Spectral Graph Theory and its Applications

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6.454
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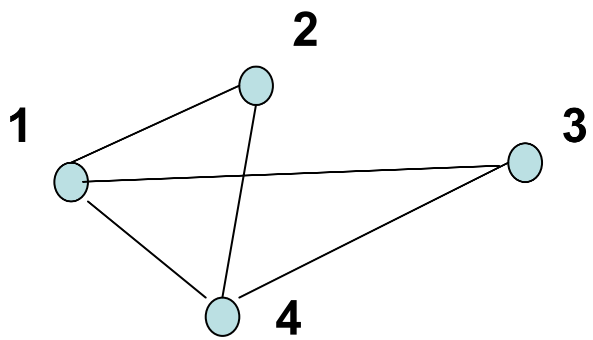
Outline

- Basic spectral graph theory

- Graph partitioning using spectral methods

D. Spielman and S. Teng, "Spectral Partitioning Works: Planar Graphs and Finite Element Meshes," 1996

Graph and Associated Matrices



$$G=(V,E)$$

$$|V|=n=4$$

$$|E|=m=5$$

Laplacian matrix

$$L_G = D_G - A_G$$
$$= B_G B_G^{\ T}$$

Adjacency matrix

$$A_G = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

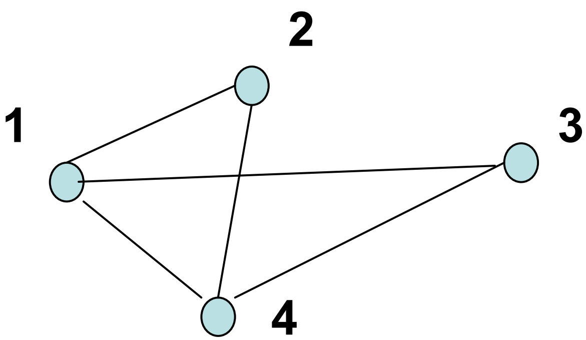
Degree matrix

$$D_G = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

Incidency matrix

$$B_G = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & -1 & -1 \end{bmatrix}$$

Properties of the Laplacian Matrix



$$L_G = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & 0 & -1 \\ -1 & 0 & 2 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix}$$

$$\lambda = \{0, 2, 4, 4\}$$

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} -1 \\ -1 \\ -1 \\ 3 \end{bmatrix} \quad \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

- Symmetric -> real eigenvalues; eigenspaces are mutually orthogonal

- Orthogonally diagonalizable -> an eigenvalue with multiplicity k has k-dimensional eigenspace

More Properties of the Laplacian Matrix

$$\lambda = \{0, 2, 4, 4\}$$

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} -1 \\ -1 \\ -1 \\ 3 \end{bmatrix} \quad \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

- Positive semidefinite -> non-negative eigenvalues

- Row sum = 0 -> singular -> at least one eigenvalue = 0, unity eigenvector (since row sum = 1)

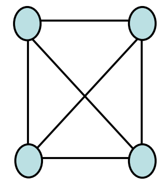
- Orthogonal eigenspaces
u = eigenvector of non-zero eigenvalue $\sum_{i=1}^{n} u_i = 0$

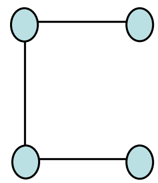
$$x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$$

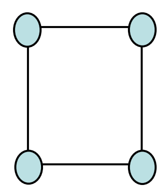
$$x^T L_G x = x^T B_G B_G{}^T x = \underbrace{\left(x^T B_G\right)}_{1\times m}\underbrace{\left(x^T B_G\right)^T}_{m\times 1} = \sum_{(i,j)\in E} \left(x_i - x_j\right)^2 \geq 0$$

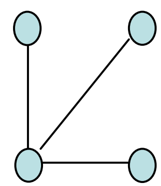
Spectrum of Some Graphs

	Eigenvalues
Complete	$\left\{0, n^{(n-1)}\right\}$
Line	$2 - 2\cos\left(\pi k / n\right)$ $k = 1, \ldots, n$
Ring	$2 - 2\cos\left(2\pi k / n\right)$ $k = 1, \ldots, n/2$
Star	$\left\{0, 1^{(n-2)}, 2\right\}$









Which graphs are determined by their spectrum?

- Complete Graphs
- Graphs with one edge
- Graphs missing 1 edge
- Regular graphs with degree 2
- Regular graphs of degree n - 3

Graph Connectedness

$$\lambda_1 \le \lambda_2 \le \ldots \le \lambda_n$$

$$\lambda = \{0, 2, 4, 4\}$$

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} -1 \\ -1 \\ -1 \\ 3 \end{bmatrix} \quad \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

For connected graphs, $\lambda_2 > 0$

Recall $\vec{x}^T L_G \vec{x} = \sum_{(i,j)\in E} \left(x_i - x_j\right)^2$

If $\vec{x}$ is eigenvector for eigenvalue 0

$L_G \vec{x} = 0 \Rightarrow x_i = x_j \quad (i,j)\in E$

Multiplicity of the 0 eigenvalue indicates # of connected components

λ_2	**Fiedler Value**
$\vec{v}$	**Fiedler Vector**

Onto Graph Partitioning …

Graph Partitioning

• Remove as little of the graph as possible to separate out a subset of vertices of some desired "size"
• "Size" may mean the number of vertices, number of edges, etc.
• Typical case is to remove as few edges as possible to disconnect the graph into two parts of almost equal size

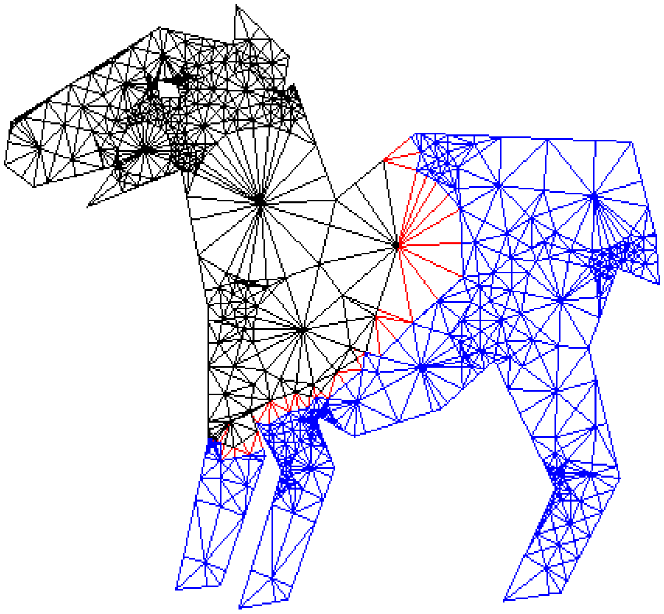
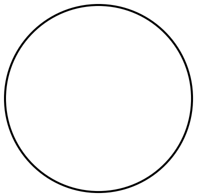


Diagram from Berkeley CS 267 lecture notes

Isoperimetric problem

One of the earliest problems in geometry – considered by the ancient Greeks: Find, among all closed curves of a given length, the one which encloses the maximum area



Stein, 1841

Applications

- Load balancing while minimizing communication
- Sparse matrix-vector multiplication
- Optimizing VLSI layout
- Communication network design

Bisection and Ratio-Partition

- Divide vertices into two disjoint subsets S and $\bar{S}$
- ***Cut Size*** $\left|E\left(S,\bar{S}\right)\right|$
- ***Cut Ratio*** $\phi_G\left(S\right)=\dfrac{\left|E\left(S,\bar{S}\right)\right|}{\min\left(\left|S\right|,\left|\bar{S}\right|\right)}$
- ***Isoperimetric Number*** $\phi_G=\min_{S\subset V}\phi_G\left(S\right)$

Bisection Minimize $\left|E\left(S,\bar{S}\right)\right|$ subject to # of nodes in each partition differ by at most 1.

NP-Complete

Ratio-Partition Minimize $\phi_G\left(S\right)$

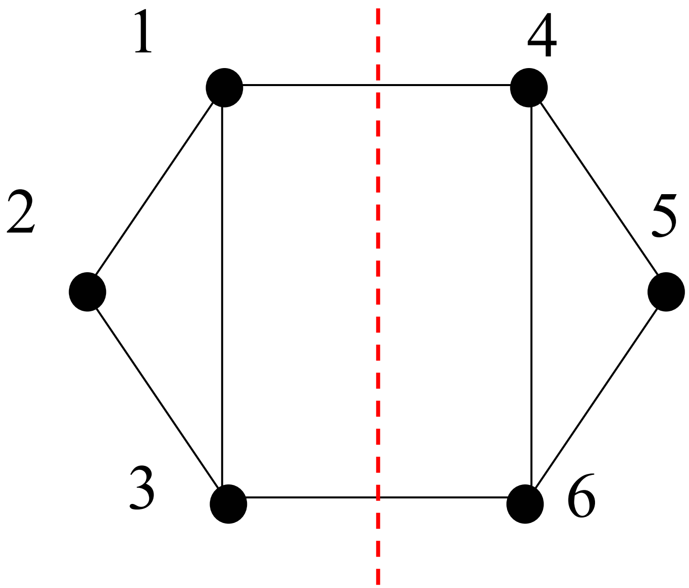
Spectral Partitioning

- Find Fiedler vector of the Laplacian matrix – map to vertices
- Choose some real number s
- Partition vertices given by $V_L = \{i : v_i \le s\}$

$$V_L = \{i : v_i > s\}$$

- **Bisection**, s = median of $\{v_1, \ldots v_n\}$
- **Ratio partition**, s is chosen to give the best cut ratio

Example



Fiedler vector [-1 -2 -1 1 2 1]

Spectral Partitioning For Planar Graphs

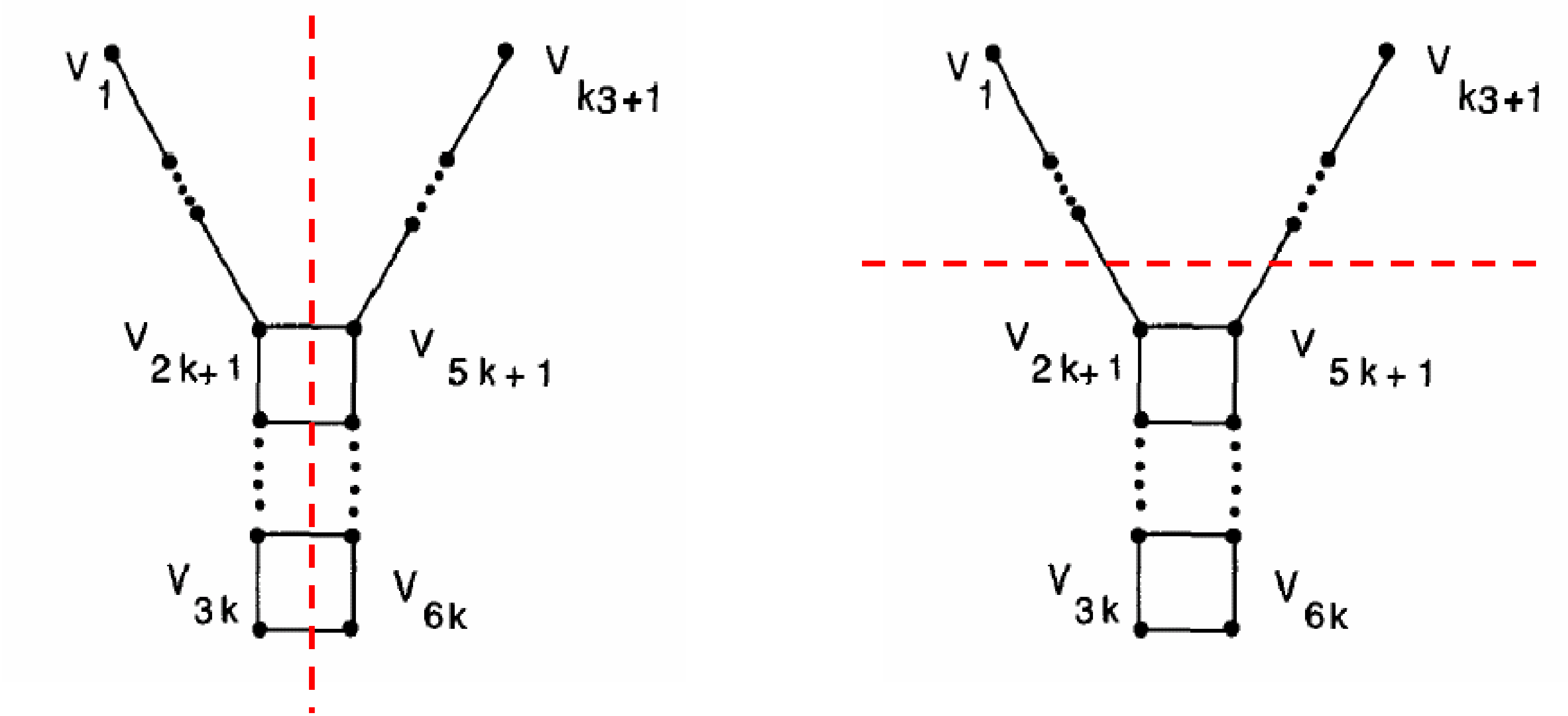
- Guattery and Miller – Performance of Spectral Graph Partitioning, 1995

- Spielman and Teng, Spectral Partitioning Works on Planar Graphs, 1996

- Kelner, Spectral Partitioning Works on Graphs with Bounded Genus, 2004

Simple Spectral Bisection May Fail

(Guattery & Miller)



The simple spectral bisection method produces cut size of $\Theta(n)$ for G_k , for any k

Optimal Bisector for Graphs with Bounded Genus
(Kelner)

Genus *g* of a graph G: smallest integer such that G can be embedded on a surface of genus g without any of its edges crossing one another. Eg. Planar graphs have genus 0

Sphere, disc, and annulus has genus 0
Torus has genus 1

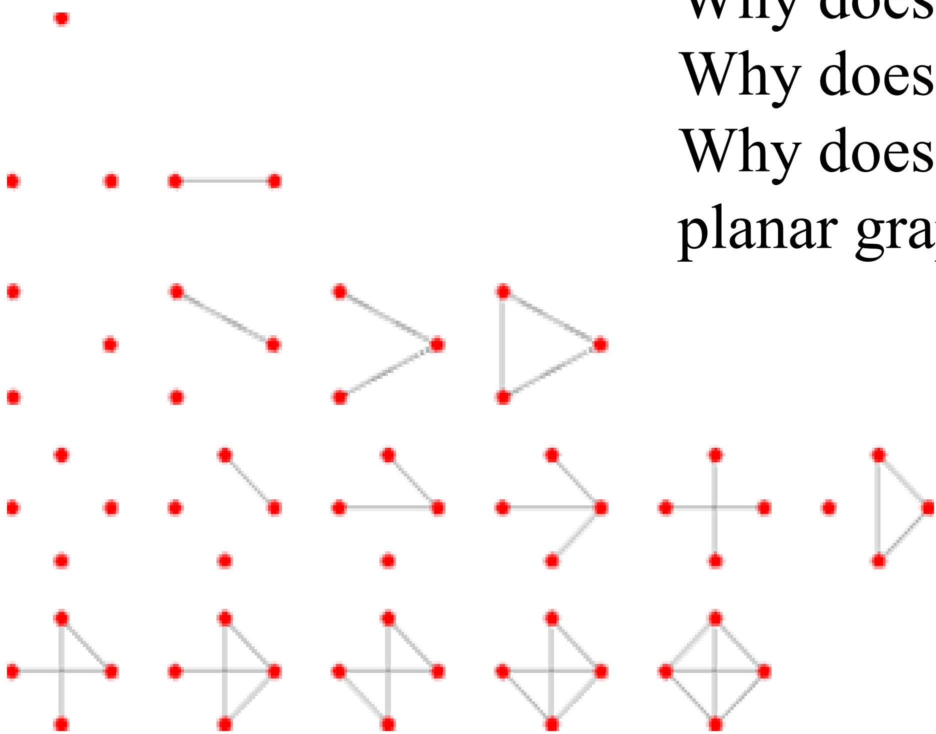
There is a spectral algorithm that produces bisector of size $O\left(\sqrt{gn}\right)$

For every *g*, there is a class of bounded degree graphs that have no bisectors smaller than $O\left(\sqrt{gn}\right)$

Improved Bisection Algorithm on Planar Graphs
(Spielman and Teng)

Bisector of size $O\left(\sqrt{n}\right)$

Why does the spectral method work?
Why does it work well on planar graphs?
Why does simple bisection fail even on planar graphs?



Another Look at Fiedler Value

Recall $x^T L_G x = \sum_{(i,j)\in E} \left(x_i - x_j\right)^2$ where $\vec{x} = \left(x_1, x_2, \ldots, x_n\right) \in \mathbb{R}^n$

Rayleigh quotient: $\phi_x = \dfrac{\vec{x}^T L_G \vec{x}}{\vec{x}^T \vec{x}} = \dfrac{\sum_{(i,j)\in E} \left(x_i - x_j\right)^2}{\sum x_i^2}$

Fiedler value satisfies $\lambda_2 = \min_{\vec{x} \perp (1,\ldots,1)} \phi_x$ with the minimum occurring only when $\vec{x}$ is a Fiedler vector.

$$\phi_x = \frac{\vec{x}^T L_G \vec{x}}{\vec{x}^T \vec{x}} = \frac{\vec{x}^T \lambda_2 \vec{x}}{\vec{x}^T \vec{x}} = \lambda_2$$

Connection Between Fiedler Value and Isoperimetric Number

Recall Isoperimetric Number
is the best ratio-partition possible

$$\phi_G = \min_{S \subset V} \frac{\left|E\left(S, \bar{S}\right)\right|}{\min\left(|S|, |\bar{S}|\right)}$$

Theorem 1 (Mihail '89) Let G be a graph on n nodes of maximum degree Δ. For any vector $\vec{x} \in \mathbb{R}^n$ such that $\sum_{i=1}^{n} x_i = 0$

$$\frac{\vec{x}^T L_G \vec{x}}{\vec{x}^T \vec{x}} \geq \frac{{\phi_G}^2}{2\Delta} \quad \Longrightarrow \quad \lambda_2 \geq \frac{{\phi_G}^2}{2\Delta}$$

Good ratio-partition can be achieved if Fiedler value is small

Moreover, there is an s so that the cut $\{i : v_i \leq s\}$ $\{i : v_i > s\}$ has ratio at most ${\phi_G}^2/(2\Delta)$

Upperbound on the Fiedler Value for Planar Graphs

Theorem 2 (Spielman & Teng '96) For all planar graphs G with n vertices and maximum degree Δ

$$\lambda_2 \leq \frac{8\Delta}{n} \quad \Rightarrow \quad O\left(\frac{1}{n}\right)$$

$$\frac{\phi_G{}^2}{2\Delta} \leq \lambda_2 \leq \frac{8\Delta}{n} \quad \Rightarrow \quad \phi_G \leq \frac{4\Delta}{\sqrt{n}} \quad \Rightarrow \quad O\left(\frac{1}{\sqrt{n}}\right)$$

By bounding Fiedler value of planar graphs, ratio-partitioning method is shown to work well

What about bisection?

Relationship Between Ratio-Partitioning and Bisection

Lemma 3 Given an algorithm that will find a cut ratio of at most $\phi(k)$ in every k-node subgraph of G, for some monotonically decreasing function ϕ. Then repeated application of this algorithm can be used to find a bisection of G of size at most

$$\int_{x=1}^{n} \phi(x)dx$$

$$\phi(x) = \frac{1}{\sqrt{x}} \Longrightarrow \int_{x=1}^{n} \phi(x)dx = 2\left(\sqrt{n}-1\right) \Longrightarrow O\left(\sqrt{n}\right)$$

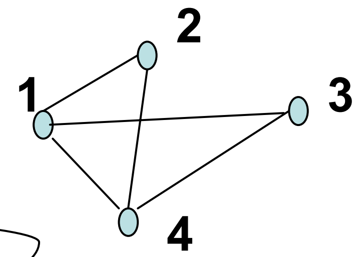
Bisection can be obtained by repeated application of ratio-partitioning

Theorem 1

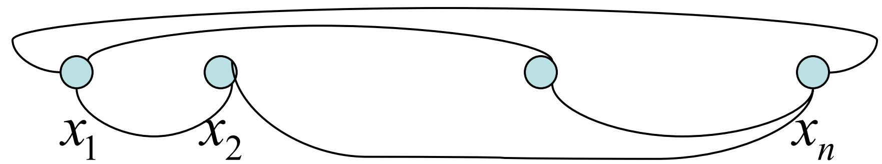
$$\frac{\vec{x}^T L_G \vec{x}}{\vec{x}^T \vec{x}} \geq \frac{\phi_G^{\ 2}}{2\Delta} \qquad \sum_{i=1}^{n} x_i = 0$$

$$\phi_G = \min_{S \subset V} \frac{\left|E\left(S, \bar{S}\right)\right|}{\min\left(|S|, |\bar{S}|\right)}$$

Map graph vertices to a line



$$x_1 \leq x_2 \leq \ldots \leq x_n$$

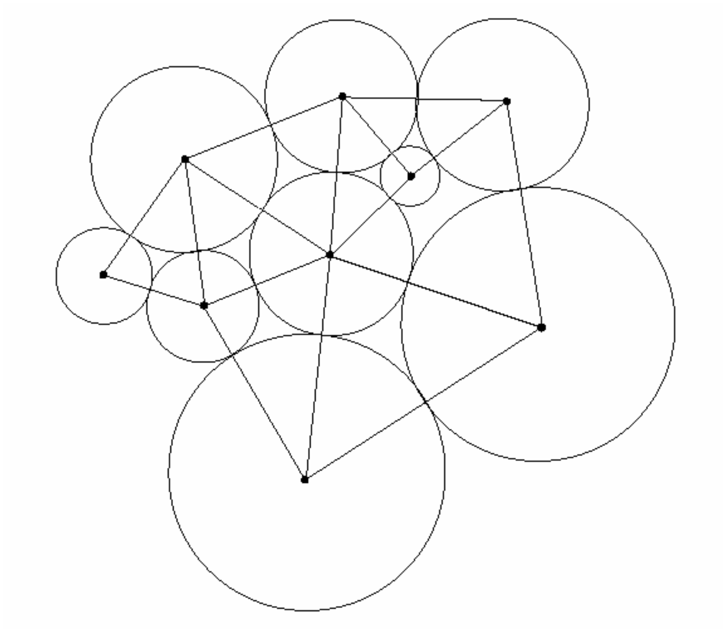


$$\frac{\vec{x}^T L_G \vec{x}}{\vec{x}^T \vec{x}} = \frac{\sum_{(i,j)\in E} \left(x_i - x_j\right)^2}{\sum x_i^{\ 2}} = \frac{\text{sum (length of edge)}^2}{\text{sum (length away from 0)}^2}$$

If $i \leq n/2$ At least $\phi_G i$ edges must cross over x_i

Proof of Theorem 2 $\lambda_2 \le \frac{8\Delta}{n}$

Theorem 4 (Koebe-Andreev-Thurston). Let G be a planar graph. Then, there exist a set of disks $\{D_1,\ldots,D_n\}$ in the plane with disjoint interiors such that D_i touches D_j iff $(i,j)\in E$

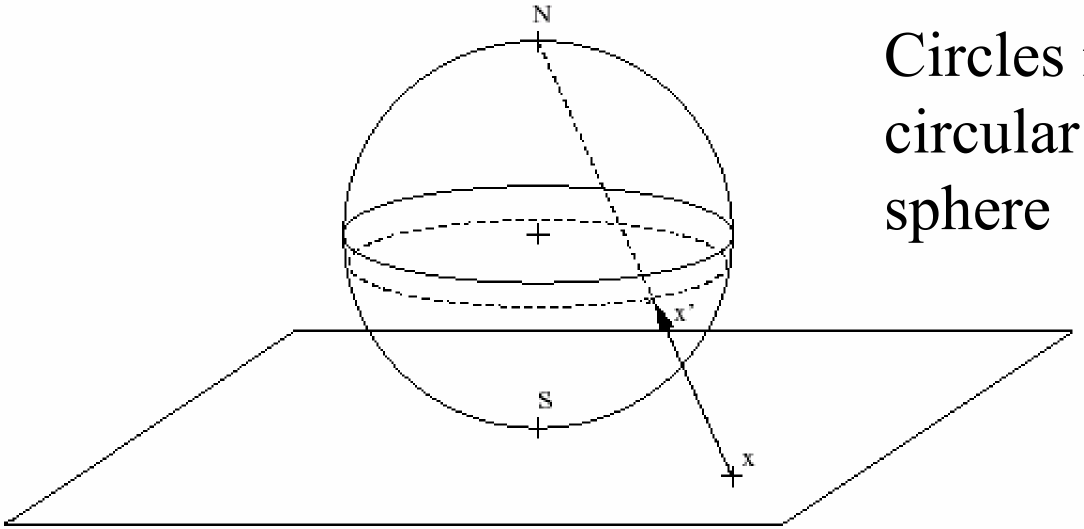


Kissing disks

Proof of Theorem 2 cont.

Stereographic Projection

$$\{\pi(D_1),\ldots,\pi(D_n)\}$$



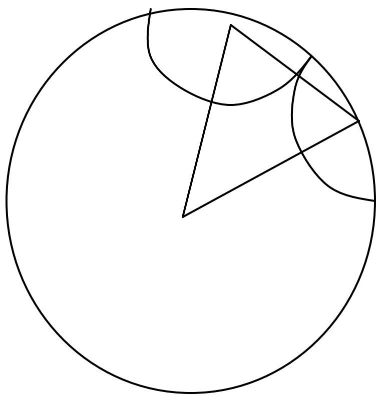
Circles in the plane -> circular caps on the sphere

Proof of Theorem 2 cont.

Let $\vec{x}_i$ be the center of $\pi(D_i)$ on the sphere.

$$\|\vec{x}_i\| = 1 \qquad \sum_{i=1}^{n} \|\vec{x}_i\|^2 = n$$

Let r_i be the radius of the cap $\pi(D_i)$



$$\|\vec{x}_i - \vec{x}_j\|^2 \le (r_i + r_j)^2 \le 2(r_i^2 + r_j^2) \qquad (i,j) \in E$$

$$\sum \pi r_i^2 \le 4\pi$$

$$\sum_{(i,j)\in E} \|\vec{x}_i - \vec{x}_j\|^2 \le 2 \sum_{(i,j)\in E} (r_i^2 + r_j^2) \le 2 \sum_i d_i r_i^2 \le 8\Delta$$

$$\lambda_2 \le \frac{\sum_{(i,j)\in E} \|\vec{x}_i - \vec{x}_j\|^2}{\sum \|\vec{x}_i\|^2} \le \frac{8\Delta}{n}$$

Conclusion

Why does the spectral method work?

- close relationship between Fiedler value and Isoperimetric number $\lambda_2 \geq \frac{\phi_G{}^2}{2\Delta}$

Why does it work well on planar graphs?

- planar graphs have nice collection of spherical cap embeddings $\lambda_2 \leq \frac{8\Delta}{n}$

Why does simple bisection fail even on planar graphs?

- even though good ratio-partitions can be found, the result may be unbalanced in the size of the partitions