Massachusetts Institute of Technology
Department of Electrical Engineering and Computer Science
6.685 Electric Machines

Problem Set 7 Solutions November 4, 2005

Problem 1: Rail Gun

Assuming the current source distributes uniformly over the depth of the track and discounting fields outside of the gun, we have a magnetic field

$$H_z = -\frac{I}{D}$$

Then, surrounding the block with a surface we evaluate the Maxwell Stress Tensor: on the normal surface facing backwards,

$$T_{xx} = -\frac{\mu_0}{2}\left(\frac{I}{D}\right)^2$$

On the normal surface facing forward all fields and so the MST are zero. Then x- directed force is

$$F_x = -DwT_{xx} = \frac{\mu_0}{2}\frac{w}{D}I^2$$

Note we could have used co-energy: inductance is

$$L = \mu_0 \frac{wx}{D}$$

To get voltage across the system we would note, that since voltage in the frame of the moving block is zero:

$$\vec{E} = -\vec{u} \times \vec{B}$$

Or,

$$E_y = u_x B_z$$

Then, voltage across the rails is

$$V = -E_y w = u_x \mu_0 \frac{w}{D} I$$

Electrical power and mechanical power are:

$$\begin{aligned} P_e &= VI &= \mu_0 \frac{w}{D} I^2 u_x \\ P_m &= F_x u_x &= \frac{\mu_0}{2}\frac{w}{D} I^2 u_x \end{aligned}$$

Since $P_e = 2P_m$, over the span of a launch we will put exactly twice as much energy into the thing, electrically, as is delivered to the projectile, leading to an efficiency of 50%.

If we can ignore friction and windage, acceleration is simply:

$$a = \frac{F}{M}$$

Velocity is:

$$u_x = at$$

And position is:

$$x = \frac{1}{2}at^2$$

Now, in order to generate the plots we must go backwards. If we have a fixed velocity u_0 at the muzzle which has position $x = L$, we reach that position at time

$$t_0 = \frac{2L}{u_0}$$

and this implies a rate of acceleration

$$a = \frac{u_0}{t_0} = \frac{u_0^2}{2L}$$

With $u_0 = 200$ m/s and $L = 10$ m, we have $a = 2000$ m/s/s (about 200 g's) and $t_0 = 0.1$ s. Current required is found by inverting the force expression:

$$I^2 = \frac{DMu_0^2}{\mu_0 wL}$$

and this evaluates to about $I = 5.64 \times 10^4$ A. The internal flux density is:

$$|B_z| = \mu_0 \frac{I}{D} \approx 1.418\text{T}$$

As a check,

$$F = \frac{B^2}{2\mu_0} wD = 2000$$

We have used these expressions to calculate the velocity and position of the slug, shown in Figure 1 and voltage and power input and output, shown in Figure 2.

Problem 2: To approach this problem, we note that $g << R$, which allows us to:

1. Use rectilinear coordinates, and
2. assume that azimuthal field is the same as stator current density.

We are concerned only with fifth and seventh space harmonic currents, so the form of stator current density will be:

$$K_z = \Re K_5 e^{j(6\omega t + 5p\theta')} + k_7 e^{j(6\omega t + 7p\theta')}$$

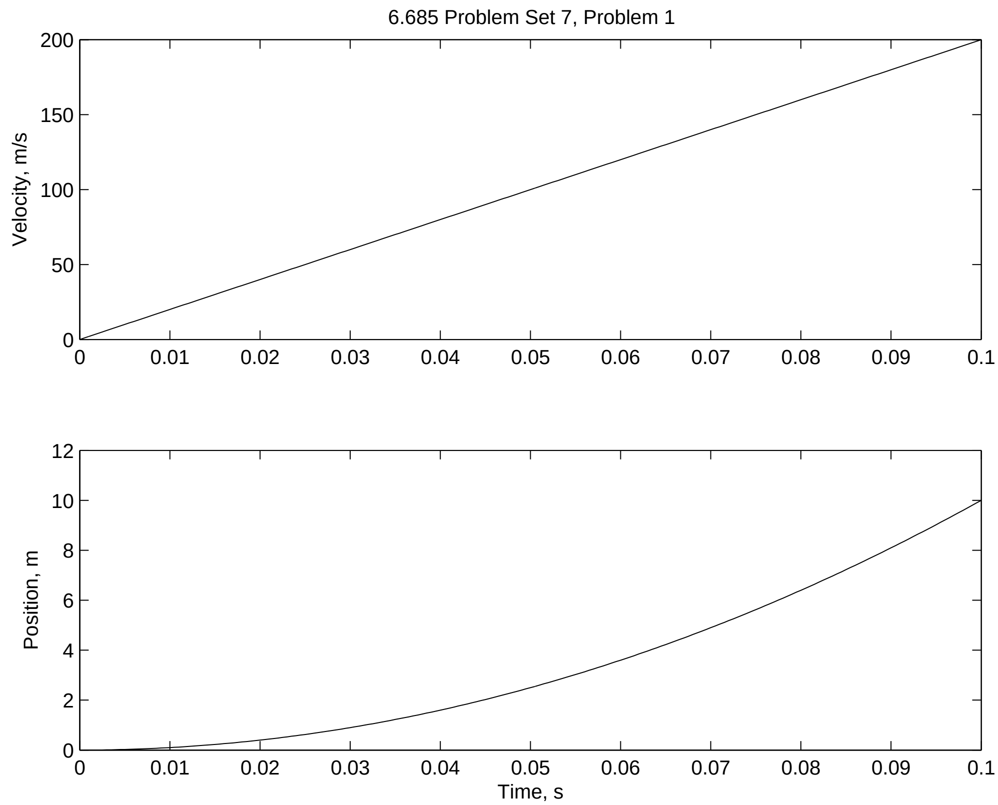

Figure 1: Slug Velocity and Position

There are several different ways of finding the harmonic current amplitudes. Probably the easiest is to note that we have already established two different ways of estimating radial flux density (in this case with the assumption of an infinitely permeable rotor and stator):

$$
\begin{aligned}
H_n &= \frac{3}{2}\frac{4}{n\pi}\frac{NI}{2pg}k_n \\
H_n &= \frac{R}{jnpg}K_n
\end{aligned}
$$

Equating these and solving for surface current density magnitude, we get:

$$|K_n| = \frac{3}{2}\frac{4}{\pi}\frac{NI}{2R}k_n$$

Noting that the winding can be viewed as $m = 2$, full pitched, we compute the breadth factors, using the slot electrical angle $\gamma = 30^\circ$,

$$
\begin{aligned}
k_5 &= \frac{\sin(2 \times 5 \times \frac{30^\circ}{2})}{2 \sin 5 \times \frac{30^\circ}{2}} \approx .2588 \\
k_7 &= \frac{\sin(2 \times 7 \times \frac{30^\circ}{2})}{2 \sin 7 \times \frac{30^\circ}{2}} \approx -.2588
\end{aligned}
$$

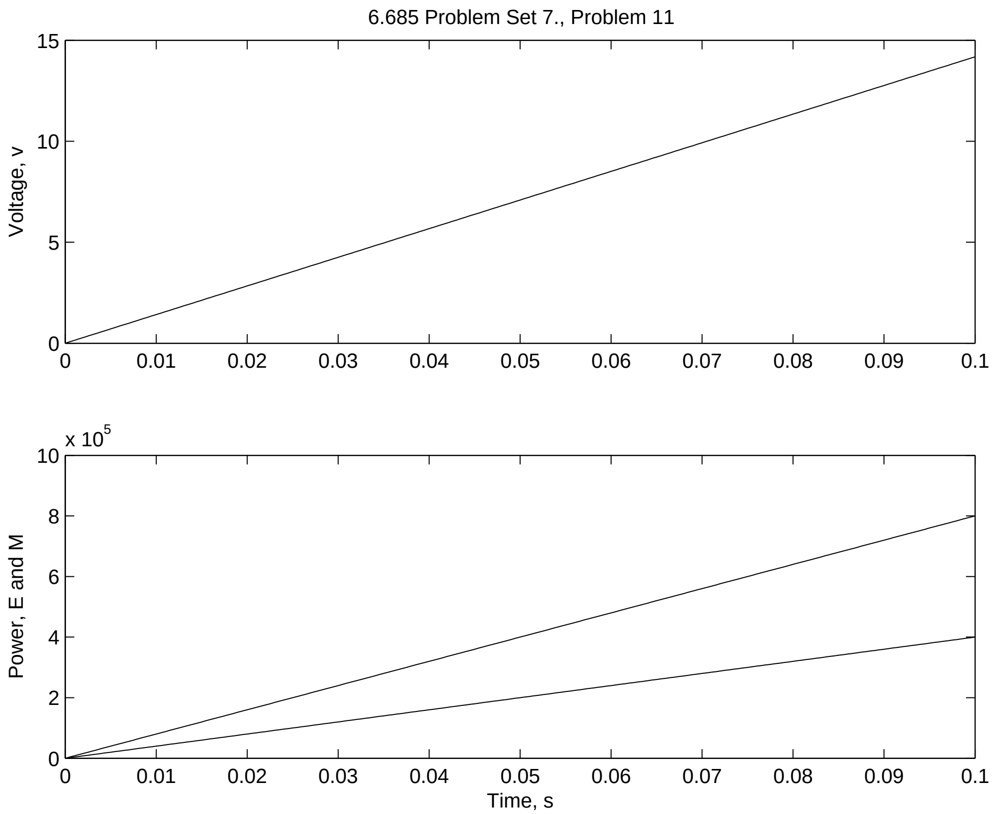

Figure 2: Voltage and Power

Harmonic current magnitudes are the same:

$$|K_5| = |K_7| = \frac{3}{2}\frac{4}{\pi}\frac{8 \times 10,000}{2} \times .2588 \approx 19771 \mathrm{A/m}$$

The two current distributions beat against each other, leading to a maximum azimuthal field strength of $H_{\mathrm{max}} = 2 \times 19771 \approx 39541$ A/m. The minimum field strength is zero. Frequency is $6\omega = 2262$ Radians/second. If we use a corner flux density of $B_0 = \frac{3}{4} \times 2 = 1.5$ T, the penetration depth is:

$$\delta = \sqrt{\frac{2 \times 39542}{2262 \times 5.5 \times 10^6 \times 1.5}} \approx 2.06\ mm.$$

Surface resistance is:

$$R_s = \frac{16}{3\pi}\frac{1}{\sigma\delta} \approx 1.5 \times 10^{-4}\Omega$$

Peak power dissipation in the rotor is:

$$P = \frac{1}{2}|K_z|^2 R_s \approx 1.7 \times 10^5 \quad W/m^2$$

We can carry out a calculation of loss as a function of azimuthal position, using the magnitude of current:

$$|K_z| = abs\left\{k_5 e^{j5\theta} + k_y e^{j7\theta}\right\}$$

Details are in the attached script.The result is shown in Figure 3.

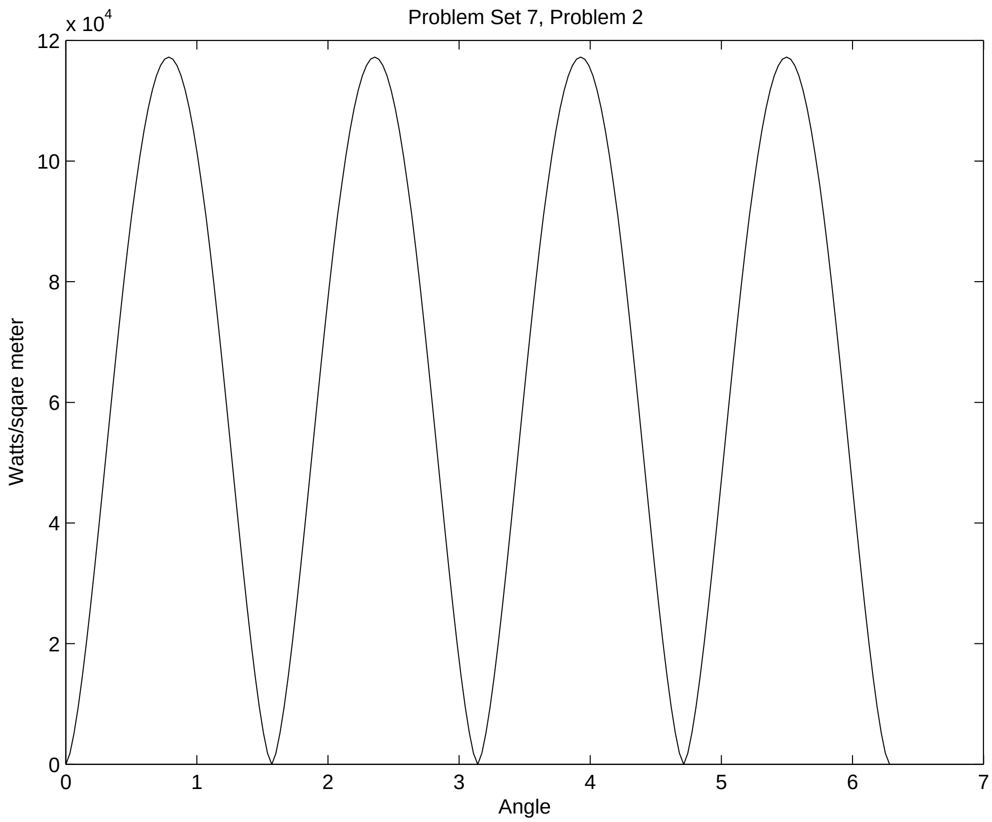

Figure 3: Voltage and Power

Problem 3: DC Traction Motor

Note that we can incorporate the gearing and wheel size into the motor characteristic. At a car velocity of 25 m/s with a terminal voltage of 600 V, the motor draws 800 A and produces a power of 400 kW. Back voltage is

$$E_b = \frac{400,000W}{800A} = 500V$$

Combined armature and field resistance must then be:

$$R = \frac{600 - 500V}{800A} = 1/8\Omega$$

The motor characteristic is then

$$G = \frac{500V}{25m/s \times 800A} = .025\Omega/m/s$$

Given that:

1. If the machine is producing 200 kW at 25 m/s it is delivering 8000 N. Current is:

$$I = \sqrt{\frac{8000}{.025}} = 566\ A$$

2. With 2000 A into the motor, force produced is:

$$F = .025 \times 2000^2 = 100,000N$$

Climbing a grade of angle θ, the component of gravity force parallel to the track is:

$$F_g = Mg\sin\theta$$

This means that

$$\sin\theta = \frac{100,000}{9.8 \times 25,000} \approx 24.1^\circ$$

With a terminal current of 2,000 A, speed voltage is $E_b = 600 - 2000 \times \frac{1}{8} = 200\ V = GIu$ Speed is then:

$$u = \frac{200}{.025 \times 2000} = 4m/s$$

3. *Jerk* is the rate of change of acceleration:

$$J = \frac{d}{dt}\frac{GI^2}{M} = \frac{G}{M}2I\frac{dI}{dt}$$

Note that we are ignoring the effects of drag for two reasons: first, drag is a slowly varying function of velocity and, second, we expect maximum jerk to occur at relatively low speed. So during the initial transient:

$$I = \frac{V}{R}\left(1 - e^{-\frac{R}{L}t}\right)$$

Then jerk is:

$$J = \frac{G}{M}2\frac{V^2}{RL}\left(e^{-\frac{R}{L}t} - e^{-2\frac{R}{L}t}\right)$$

The maximum value of this is found by finding the zero of its derivative, and that is found to be:

$$2e^{-\frac{R}{L}t} = 1$$

, which implies that, if $L = 1/8$,

$$t = \frac{L}{R}\log 2 \approx .693s$$

The maximum jerk is

$$J_{\max} = 2\frac{G}{M}\frac{V^2}{LR}\frac{1}{2}\left(1 - \frac{1}{2}\right)$$

The principal number here:

$$2\frac{g}{m}\frac{V^2}{LR} \approx 46.08$$

So the jerk rate is $46.08 \times \frac{1}{4} = 11.52 m/s^3$. Note that current at this point is greater than the limit:

$$I = \frac{600}{\frac{1}{8}} \times \frac{1}{2} = 2400A$$

With the current limit in place, the jerk rate will hit its maximum value as the current hits the limit, or when:

$$\frac{V}{R}\left(1 - e^{-\frac{R}{L}t}\right) = 2000$$

This implies that

$$e^{-\frac{R}{L}t} = 1 - \frac{2000}{4800} = \frac{2800}{4800} = \frac{7}{12}$$

The maximum jerk is

$$J\text{max} = 2\frac{G}{M}\frac{V^2}{LR}\frac{7}{12}\left(1 - \frac{7}{12}\right)$$

The difference is not large:

$$\frac{7}{12}\left(1 - \frac{7}{12}\right) \approx .243$$

so the maximum (current limited) jerk is about $11.2 m/s^3$

4. Simulation of the transient is fairly straightfoward as shown in the attached scripts. The obvious choice of state variables is motor current and vehicle speed. We have run this with the current not limited and with the current limited to 2,000 A and the results are shown below.

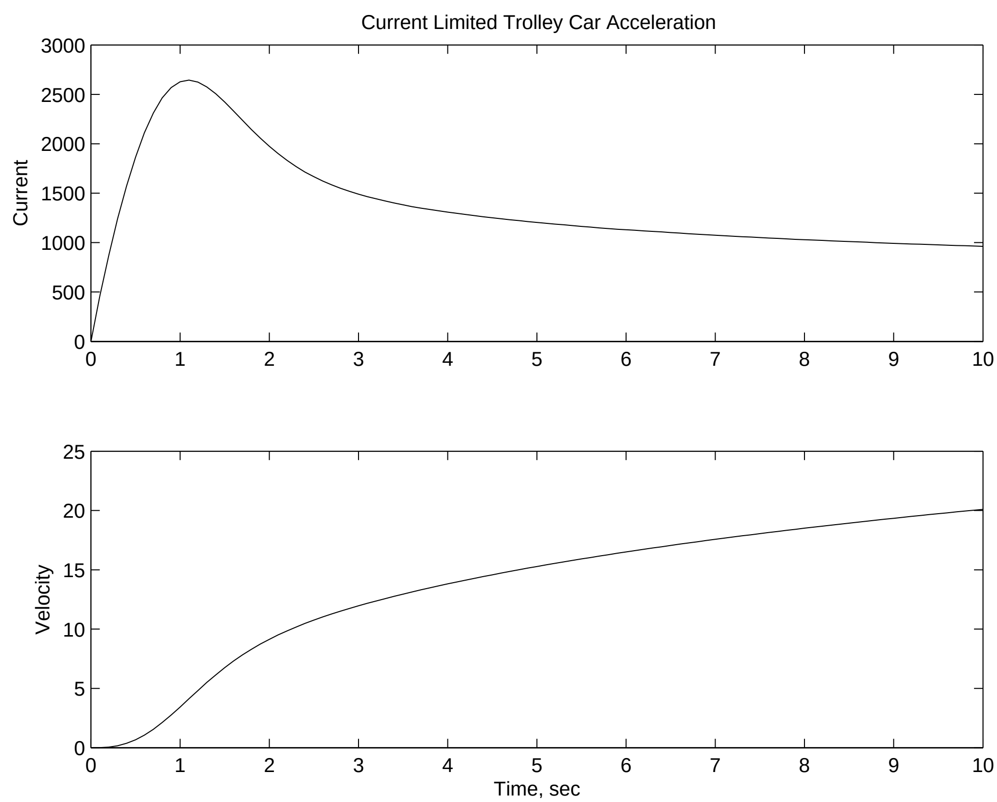

Figure 4: Current and motor speed, current unlimited

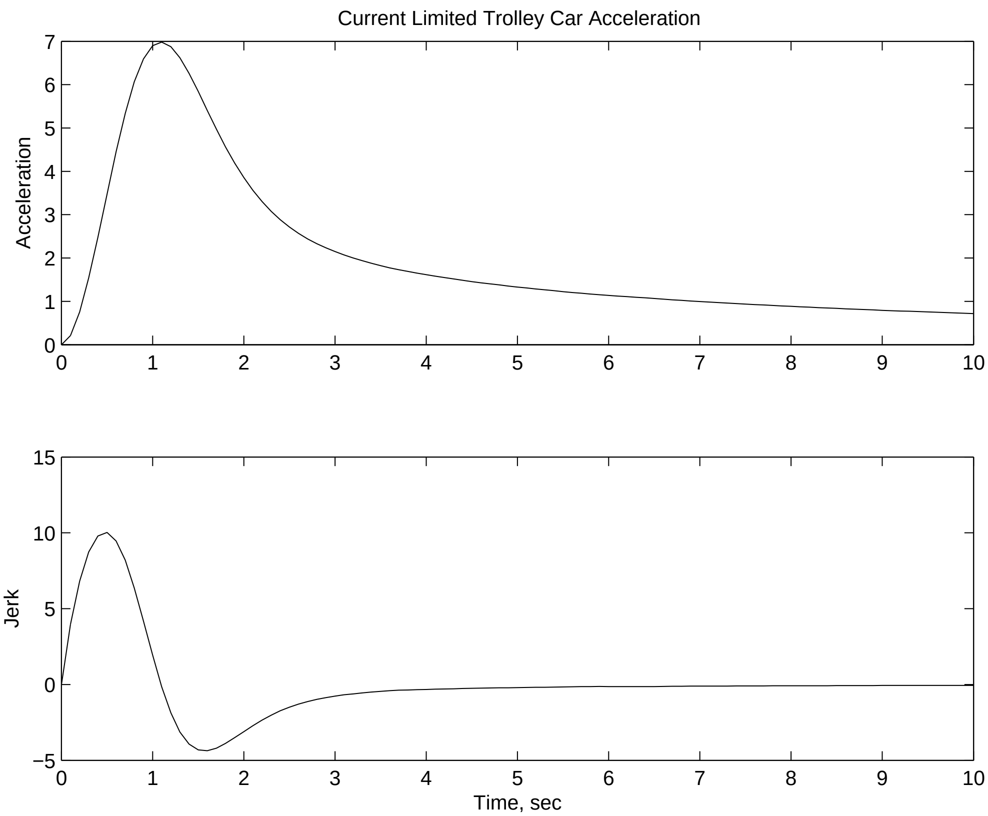

Figure 5: Acceleration and Jerk Rate, current unlimited

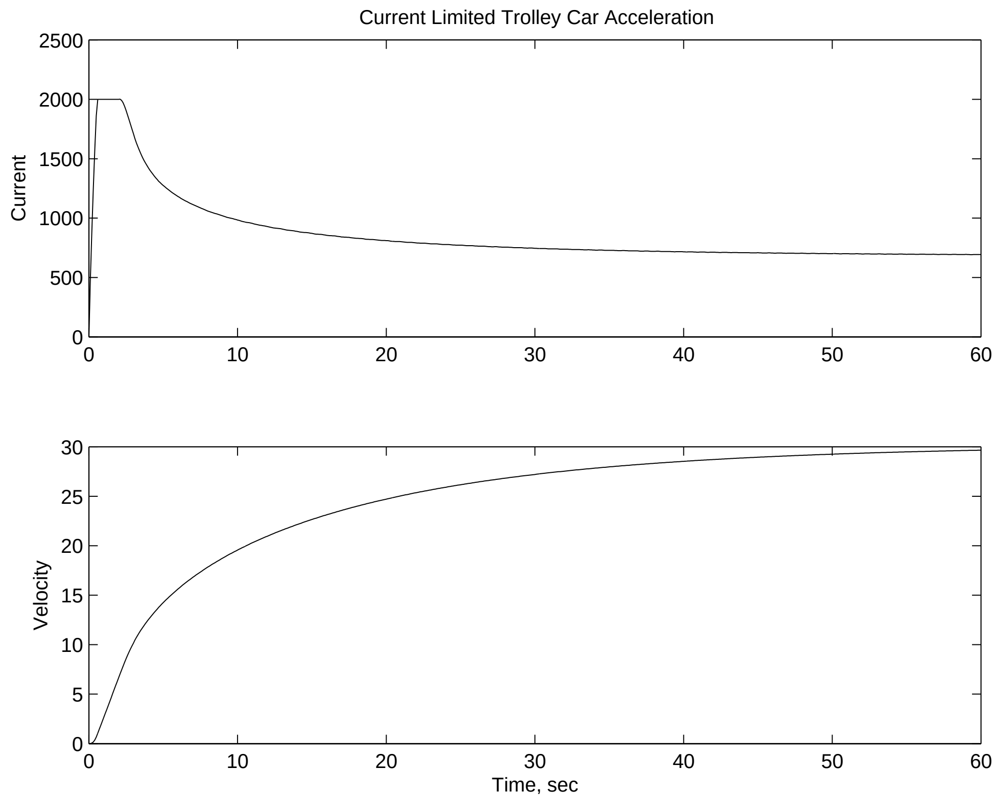

Figure 6: Current and motor speed, current limited

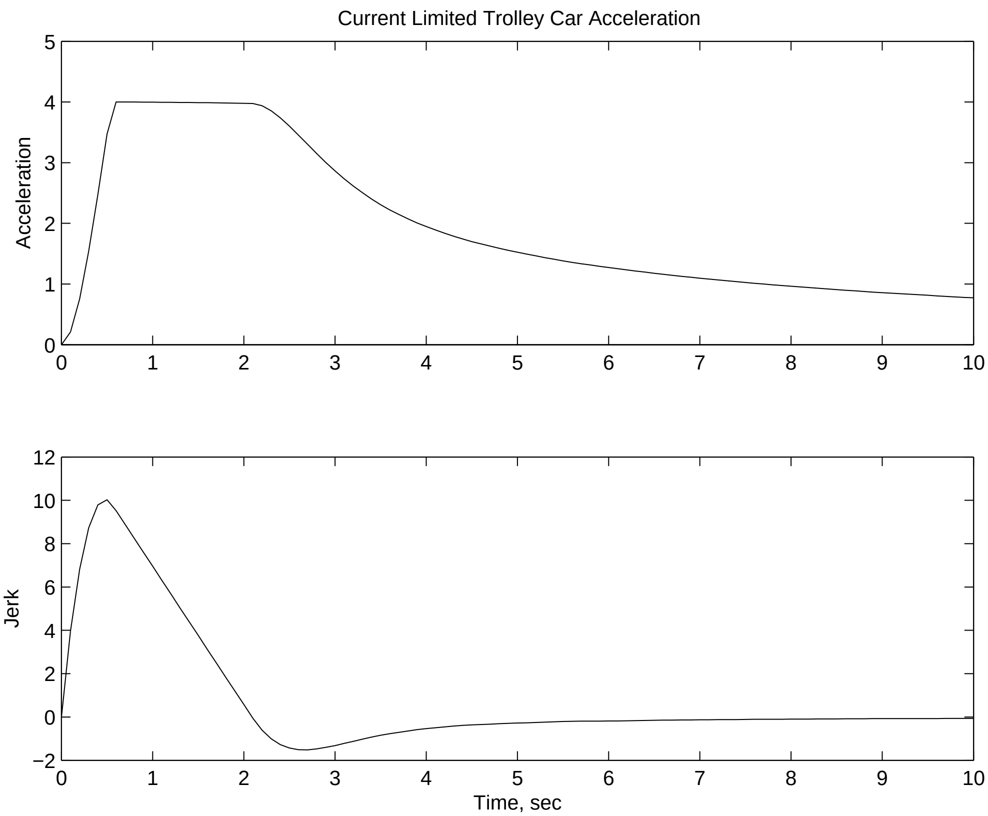

Figure 7: Acceleration and Jerk Rate, current limited

```
% Solution to 6.685 Problem Set 7, Problem 1
% This is a rail gun

M = 1;              % projectile mass
w = .05;            % barrel width
D = .05;            % barrel depth
L = 10;             % barrel length
u_0 = 200;          % muzzle velocity
muzero=pi*4e-7;

t_0 = 2*L/u_0;      % this is time to launch
a = u_0/t_0;        % acceleration

t=0:t_0/200:t_0;    % setup time
u = a .* t;         % velocity vs. time
x = .5*a .* t .^2;  % x location

F = .5*(M/L)*u_0^2;             % required force
I = sqrt(D*M/(muzero*w*L))*u_0;  % required current

V = (muzero*I*w/D) .* u;        % voltage
P_e = V .* I;                   % electrical power
P_m = F .* u;                   % mechanical power

% now for output

fprintf('6.685 Problem 6_1\n')
fprintf('Acceleration         = %10.3g    Launch Time     = %10.3g\n',a, t_0);
fprintf('Force                = %10.3g    Current         = %10.3g\n',F, I);

figure(1)
subplot 211
plot(t, u)
title('6.685 Problem 7.1')
ylabel('Velocity, m/s')
subplot 212
plot(t, x)
ylabel('Position, m')
xlabel('Time, s')

figure(2)
subplot 211
plot(t, V)
title('6.685 Problem 7.1')
ylabel('Voltage, v')
```

```
subplot 212
plot(t, P_e, t, P_m)
ylabel('Power, E and M')
xlabel('Time, s')

% 6.685 Problem Set 7, Problem 2
% Parameters
p = 2;                 % 4 pole machine
m = 2;                 % slots per pole per phase
Nc = 1;                % number of turns per coil
I = 10000;             % PEAK current in each coil
R = 1;                 % rotor radius
sig = 5.5e6;           % material conductivity
B0 = 1.5;              % corner flux density for Agarwal model
om = 6*377;            % electrical frequency

gama = pi/(3*m);       % electrical slot angle
k5 = sin(m*5*gama/2)/(m*sin(5*gama/2));   % breadth factors
k7 = sin(m*7*gama/2)/(m*sin(7*gama/2));

N = Nc*2*m*p;          % turn count

K_5 = (6/pi)*(N*I/(2*R))*k5;
K_7 = (6/pi)*(N*I/(2*R))*k7;

th = 0:pi/100:2*pi;    % full circle
K = abs(K_5 .* exp(j*5*p .* th) + K_7 .* exp(j*7*p .* th));

delt = sqrt((2/(om*sig*B0)) .* K);
Rs = (16/(3*pi*sig)) ./ delt;

P = .5*Rs .* K .^2;

figure(1)
clf
plot(th, P)
title('Problem Set 7, Problem 2');
ylabel('Watts/sqare meter');
xlabel('Angle')
```

```
% Problem Set 7, Problem 3 simulation
% put the constants in here
M = 25000;                % mass
G = .025;                 % motor constant
R = 1/8;                  % motor resistance
L = 1/8;                  % motor inductance
V = 600;

% parameters are in ddt.m so we don't have to worry about them

t0 = 0:.1:10;
X0 = [0 0];
[t, X] = ode23('ddt', t0, X0);
i = X(:,1);
u = X(:,2);

figure(1)
clf
subplot 211
plot(t, i)
title('Current Limited Trolley Car Acceleration')
ylabel('Current')
subplot 212
plot(t, u)
ylabel('Velocity')
xlabel('Time, sec')

% calculation of jerk rate
didt = (V - R .* i - G .* u .* i) ./ L;

J = 2*(G/M) .* i .* didt;

A = (G .* i .^2-8000 .* (u ./25) .^2) ./ M;

figure(2)
clf
subplot 211
plot(t, A)
title('Current Limited Trolley Car Acceleration')
ylabel('Acceleration')
subplot 212
plot(t, J)
ylabel('Jerk')
xlabel('Time, sec')
---------------------------
```

```
function dX = ddt(t, X0)
% put the constants in here
M = 25000;                % mass
G = .025;                 % motor constant
R = 1/8;                  % motor resistance
L = 1/8;                  % motor inductance
V = 600;

i = X0(1);
u = X0(2);
idotp = (V-R*i-G*u*i)/L;

if (i>= 2000 && idotp > 0),
    idot = 0;
else
    idot = idotp;
end

udot = (G*i^2-8000*(u/25)^2)/M;
dX = [idot udot]';
```