, which implies:

$$f_r(\overline{\boldsymbol{x}}_r; \boldsymbol{\sigma}_{-r}) > f_r(\boldsymbol{x}_r(\boldsymbol{\sigma}); \boldsymbol{\sigma}_{-r}).$$

But this violates Condition 2 in the statement of the lemma, a contradiction. So $\boldsymbol{\sigma}$ must have been a Nash equilibrium, as required. □

Now suppose that $\boldsymbol{\sigma}$ is a Nash equilibrium. Our goal will be to construct, for each user r, a vector $\boldsymbol{\alpha}_r = (\alpha_{jr}, j \in J)$, with the interpretation that a "virtual agent" for user r plays a single link game at each link j with linear utility function $U_{jr}(x_{jr}) = \alpha_{jr} x_{jr}$. We want to choose the vectors $\boldsymbol{\alpha}_r$ so that the Nash equilibrium at each single link game is also given by $\boldsymbol{\sigma}$; we can then apply the result of Theorem 3 for the single link model to complete the proof of the theorem.

A technical difficulty arises here because the function $U_r(d_r(\cdot))$ may not be differentiable. If the composite function $g_r = U_r(d_r(\cdot))$ were differentiable, then as in the proof of Theorem 3, we could find an appropriate vector $\boldsymbol{\alpha}_r$ by choosing $\boldsymbol{\alpha}_r = \nabla g_r(\boldsymbol{x}_r(\boldsymbol{\sigma}))$. However, in general $U_r(d_r(\cdot))$ is not differentiable; instead, we must choose $\boldsymbol{\alpha}_r$ to be a *supergradient* of $U_r(d_r(\cdot))$.

We now introduce the required notions from the theory of supergradients. We recall the following definitions from convex analysis [28]. Suppose we are given a function $g : \mathbb{R}^J \to [-\infty, \infty)$; in this case we say g is an *extended real-valued function*. If $g(\boldsymbol{x}) > -\infty$ for at least one $\boldsymbol{x} \in \mathbb{R}^J$, we say g is *proper*. A vector $\boldsymbol{\gamma} = (\gamma_j, j \in J) \in \mathbb{R}^J$ is a *supergradient* of g at $\boldsymbol{x} \in \mathbb{R}^J$ if the following relation holds for all $\overline{\boldsymbol{x}} \in \mathbb{R}^J$:

$$g(\overline{\boldsymbol{x}}) \le g(\boldsymbol{x}) + \boldsymbol{\gamma}^T(\overline{\boldsymbol{x}} - \boldsymbol{x}).$$

We say that g is *superdifferentiable* at $\boldsymbol{x}$ if g possesses at least one supergradient at $\boldsymbol{x}$; in this case we denote the superdifferential, i.e. the set of supergradients, of g at $\boldsymbol{x}$ by $\partial g(\boldsymbol{x}) \subset \mathbb{R}^J$.

Lemma 8 allows us to characterize the Nash equilibrium $\boldsymbol{\sigma}$ as a choice of optimal rate allocation $\overline{\boldsymbol{x}}_r$ by each user r, given the strategy vector $\boldsymbol{\sigma}_{-r}$ of all other users. We recall the definition of f_r

in (56); we will now view f_r as an extended real valued function, by defining $f_r(\overline{\boldsymbol{x}}_r) = -\infty$ for $\overline{\boldsymbol{x}}_r \notin \mathcal{C}$. We also define extended real-valued functions g_r and h_r on $\mathbb{R}^J$ as follows:

$$g_r(\overline{\boldsymbol{x}}_r) = \begin{cases} U_r(d_r(\overline{\boldsymbol{x}}_r)), & \text{if } \overline{\boldsymbol{x}}_r \in \mathcal{C}; \\ -\infty, & \text{otherwise.} \end{cases}$$

and

$$h_r(\overline{\boldsymbol{x}}_r; \boldsymbol{\sigma}_{-r}) = \begin{cases} -\infty, & \text{if } \overline{x}_{jr} \geq C_j \text{ for some } j \text{ with } W_{jr} > 0; \\ & \\ -\displaystyle\sum_{j: W_{jr}>0} \frac{W_{jr}\overline{x}_{jr}}{C_j - \overline{x}_{jr}}, & \text{otherwise.} \end{cases}$$

Then we have $f_r = g_r + h_r$ on $\mathbb{R}^J$. We observe that g_r is a concave function of $\overline{\boldsymbol{x}}_r \in \mathbb{R}^J$. This follows because d_r is a concave function of its argument (as it is the solution to the linear program (39)-(41)), and U_r is nondecreasing and concave. We also note that h_r is a concave function of $\overline{\boldsymbol{x}}_r \in \mathbb{R}^J$, since $(W_{jr}\overline{x}_{jr})/(C_j - \overline{x}_{jr})$ is a strictly convex function of $\overline{x}_{jr} \in (-\infty, C_j)$ whenever $W_{jr} > 0$. Consequently, f_r is a concave function of $\overline{\boldsymbol{x}}_r \in \mathbb{R}^J$. Furthermore, the functions f_r, g_r, and h_r are obviously proper—e.g., $g_r(\mathbf{0}) = U_r(0)$, $h_r(\mathbf{0}) = 0$, and $f_r(\mathbf{0}; \boldsymbol{\sigma}_{-r}) = U_r(0)$. We now have the following lemma.

Lemma 9 *Let $\boldsymbol{\sigma}$ be a Nash equilibrium. Then for each user r, there exists a vector $\boldsymbol{\alpha}_r = (\alpha_{jr}, j \in J)$ such that:*

1. $\boldsymbol{\alpha}_r \in \partial g_r(\boldsymbol{x}_r(\boldsymbol{\sigma}))$.
2. *If $W_{jr} = 0$, then $\alpha_{jr} = 0$.*
3. *If $W_{jr} > 0$, then $\alpha_{jr} > 0$.*
4. *The following relation holds:*

$$\boldsymbol{x}_r(\boldsymbol{\sigma}) \in \arg\max_{\overline{\boldsymbol{x}}_r \in \mathcal{C}} \left[\boldsymbol{\alpha}_r^T \overline{\boldsymbol{x}}_r - \sum_{j: W_{jr}>0} \frac{W_{jr}\overline{x}_{jr}}{C_j - \overline{x}_{jr}} \right]. \tag{60}$$

Proof of Lemma. Fix a user r. Observe that with the definitions we have made, the domain of g_r is equal to $\mathcal{C}$ (that is, $-\infty < g_r(\overline{\boldsymbol{x}}_r) < \infty$ for all $\overline{\boldsymbol{x}}_r \in \mathcal{C}$). Furthermore, for any $\overline{\boldsymbol{x}}_r$ such that $\overline{x}_{jr} < C_j$ for all j, we have $-\infty < h_r(\overline{\boldsymbol{x}}_r; \boldsymbol{\sigma}_{-r}) < \infty$. Thus, the relative interior of the domain of g_r (denoted $\mathrm{ri}(\mathrm{dom}(g_r))$) has nonempty intersection with the relative interior of the domain of h_r: $\mathrm{ri}(\mathrm{dom}(g_r)) \cap \mathrm{ri}(\mathrm{dom}(h_r)) \neq \emptyset$. From Theorem 23.8 in [28], this is is sufficient to ensure that at $\boldsymbol{x}_r(\boldsymbol{\sigma})$, we have:

$$\partial f_r(\boldsymbol{x}_r(\boldsymbol{\sigma}); \boldsymbol{\sigma}_{-r}) = \partial g_r(\boldsymbol{x}_r(\boldsymbol{\sigma})) + \partial h_r(\boldsymbol{x}_r(\boldsymbol{\sigma}); \boldsymbol{\sigma}_{-r}). \tag{61}$$

(The summation here of the two superdifferentials on the right hand side is a summation of sets, where $A + B = \{\boldsymbol{x} + \boldsymbol{y} : \boldsymbol{x} \in A, \boldsymbol{y} \in B\}$; if either A or B is empty, then $A + B$ is empty as well.)

From Condition 2 in Lemma 8, we have for all $\overline{\boldsymbol{x}}_r \in \mathcal{C}$ that:

$$f_r(\boldsymbol{x}_r(\boldsymbol{\sigma}); \boldsymbol{\sigma}_{-r}) \geq f_r(\overline{\boldsymbol{x}}_r; \boldsymbol{\sigma}_{-r}).$$

Since $f_r(\overline{\boldsymbol{x}}_r; \boldsymbol{\sigma}_{-r}) = -\infty$ for $\overline{\boldsymbol{x}}_r \notin \mathcal{C}$, we conclude $\mathbf{0}$ is a supergradient of f_r at $\boldsymbol{x}_r(\boldsymbol{\sigma})$. As a result, we know from (61) that there exists $\boldsymbol{\alpha}_r \in \partial g_r(\boldsymbol{x}_r(\boldsymbol{\sigma}))$ and $\boldsymbol{\beta}_r \in \partial h_r(\boldsymbol{x}_r(\boldsymbol{\sigma}); \boldsymbol{\sigma}_{-r})$ such that $\boldsymbol{\alpha}_r = -\boldsymbol{\beta}_r$.

We will explicitly compute $\boldsymbol{\beta}_r$. We first note that from Condition 2 of Lemma 8, we must have $0 \leq x_{jr}(\boldsymbol{\sigma}) < C_j$ if $W_{jr} > 0$; otherwise the objective function in (57) is equal to $-\infty$, which cannot be optimal for user r (e.g., choosing $\overline{\boldsymbol{x}}_r = \mathbf{0}$ yields an objective function value of $U_r(0) > -\infty$). Now at any point $\overline{\boldsymbol{x}}_r \in \mathcal{C}$ such that $\overline{x}_{jr} < C_j$ if $W_{jr} > 0$, we note that h_r is in fact differentiable, with:

$$\frac{\partial h_r}{\partial \overline{x}_{jr}}(\overline{\boldsymbol{x}}_r; \boldsymbol{\sigma}_{-r}) = \begin{cases} -\dfrac{W_{jr} C_j}{(C_j - \overline{x}_{jr})^2}, & \text{if } W_{jr} > 0; \\ 0, & \text{otherwise.} \end{cases}$$

Since h_r is differentiable at $\boldsymbol{x}_r(\boldsymbol{\sigma})$, we conclude that in fact $\partial h_r(\boldsymbol{x}_r(\boldsymbol{\sigma}); \boldsymbol{\sigma}_{-r})$ is a singleton, containing only $\nabla h_r(\boldsymbol{x}_r(\boldsymbol{\sigma}); \boldsymbol{\sigma}_{-r})$, which is defined by the previous equation. So we must have $\boldsymbol{\beta}_r = \nabla h_r(\boldsymbol{x}_r(\boldsymbol{\sigma}); \boldsymbol{\sigma}_{-r})$, and thus:

$$\alpha_{jr} = -\beta_{jr} = \begin{cases} \dfrac{W_{jr} C_j}{(C_j - x_{jr}(\boldsymbol{\sigma}))^2}, & \text{if } W_{jr} > 0; \\ 0, & \text{otherwise.} \end{cases}$$

We have established conclusions 1, 2, and 3 of the lemma. To establish conclusion 4, we observe that $\mathbf{0}$ is a supergradient of the following function at $\boldsymbol{x}_r(\boldsymbol{\sigma})$:

$$\hat{f}_r(\overline{\boldsymbol{x}}_r; \boldsymbol{\sigma}_{-r}) = \begin{cases} -\infty, & \text{if } \overline{\boldsymbol{x}}_r \notin \mathcal{C} \\ & \text{or if } \overline{x}_{jr} = C_j \text{ for some } j \text{ with } W_{jr} > 0; \\ \boldsymbol{\alpha}_r^T \overline{\boldsymbol{x}}_r - \displaystyle\sum_{j: W_{jr} > 0} \frac{W_{jr} \overline{x}_{jr}}{C_j - \overline{x}_{jr}}, & \text{otherwise.} \end{cases}$$

This observation follows by replacing $g_r(\overline{\boldsymbol{x}}_r)$ with the following function $\hat{g}_r$ on $\mathbb{R}^J$:

$$\hat{g}_r(\overline{\boldsymbol{x}}_r) = \begin{cases} \boldsymbol{\alpha}_r^T \overline{\boldsymbol{x}}_r, & \text{if } \overline{\boldsymbol{x}}_r \in \mathcal{C}; \\ -\infty, & \text{otherwise.} \end{cases}$$

Then we have $\hat{f}_r = \hat{g}_r + h_r$; and as before, $\mathrm{ri}(\mathrm{dom}(\hat{g}_r)) \cap \mathrm{ri}(\mathrm{dom}(h_r)) \neq \emptyset$, so we have:

$$\partial \hat{f}_r(\boldsymbol{x}_r(\boldsymbol{\sigma}; \boldsymbol{\sigma}_{-r})) = \partial \hat{g}_r(\boldsymbol{x}_r(\boldsymbol{\sigma})) + \partial h_r(\boldsymbol{x}_r(\boldsymbol{\sigma}); \boldsymbol{\sigma}_{-r}).$$

The vector $\boldsymbol{\alpha}_r$ is a supergradient of $\hat{g}_r$ for all $\overline{\boldsymbol{x}}_r \in \mathcal{C}$; in particular, $\boldsymbol{\alpha}_r \in \partial \hat{g}(\boldsymbol{x}_r(\boldsymbol{\sigma}))$. We have already shown $\{-\boldsymbol{\alpha}_r\} = \partial h(\boldsymbol{x}_r(\boldsymbol{\sigma}); \boldsymbol{\sigma}_{-r})$. Thus $\mathbf{0} \in \partial \hat{f}_r(\boldsymbol{x}_r(\boldsymbol{\sigma}); \boldsymbol{\sigma}_{-r})$. This implies conclusion 4 of the lemma, as required. □

For each user r, fix the supergradient $\boldsymbol{\alpha}_r$ given by the preceding lemma. We start by observing that for each user r, since $\boldsymbol{\alpha}_r$ is a supergradient of $g_r(\boldsymbol{x}_r(\boldsymbol{\sigma}))$, we have:

$$U_r(d_r(\boldsymbol{x}_r^S)) \leq U_r(d_r(\boldsymbol{x}_r(\boldsymbol{\sigma}))) + \boldsymbol{\alpha}_r^T(\boldsymbol{x}_r^S - \boldsymbol{x}_r(\boldsymbol{\sigma})). \tag{62}$$

Now note that if $\boldsymbol{\alpha}_r = 0$ for all r, then we have the following trivial inequality:

$$\sum_r U_r(d_r(\boldsymbol{x}_r(\boldsymbol{\sigma}))) \geq \sum_r U_r(d_r(\boldsymbol{x}_r^S)) \geq \frac{3}{4} \sum_r U_r(d_r(\boldsymbol{x}_r^S)).$$

Thus the theorem holds in this case; so we may assume without loss of generality that $\boldsymbol{\alpha}_r \neq 0$ for at least one user r. This implies that $\alpha_{jr} > 0$ for at least one link j and user r; by the preceding lemma, we must have $W_{jr} > 0$. In particular, we conclude that at least two users are competing for resources at link j.

Since $\alpha_{jr} = 0$ if $W_{jr} = 0$, we have the following simplification of (60):

$$\begin{aligned}\boldsymbol{x}_r(\boldsymbol{\sigma}) &\in \arg\max_{\overline{\boldsymbol{x}}_r \in \mathcal{C}} \left[\boldsymbol{\alpha}_r^T \overline{\boldsymbol{x}}_r - \sum_{j: W_{jr} > 0} \frac{W_{jr} \overline{x}_{jr}}{C_j - \overline{x}_{jr}} \right] \\ &= \arg\max_{\overline{\boldsymbol{x}}_r \in \mathcal{C}} \left[\sum_{j: W_{jr} > 0} \left(\alpha_{jr} \overline{x}_{jr} - \frac{W_{jr} \overline{x}_{jr}}{C_j - \overline{x}_{jr}} \right) \right].\end{aligned}$$

The maximum on the right hand side of the preceding expression decomposes into separate maximizations for each link j with $W_{jr} > 0$. We conclude that for each link j with $W_{jr} > 0$, we in fact have:

$$x_{jr}(\boldsymbol{\sigma}) \in \arg\max_{0 \leq \overline{x}_{jr} \leq C_j} \left[\alpha_{jr} \overline{x}_{jr} - \frac{W_{jr} \overline{x}_{jr}}{C_j - \overline{x}_{jr}} \right].$$

Fix now a link j with $\sum_r w_{jr} > 0$. We view the users as playing a single link game at link j, with utility function for user r given by $U_{jr}(x_{jr}) = \alpha_{jr} x_{jr}$. The preceding expression states that Condition 2 of Lemma 8 is satisfied. Furthermore, since $\sum_r w_{jr} > 0$ and $\boldsymbol{\sigma}$ is a Nash equilibrium for the network game, from Condition 1 in Lemma 8 there must exist at least two users r_1, r_2 such that $W_{jr_1}, W_{jr_2} > 0$, so in particular, $W_{jr} > 0$ for all users r. Thus Condition 1 of Lemma 8 is vacuously satisfied for the single link game; and we conclude that $\boldsymbol{\sigma}$ is a Nash equilibrium for this single link game at link j. More precisely, we have that $(w_{j1}, \ldots, w_{jR})$ is a Nash equilibrium for the single link game at link j, when R users with utility functions $(U_{j1}, \ldots, U_{jR})$ compete for link j. Since $W_{jr} > 0$ for all r, we know $\alpha_{jr} > 0$ for all users r from the preceding lemma, so U_{jr} is strictly increasing for each r; and since $R > 1$, we apply Theorem 3 to conclude that:

$$\sum_r \alpha_{jr} x_{jr}(\boldsymbol{\sigma}) \geq \frac{3}{4} \left(\max_r \alpha_{jr} \right) C_j. \tag{63}$$

(The right hand side is $3/4$ of the optimal value of *SYSTEM* for a single link of capacity C_j, when each user r has linear utility $U_r(x_{jr}) = \alpha_{jr} x_{jr}$.)

We now complete the proof of the theorem, by following the proof of Lemma 4. Note that since $W_{jr} = 0$ implies $\alpha_{jr} = 0$ from Lemma 9, the following relation holds:

$$\sum_r \sum_{j: W_{jr} > 0} \alpha_{jr} x_{jr}^S = \sum_j \sum_r \alpha_{jr} x_{jr}^S.$$

Thus we have:

$$\sum_r \boldsymbol{\alpha}_r^T \boldsymbol{x}_r^S = \sum_r \sum_{j:W_{jr}>0} \alpha_{jr} x_{jr}^S = \sum_j \sum_r \alpha_{jr} x_{jr}^S \leq \sum_j \left(\max_r \alpha_{jr} \right) C_j. \tag{64}$$

We reason as follows, using (62) for the first inequality, and (64) for the second:

$$\frac{\sum_r U_r(d_r(\boldsymbol{x}_r(\boldsymbol{\sigma})))}{\sum_r U_r(d_r(\boldsymbol{x}_r^S))} \geq \frac{\sum_r \left(U_r(d_r(\boldsymbol{x}_r(\boldsymbol{\sigma}))) - \boldsymbol{\alpha}_r^T \boldsymbol{x}_r(\boldsymbol{\sigma})\right) + \sum_r \boldsymbol{\alpha}_r^T \boldsymbol{x}_r(\boldsymbol{\sigma})}{\sum_r \left(U_r(d_r(\boldsymbol{x}_r(\boldsymbol{\sigma}))) + \boldsymbol{\alpha}_r^T (\boldsymbol{x}_r^S - \boldsymbol{x}_r(\boldsymbol{\sigma}))\right)}$$

$$= \frac{\sum_r \left(U_r(d_r(\boldsymbol{x}_r(\boldsymbol{\sigma}))) - \boldsymbol{\alpha}_r^T \boldsymbol{x}_r(\boldsymbol{\sigma})\right) + \sum_r \boldsymbol{\alpha}_r^T \boldsymbol{x}_r(\boldsymbol{\sigma})}{\sum_r \left(U_r(d_r(\boldsymbol{x}_r(\boldsymbol{\sigma}))) - \boldsymbol{\alpha}_r^T \boldsymbol{x}_r(\boldsymbol{\sigma})\right) + \sum_r \boldsymbol{\alpha}_r^T \boldsymbol{x}_r^S}$$

$$\geq \frac{\sum_r \left(U_r(d_r(\boldsymbol{x}_r(\boldsymbol{\sigma}))) - \boldsymbol{\alpha}_r^T \boldsymbol{x}_r(\boldsymbol{\sigma})\right) + \sum_j \sum_r \alpha_{jr} x_{jr}(\boldsymbol{\sigma})}{\sum_r \left(U_r(d_r(\boldsymbol{x}_r(\boldsymbol{\sigma}))) - \boldsymbol{\alpha}_r^T \boldsymbol{x}_r(\boldsymbol{\sigma})\right) + \sum_j \left(\max_r \alpha_{jr}\right) C_j}. \tag{65}$$

Since $U_r(d_r(\mathbf{0})) = U_r(0) = 0$, applying the fact that $\boldsymbol{\alpha}_r$ is a supergradient we have:

$$U_r(d_r(\boldsymbol{x}_r(\boldsymbol{\sigma}))) - \boldsymbol{\alpha}_r^T \boldsymbol{x}_r(\boldsymbol{\sigma}) \geq 0.$$

We also have:

$$0 \leq \sum_j \sum_r \alpha_{jr} x_{jr}(\boldsymbol{\sigma}) \leq \sum_j \left(\max_r \alpha_{jr} \right) C_j.$$

So we conclude from relations (63) and (65) that:

$$\frac{\sum_r U_r(d_r(\boldsymbol{x}_r(\boldsymbol{\sigma})))}{\sum_r U_r(d_r(\boldsymbol{x}_r^S))} \geq \frac{\sum_j \sum_r \alpha_{jr} x_{jr}(\boldsymbol{\sigma})}{\sum_j \left(\max_r \alpha_{jr}\right) C_j} \geq \frac{3}{4}.$$

Observe that all denominators in this chain of inequalities are nonzero, since $\boldsymbol{\alpha}_r \neq 0$ for at least one user r implies that:

$$\sum_j (\max_r \alpha_{jr}) C_j > 0.$$

Since $\boldsymbol{\sigma}$ was assumed to be a Nash equilibrium, this completes the proof of the theorem. $\square$

The preceding theorem uses the single link result to establish the price of anarchy for general networks. Note that since we knew from Theorem 3 that the bound of $3/4$ was essentially tight for single link games, and a single link is a special case of a general network, the $3/4$ bound is also tight in this setting. In particular, note that a single link yields the worst price of anarchy. This is similar to a result observed by Roughgarden for traffic routing games [29], where the worst price of anarchy occurs in very simple networks.

5 A General Resource Allocation Game

In this section we consider an extension of the development of the previous section to more general resource allocation games. Suppose that there are J infinitely divisible scarce resources, and R

users require these resources. As before, let C_j be the total available amount of resource j, let x_{jr} denote the amount of resource j allocated to user r. The key assumption which drives the model of this section is that user r receives a utility $V_r(\boldsymbol{x}_r)$ from the allocation $\boldsymbol{x}_r = (x_{jr}, j \in J)$, where $V_r(\boldsymbol{x}_r)$ is a concave and continuous function of the vector $\boldsymbol{x}_r \geq 0$. We also assume V_r is *nondecreasing*; that is, if $x_{jr} \geq \overline{x}_{jr}$ for all $j \in J$, then $V_r(\boldsymbol{x}_r) \geq V_r(\overline{\boldsymbol{x}}_r)$.

Of course, one example where these conditions are satisfied is given by the model of this paper, where the resources represent links in a communication network, and each user requires a subset of these resources. User r receives a nondecreasing, concave, continuous utility $U_r(d_r)$ as a function of the total rate d_r obtained from the network; and the rate $d_r(\boldsymbol{x}_r)$ is determined by solving the max-flow problem (39)-(41). In this case, the composite function $U_r(d_r(\boldsymbol{x}_r))$ is concave and nondecreasing in the argument $\boldsymbol{x}_r$.

Another example may be described by interpreting each resource j as a distinct raw material, and $V_r(\boldsymbol{x}_r)$ as the profits of a firm r which has access to x_{jr} units of raw material j for each $j \in J$. In this case, the assumption that V_r is concave corresponds to decreasing marginal returns; and the assumption that V_r is nondecreasing implies profits should not fall as the raw materials available increase.

We suppose now that the users play a game to acquire resources exactly as described in Section 4.1. In particular, each user r chooses a requested resource allocation ϕ_{jr} and makes a bid w_{jr} to each resource $j \in J$. Given the composite strategy vector $\boldsymbol{\sigma}$, resource j then allocates an amount $x_{jr}(\boldsymbol{\sigma})$ to user r, where $x_{jr}(\cdot)$ is defined by (44)-(46). The payoff to user r is then:

$$Y_r(\boldsymbol{\sigma}_r; \boldsymbol{\sigma}_{-r}) = V_r(\boldsymbol{x}_r(\boldsymbol{\sigma})) - \sum_j w_{jr}.$$

Following the proof of Theorem 6, but replacing $U_r(d_r(\cdot))$ with $V_r(\cdot)$ for each r, we may prove the following theorem.

Theorem 10 *Assume that for each r, the utility function $V_r(\overline{\boldsymbol{x}}_r)$ is concave, nondecreasing, and continuous as a function of $\overline{\boldsymbol{x}}_r \geq 0$. Then a Nash equilibrium exists for the game defined by $(Y_1, \ldots, Y_R)$.*

More importantly, we would like to compare the performance at any Nash equilibrium of this game with an "efficient" allocation. As in the preceding development, we define the problem *SYSTEM* as follows:

SYSTEM:

$$\text{maximize} \quad \sum_r V_r(\overline{\boldsymbol{x}}_r) \tag{66}$$

$$\text{subject to} \quad \sum_r \overline{x}_{jr} \leq C_j, \quad j \in J; \tag{67}$$

$$\overline{x}_{jr} \geq 0, \quad j \in J,\ r \in R. \tag{68}$$

Since the objective function is continuous and the feasible region is compact, an optimal solution exists for this problem. Again, following the proof of Theorem 7, we may prove the following result.

Theorem 11 *Assume that for each r, the utility function $V_r(\overline{\boldsymbol{x}}_r)$ is concave, nondecreasing, and continuous as a function of $\overline{\boldsymbol{x}}_r \geq 0$. Assume also that $V_r(\mathbf{0}) \geq 0$ for all users r. If $\boldsymbol{x}^G = (\boldsymbol{x}_r^G, r \in r)$ is any Nash equilibrium allocation for the game defined by $(Y_1, \ldots, Y_R)$, and $\boldsymbol{x}^S = (\boldsymbol{x}_r^S, r \in R)$ is any SYSTEM optimal allocation, then:*

$$\sum_r V_r(\boldsymbol{x}_r^G) \geq \frac{3}{4} \sum_r U_r(\boldsymbol{x}_r^S).$$

The preceding theorem shows that the essential structure in the network context is the bidding scheme which allows each resource to operate its own "market." Each user then decides how to employ allocated resources, resulting in the utility $V_r(\boldsymbol{x}_r(\boldsymbol{\sigma}))$. This decoupling between the pricing mechanism employed at each resource and the eventual use of the resources by the end users allows the extension of the result of Theorem 3 from a single resource context to a general multiple resource context.

6 Conclusion

Results such as those provided in Theorems 3 and 7 suggest that selfish behavior by network users need not lead to arbitrarily inefficient outcomes. This conceptual issue is one which only grows in importance as the Internet becomes more decentralized and commercialized, with individual users typically desiring to optimize their own performance.

We emphasize here that the model considered in this paper is a static model of network behavior. In practice, users will dynamically interact with the network, updating their strategies over time to perform their individual payoff optimization. In general, convergence of such dynamics is not very well understood; and further, the information available to each user in making their decisions is frequently quite limited. One advantage of the results proposed in Theorems 3 and 7 is that although they refer to Nash equilibrium, which generally requires that each user have perfect knowledge of the strategies of all other users, in fact each user is only required to know the prices of the links he wishes to use. From this information, together with his own strategy, the user can compute whether or not a profitable deviation exists. Such an observation suggests that it may be possible to model the dynamics of users in such a setting, without the requirement that complete network state be available to all users; in particular, distributed algorithms which feed back only prices to users (such as those proposed in [30]) may provide the foundation for a tractable dynamic analysis of this network congestion game. Ultimately, the goal of such an agenda is a model which addresses both the decentralized engineering design issues inherent in a large scale network, as well as the self interested behavior of the users which share that network.

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