6.728 Applied Quantum and Statistical Physics

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PROBLEM SET 5

Issued: 10-10-02 Due: 10-16-02, in-class

Problem 5.1 Given an unnormalized wavepacket made up of a superposition of harmonic oscillator states:

$$\Psi(x,t) = \psi_0(x)e^{-iE_0t/\hbar} - \frac{1}{2}\psi_1(x)e^{-iE_1t/\hbar} + i\psi_2(x)e^{-iE_2t/\hbar}$$

where $\psi_n(x)$ is a normalized harmonic oscillator stationary state wavefunction with energy $E_n = (n+1/2)\hbar\omega_o$.

(a) Show by explicit calculation with creation and annihilation operators that $\langle x\rangle$ and $\langle p\rangle$ are oscillatory functions of time, and determine the frequency of oscillation.

(b) Make a phase plot of $\langle p\rangle$ vs. $\langle x\rangle$ with time as a parameter. How does this compare with the phase plot for the classical harmonic oscillator? Explain (carefully) using the Ehrenfest relation why the result is "obvious".

(c) How would your results differ if there was only one stationary state in $\Psi(x,t)$. What does this tell you about building moving wavepackets from stationary states in confined systems?

Problem 5.2 *Quantum LC Circuit II.*

(a) The Hamiltonian of the LC circuit can also be written as

$$H = \frac{1}{2}CV^2 + \frac{1}{2}\frac{\Phi^2}{L}$$

where Φ is the flux in the inductor.

(b) Show that the Hamiltonian can be written with the flux as the variable as

$$H = \frac{P_\Phi^2}{2M_\Phi} + \frac{1}{2}M_\Phi\omega^2\Phi^2$$

Find M_Φ and $P_\Phi = M_\Phi\, d\Phi/dt$ and show that $P_\Phi = Q$, where Q is the charge on the capacitor.

(c) Express Q and Φ in terms of creation and annihilation operators.

(d) Show that the uncertainty principle can be written as

$$\Delta Q\,\Delta\Phi \geq \frac{\hbar}{2}$$

(e) Capacitors of niobium can be made 100nm × 100nm and have a specific capacitance of about 50 fF/μm^2. The circuit loop can be about 5 μm in diameter p. Assume $L \approx \mu_o p$. At what temperature would you have to operate to see the circuit behave quantum mechanically. (Note niobium is a superconductor and the resistance can be made negligibley small at low temperatures.)

The following two problems are short demonstrations of the gaussian wave packet in the simple harmonic oscillator potential.

Problem 5.3

Harmonic oscillator with MATLAB. Copy the the MATLAB programs `sho.m` and `herm.m` from the directory `/mit/6.728/matlab` to your local directory. This program places a gaussian wave packet in a simple harmonic oscillator potential. You can control the width of the initial packet as well as its displacement. The potential is such that the first eigenstate (which is gaussian) has a width of 1nm.

1. Define our initial wave packet width, $L = 1\,\text{nm}$. Let the wave packet be initially centered in the potential. Describe and explain the time evolution. What would a plot of $|C_n|^2$ vs. n, where C_n are the coefficients of the eigenfunction expansion look like?

2. Now, consider our initial gaussian wave packet to be a width of 1nm, and initially displaced 2nm to the right of the well. What, in general terms, will happen to the plot of $|C_n|^2$ vs. n?

3. Plot $\langle x \rangle$ vs. t. Graphically determine the frequency of oscillation. Compare to the results obtained by using Ehrenfest's Theorem.

4. Plot $(\Delta x)^2$ vs. t. Comment on and explain this plot.

Problem 5.4 Now, we want to consider a gaussian wave packet whose width is wider than the first eigenfunction of the system. Define our initial wave packet width, $L = 1.5\,\text{nm}$. Let the initial center of the wavepacket be displaced 3nm to the right of center.

1. Argue what the time evolution of the packet should look like. Explain your reasoning.

2. Plot $\langle x \rangle$ vs. t. Graphically determine the frequency of oscillation. Compare to the results obtained by using Ehrenfest's Theorem, and that of the previous problem . Comment.

3. Plot $(\Delta x)^2$ vs. t. Comment on this plot. Explain any differences from the corresponding plot in problem 7.5.

4. Plot 2-D `surf` plot of the probability density as the x vs. t graph. Can you identify the effects shown in parts 2 and 3 in this graph?