6.728 Applied Quantum and Statistical Physics

Department of Electrical Engineering and Computer Science
Massachusetts Institute of Technology

PROBLEM SET 2

Issued: 9-12-02 Due: 9-18-02, in-class

Problem 2.1

We are interested in the masses, (deBroglie) wavelengths, energies, and frequencies of various kinds of particles.

Fill in the following matrix, using kg-m/sec as the unit of momentum, p, electron-volts as the unit of energy, E, meters (cm, nm, etc.) as the unit of wavelength, λ, and (for photons only), sec^{-1} as the unit of frequency, ν. The energies are the kinetic energies for the particles.

Particle	p	E	λ	ν
Photon	-	-	1 m	?
	-	-	1 mm	?
	-	-	1 nm	?
Electron	-	25 meV	-	n/a
	-	100 eV	-	n/a
Neutron	-	1 meV	-	n/a
	-	1 eV	-	n/a
Pollen grain (10 μg)	-	25 meV	-	n/a
Golf ball (100 g)	1	-	-	n/a

Problem 2.2

In signal theory, there are limitations on the time-bandwidth product imposed by Fourier transform theory. One such relation can be written as

$$\Delta\omega\, \Delta t \geq \frac{1}{2}$$

In this problem we will explore the spatial version of this result

$$\Delta x\, \Delta k \geq \frac{1}{2}$$

which is essentially equivalent.

Assume in what follows that the probability amplitude $\Psi(x)$ is given by

$$\Psi(x) = Ce^{-x^2/2}$$

where C is a normalization constant. The Fourier transform of this function is given by

$$\Psi(k) = De^{-k^2/2}$$

where D is a normalization constant.

(a) Find the expression for the normalization constant C and write out the probability distribution in space

$$P(x) = |\Psi(x)|^2$$

(b) Determine Δx from

$$(\Delta x)^2 = \int_{-\infty}^{\infty} P(x)(x - \langle x \rangle)^2 \, dx$$

(c) Find the probability distribution in momentum space

$$P(k) = |\Psi(k)|^2$$

(d) Determine Δk from

$$(\Delta k)^2 = \int_{-\infty}^{\infty} P(k)(k - \langle k \rangle)^2 \, dk$$

(e) What is the product $\Delta x \, \Delta k$ in this case?

(f) If we had instead selected $\Psi(x) = Ce^{-2x^2}$ what would we have found for Δx, Δk and the product $\Delta x \, \Delta k$?

Problem 2.3

In this problem we are interested in the non-relativistic limit of Compton scattering. A photon with wavelength λ is assumed to scatter from an electron with mass m_e in free space. We are interested in the recoil of the electron and in the frequency shift of the photon after scattering. Assume that the electron is at rest prior to any scattering. You may assume in the following that the scattering events under consideration are back-scattering at 180°.

(a) What is the total momentum of the electron and photon before and after scattering? Is momentum conserved?

(b) What is the total energy of the electron and photon before and after scattering? Is energy conserved?

(c) Estimate the electron recoil energy.

(d) Estimate the change in wavelength λ of the scattered photon. Compare this wavelength shift with the Compton wavelength.

(e) Is this shift observable for free electrons in a plasma using a He:Ne laser source (He:Ne emission wavelength is λ = 632.8 nm).

(f) What problems might be involved with attempting to observe the analog of this effect for scattering from an atom or molecule (use H_2O as an example).

Problem 2.4

Let $\hat{A}$, $\hat{B}$, and $\hat{C}$ be operators.
A comutator of two operators that act on wavefunction Ψ is the expression

$$[\hat{A},\hat{B}]\,\Psi = \hat{A}\hat{B}\,\Psi - \hat{B}\hat{A}\,\Psi$$

(a) Show that

$$[\hat{A},\hat{B}\hat{C}]\,\Psi = [\hat{A},\hat{B}]\hat{C}\,\Psi + \hat{B}[\hat{A},\hat{C}]\,\Psi$$

also written as

$$[\hat{A},\hat{B}\hat{C}] = [\hat{A},\hat{B}]\hat{C} + \hat{B}[\hat{A},\hat{C}]$$

(b) Let $\hat{p}$ and $\hat{x}$ be operators satisfying $[\hat{x},\hat{p}] = i\hbar$ and let the Hamiltonian be

$$H = \frac{\hat{p}^2}{2m} + \frac{1}{2}k\hat{x}^2$$

Use the commutation relations and Ehrenfest's Theorem, $\frac{d}{dt}\langle \hat{A}\rangle = \frac{1}{i\hbar}\langle[\hat{A},\hat{H}]\rangle + \left\langle \frac{d\hat{A}}{dt}\right\rangle$

to find

$$\frac{d}{dt}\langle \hat{x}\rangle, \qquad \frac{d}{dt}\langle \hat{p}\rangle$$

and

$$\frac{d}{dt}\langle \hat{x}^2\rangle, \qquad \frac{d}{dt}\langle \hat{p}^2\rangle$$