Efficiency Loss in a Network Resource Allocation Game

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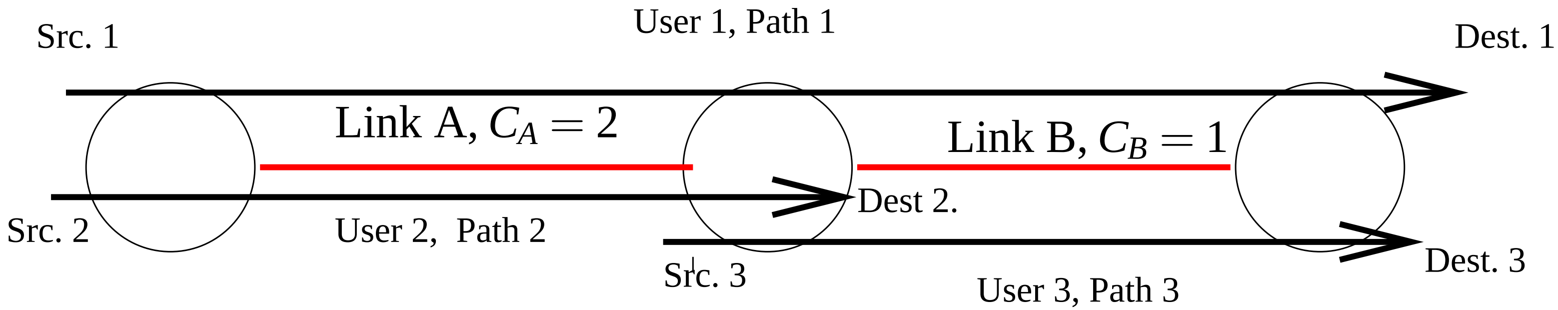
Resource Allocation in Networks

Basic Question: How should the bandwidth on each link be divided among users?



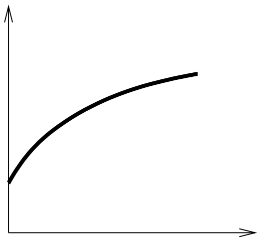
Max-min fair allocaton: $d_1 = 0.5, d_2 = 1.5, d_3 = 0.5$. Is this always desirable?

Let $U_1(\cdot), U_2(\cdot), \ldots U_R(\cdot)$, denote the utility functions of user $1, 2 \ldots R$ respectively.

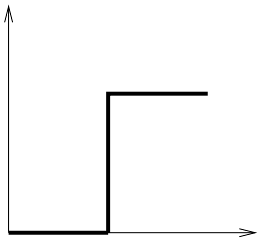
$$\max_{\{d_1, d_2, \ldots, d_R\} \in \mathcal{F}} \sum_{r=1}^{R} U_r(d_r)$$

Utility Functions

Assumption 1 The utility function $U_r(x_r)$, for each r is a continuously differentiable, strictly increasing and concave function with $U_r(0) \geq 0$.



(a) Permissible Utility Function



(b) Not Permissible Utility Function

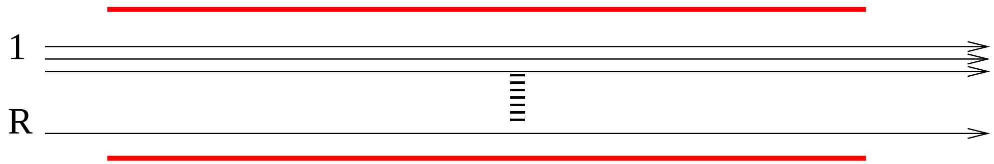
- The optimization problem for resource allocation problem can be efficiently solved.
- But.. The solution requires the Network manager to know the utility function of the users. This may not be possible.
- **Focus :** We will study distributed pricing mechanism for efficient resource allocation using a game theory approach.

Presentation Outline

- Single Link Case
  - Pricing Mechanism
  - Competitive Equilibrium: Global Optimality
  - Nash Equilibrium
  - Efficiency Loss at Nash Equilibrium
- General Network Case
  - Pricing Mechanism
  - Competitive Equilibrium: Global Optimality
  - Existence Of Nash Equilibrium
  - Efficiency Loss at Nash Equilibrium
- Subsequent and RelatedWork
- Conclusions

Single Link Setup

R users communicate over a single link with capacity C. Each user is assigned a rate d_r.



Link Capacity = C

SYSTEM:

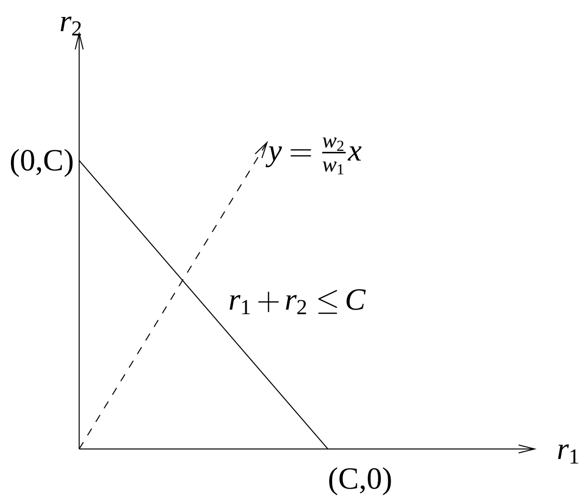
$$\text{maximize} \quad \sum_r U_r(d_r)$$

$$\text{subject to} \quad \sum_r d_r \leq C$$

$$d_r \geq 0, r = 1, 2, \ldots R$$

Pricing Mechanism

- Users submit the bids $w_1, w_2, \ldots w_R$ to the network manager.
- The network manager computes the price $\mu = \frac{\sum_r w_r}{C}$
- Each user receives bandwidth $d_r = \frac{w_r}{\mu}$ (Market Clearing Allocation)



The network does not price discriminate between the users.

Competitive Equilibrium

Bidding is an Iterative process...

- Given the bids $w_1, w_2, \ldots w_R$, the network calculates the price $\mu = \frac{\sum_r w_r}{C}$. (Proportional Fairness Criterion).
- Given price μ, user r chooses w_r that maximizes the *payoff* function: $P_r(w_r, \mu) = U_r\left(\frac{w_r}{\mu}\right) - w_r$.

Theorem 1 *(Kelly '97) There exists a unique* competitive *equilibrium for the above pricing mechanism. Furthermore, if* $(\mathbf{w}, \mu)$ *achieves this equilibrium then the rate vector* $\mathbf{d} = \frac{\mathbf{w}}{\mu}$ *solves the SYSTEM optimization problem.*

Note: The users take the price μ as given. They do not anticipate that it depends on their bid. So they are called *price taking*.

Proof Outline - Theorem 1

SYSTEM Problem: Lagrangian Optimization

$$\mathcal{L}(\mathbf{d}, \mu) = \sum_r U_r(d_r) - \lambda \left(\sum_r d_r - C \right)$$

Differentiating w.r.t. d_r:

$$\begin{aligned} U_r'(d_r) &= \lambda && \text{if } d_r > 0 \\ U_r'(d_r) &\leq \lambda && \text{if } d_r = 0 \end{aligned}$$

Maximizing Payoff Function: $P(w_r, \mu) = U_r \left(\frac{w_r}{\mu} \right) - w_r$ gives:

$$\begin{aligned} U_r'(w_r/\mu) &= \mu && \text{if } w_r > 0 \\ U_r'(w_r/\mu) &\leq \mu && \text{if } w_r = 0 \end{aligned}$$

The equations are identical, if we set $d_r = w_r/\mu$ and $\lambda = \mu$.

Remarks - Competitive Equilibrium

- This Theorem is a special case of the *fundamental theorem* on Social welfare. Pareto efficient solution maximizes the aggregate utility of the system.
- The Theorem only asserts that there exists a unique competitive equilibrium. It does not say anything about the dynamics of reaching the equilibrium.
- The choice payoff function is quite natural. It appears unique upto a scaling constant. The Theorem also holds for $P(w_r, \mu) = U_r\left(\frac{w_r}{\mu}\right) - 0.5 w_r$, for example.

Nash Equilibrium - Example

Prisoner's dilemma:

Strategy	You deny	You confess
He denies	Both 6 months	He:10 yrs; You:free
He confesses	He:free; you:10 yrs.	both 6 yrs

Not knowing what your accomplice is going to do you act selfishly.

- He decides to confess $\Rightarrow$ You should confess
- He decides to deny $\Rightarrow$ You should confess

At Nash equilibrium: both decide to confess. This is NOT the global optimum.
No single user can have a profitable deviation away from the Nash equilibrium, if all the other users remain unchanged.

Nash Equilibrium: Networks

Bidding process is as before, but the payoff function is given by ($\mathbf{w}_{-r} = (w_1, \ldots w_{r-1}, w_{r+1}, \ldots w_R)$):

$$Q_r(w_r; \mathbf{w}_{-r}) = \begin{cases} U_r\left(\frac{w_r}{\sum_s w_s} C\right) - w_r & \text{if } w_r > 0 \\ U_r(0) & \text{if } w_r = 0 \end{cases}$$

If the bid vector $\mathbf{w}$ is a Nash equilibrium, then

$$Q_r(w_r; \mathbf{w}_{-r}) \geq Q_r(\overline{w}_r; \mathbf{w}_{-r}) \qquad \text{for all } \overline{w}_r \geq 0$$

Theorem 2 *(Hajek and Gopalkrishnan, 2002) There exists a unique* $\mathbf{w}$ *that achieves the Nash equilibrium. Furthermore the corresponding rate assignments* $d_r = \frac{w_r}{\sum_s w_s}$ *maximize the SYSTEM problem with the following modified utility functions.*

Nash Eqm to System with $U_1(r) \ldots U_R(r)$ ↔ Global Optimal of another System: $\hat{U}_1(r) \ldots \hat{U}_R(r)$

Price of Anarchy

Theorem 3 *Suppose that the utility function U_r satisfy assumption 1. If $\mathbf{d}^G$ and $\mathbf{d}^S$ are the rate allocations for the Nash equilibrium and Competitive equilibrium respectively then*

$$\sum_r U_r(d_r^G) \geq \frac{3}{4} \sum_r U_r(d_r^S)$$

Moreover, for every $\varepsilon > 0$ there exists an $R > 1$ and a choice of utility functions $U_r(\cdot)$ such that

$$\sum_{r=1}^{R} U_r(d_r^G) \leq \left(\frac{3}{4} + \varepsilon\right) \sum_{r=1}^{R} U_r(d_r^S)$$

Proof Outline - 1

Lemma 1

$$\frac{\sum_r U_r(d_r^G)}{\sum_r U_r(d_r^S)} \geq \frac{\sum_r U_r'(d_r^G) d_r^G}{(\max_r U_r'(d_r^G))C}$$

The equality occurs if $U_r(\cdot)$ *are linear functions with* $U_r(0) = 0$ *(i.e. linear utility functions are the worst case scenario).*

Note: $U_r(d_r^S) \leq U_r(d_r^G) + U_r'(d_r^G)(d_r^S - d_r)$.

$$\begin{aligned} \text{LHS} = \frac{\sum_r U_r(d_r^G)}{\sum_r U_r(d_r^S)} &\geq \frac{\sum_r (U_r(d_r^G) - U_r'(d_r^G) d_r^G) + \sum_r U_r'(d_r^G) d_r^G}{\sum_r (U_r(d_r^G) - U_r'(d_r^G) d_r^G) + \sum_r U_r'(d_r^G) d_r^S} \\ &\geq \frac{\sum_r U_r'(d_r^G) d_r^G}{\sum_r U_r'(d_r^G) d_r^S} \\ &\geq \frac{\sum_r U_r'(d_r^G) d_r^G}{(\max_r U_r'(d_r^G))C} = \text{RHS} \end{aligned}$$

Proof Outline - 2

Let $U_r(d_r) = \alpha_r d_r$. Let $\alpha_1 = 1$ and $\alpha_r \leq 1$ for $r > 1$. Assume $d_r > 0$ for each r. Search over all $(\alpha_2, \alpha_3, \ldots)$ and $(d_1^G, d_2^G, d_3^G, \ldots)$:

$$\begin{aligned} \text{minimize} \quad & d_1^G + \sum_{r=2}^{R} \alpha_r d_r^G \\ \text{subject to} \quad & \alpha_r(1 - d_r^G) = 1 - d_1^G, \\ & \sum_r d_r^G = 1, \quad \alpha_r \leq 1 \quad d_r^G > 0 \end{aligned}$$

The above optimization problem has a minimum at 3/4 when

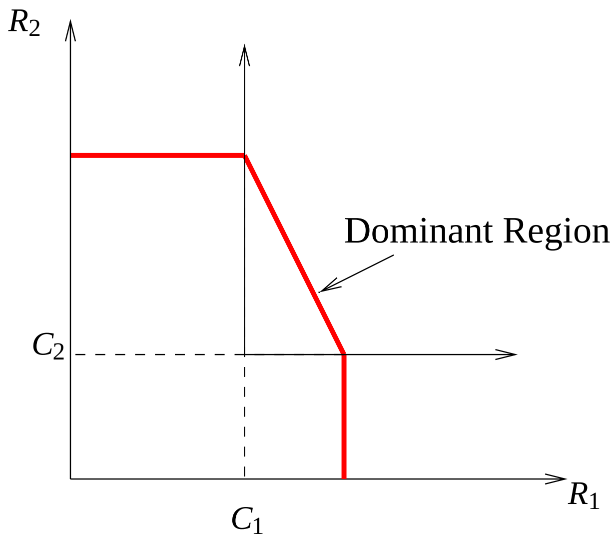
- $U_1(d_1) = d_1$, $U_r(d_r) \approx d_r/2$ for $r > 1$.
- $d_1^G = \frac{1}{2}$, $d_r^G = \frac{1}{2(R-1)}$ for $r > 1$
- $R \to \infty$

Discussion

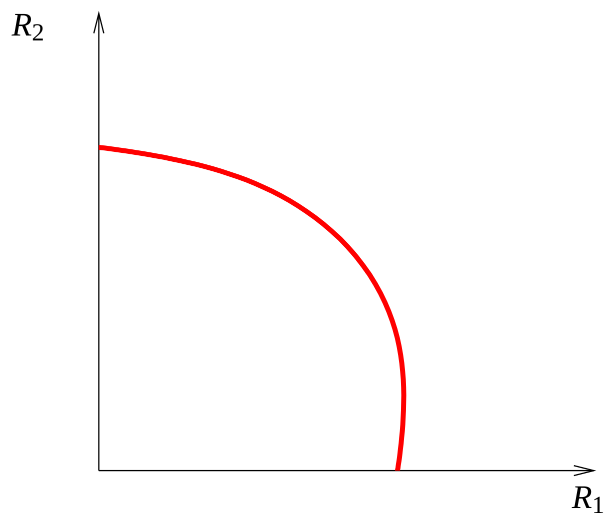
- Intuition: Linear utility functions are worst case scenario.
- The result holds for the specific pricing mechanism introduced by Kelly. Subsequent work shows that the loss 3/4 holds for a broader class of pricing mechanism.
- Sanghvi and Hajek (2004) show that for the 2 user case we can achieve within 7/8 of the optimal value by using a different pricing mechanism. This mechanism does not achieve global optimum though.
- The result can be easily generalized to the case to the case when $\sum_r \beta_r d_r < C$ but it does not generalize to arbitrary convex curves.
- The case of profit maximizing link managers is still open.

Discussion(cont'd)

The multiple access channel (MAC) can be reduced to the single link case. Only the sum constraint is active while bidding, so do a change in co-ordinates.



MAC Channel

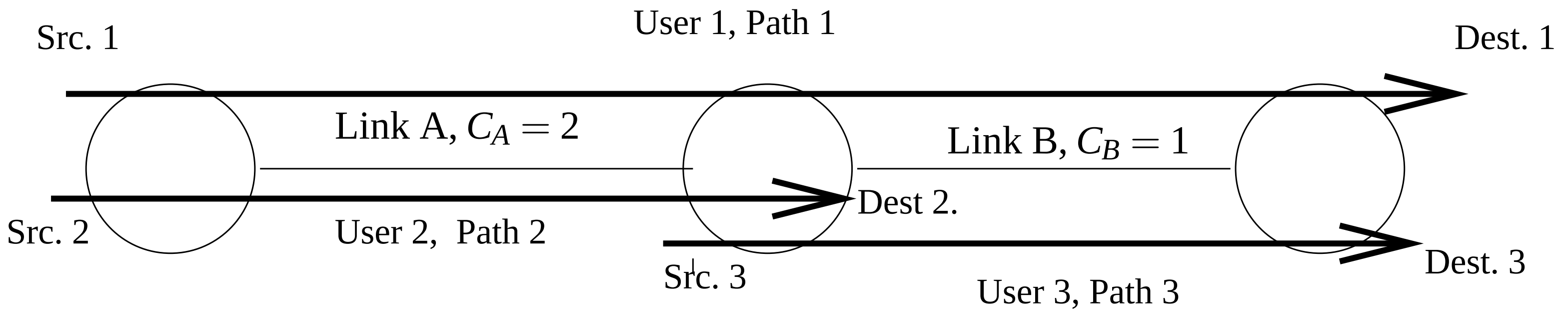


Broadcast Channel

General Network Case - Notation

There are J links, Capacities $(C_1, C_2 \ldots C_J)$ and P paths.

$$A_{jp} = \begin{cases} 1 & \text{if j} \in \text{p} \\ 0 & \text{otherwise} \end{cases} \quad H_{rp} = \begin{cases} 1 & \text{if p} \in \text{r} \\ 0 & \text{otherwise} \end{cases}$$



In the above figure, $A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$, and $H = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Problem Setup

SYSTEM

$$\text{maximize} \quad \sum_r U_r(d_r)$$

$$\text{subject to} \quad Ay \le C, Hy = d, y_p \ge 0, p \in P$$

Pricing Mechanism: Bid vector $\mathbf{w}_r = (w_{1r}, w_{2r} \ldots)$

$$x_{jr} = \begin{cases} \frac{w_{jr}}{\mu_j} & \text{if } w_{jr} > 0 \\ 0 & \text{otherwise} \end{cases}$$

Use Max-Flow algorithm to determine the rate $d_r(\mathbf{x}_r)$:

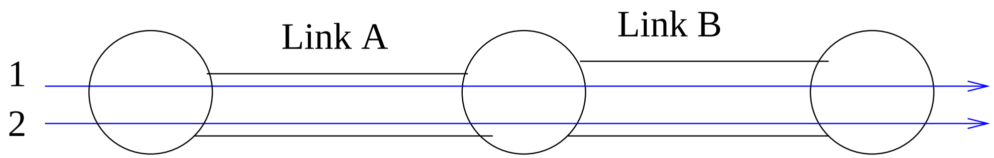
$$\text{maximize} \quad \sum_{p \in r} y_p$$

$$\text{subject to} \quad \sum_{p \in r : j \in p} y_p \le x_{jr} \qquad y_p \ge 0, p \in r$$

Nash Equilibrium

Kelly('97) showed that the competitive equilibrium achieves global optimum.

Nash equilibrium may not exist for the pricing mechanism.



Proof: Let $C_A < C_B$ Suppose $\mathbf{w} = (\{w_{11}, w_{21}\}, \{w_{21}, w_{22}\})$ be a Nash equilibrium. $\mathbf{x} = (\{x_{11}, x_{21}\}, \{x_{21}, x_{22}\})$ be its rate vector. Then $x_{1j} < x_{2j}$ for atleast one j. This user can reduce his bid and have a profitable deviation. **Contradition**. $\square$

This problem can be easily fixed with a modified pricing mechanism that allows 0 bids on “surplus" links.

Nash Equilibrium

Nash Equilibrium Payoff Functions:

$$Q_r(\mathbf{w}_r; \mathbf{w}_{-r}) = U_r(d_r(\mathbf{x}(\mathbf{w}))) - \sum_j w_{jr}$$

Claim 1 *If* $\mathbf{w}$ *is a Nash equilibrium for the original problem then the corresponding* $\mathbf{x}_r$ *also satisfies (with* $\alpha_r = \nabla U_r(d_r(\mathbf{x}_r))$*)*

$$\mathbf{x}_r = \arg\max_{\mathbf{x}'_r} \left[\alpha_r^T \mathbf{x}'_r - \sum_j w_{jr}(\mathbf{x}'_r) \right]$$

This shows that if the system is in Nash equilibrium then *each link* is in Nash equilibrium with modified utility function $\alpha_{jr} x_{jr}$.

Efficiency Loss

Theorem 4 *The efficiency loss for the network case is atmost 1/4.*

Proof: Suppose $\mathbf{x}_r^G$ be a Nash equilibrium and $\mathbf{x}_r^S$ be the global optimum point. We derive the following set of inequalities as in the single link case.

$$\frac{\sum_r U_r(d_r(\mathbf{x}_r^G))}{\sum_r U_r(d_r(\mathbf{x}_r^S))} \geq \frac{\sum_r (U_r(d_r(\mathbf{x}_r^G)) - \alpha_r^T \mathbf{x}_r^G) + \sum_r \alpha_r^T \mathbf{x}_r^G}{\sum_r (U_r(d_r(\mathbf{x}_r^S)) - \alpha_r^T \mathbf{x}_r^G) + \sum_r \alpha_r^T \mathbf{x}_r^S}$$

$$\geq \frac{\sum_j \sum_r \alpha_{jr} x_{jr}^G}{\sum_j (\max_r \alpha_{jr}) C_j}$$

If the overall system is at the Nash equilibrium each single link has a Nash equilibrium and we can invoke the single link result that $\sum_r \alpha_{jr} x_{jr}^G \geq \frac{3}{4}(\max_r \alpha_{jr})C$. Substituting this we get the desired result.

Related Work and Conclusions

- Sanghvi and Hajek (2004)- Pricing Mechanisms where efficiency loss is smaller than 3/4.
- Johari, Mannor and Tsitsiklis (2004): The case of elastic supply.
- Johari and Tsitsiklis (2004): Cournot Mechanism and efficiency loss.
- Roughgarden and Tardös(2002)-How bad is Selfish routing?
- Mandyam et. al. (2004): Distributed Power control in CDMA systems.