6.728 Applied Quantum and Statistical Physics

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PROBLEM SET 9

Issued: 11-08-02 Due: 11-13-02, in-class

Problem 9.1

A two-level system is coupled to a bath of simple harmonic oscillators. This situation is described by the Hamiltonian

$$\hat{H} \;=\; \sum_{\mathbf{k}} \hbar\omega(\mathbf{k})(\hat{a}^{\dagger}_{\mathbf{k}}\hat{a}_{\mathbf{k}} + \frac{1}{2}) \;+\; \sum_{n=1}^{2} |\phi_n\rangle E_n \langle\phi_n| \;+\; \sum_{\mathbf{k}} V_0(\hat{a}^{\dagger}_{\mathbf{k}} + \hat{a}_{\mathbf{k}})\Big[|\phi_1\rangle\langle\phi_2| \;+\; |\phi_2\rangle\langle\phi_1|\Big]$$

Question: What is the decay rate for a particle that is initially in ϕ_2 and decays to ϕ_1 (by exciting the oscillators)? Assume that the wavenumbers satisfy

$$k_x L \;=\; \pi n_1$$

$$k_y L \;=\; \pi n_2$$

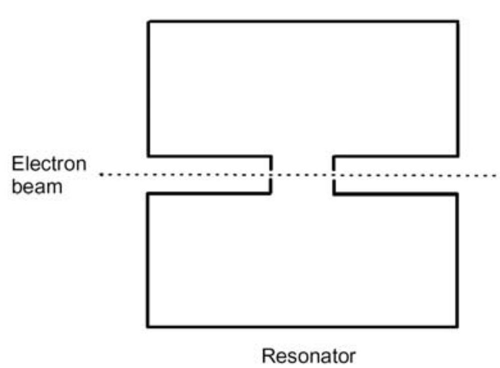
$$k_z L \;=\; \pi n_3$$

and that $\omega(\mathbf{k})$ is arbitrary.

Problem 9.2

Many years ago a microwave oscillator termed the Monotron was developed and studied. This oscillator consists of a microwave cavity through which an electron beam is passed. The electric field of the cavity interacts with the electrons, and energy is transferred from the beam to the oscillator if the electrons are timed correctly relative to the field.

We consider here a quantum version of this device. We consider a cavity and beam configuration as indicated below. We are interested in the electron interaction with a cavity mode at frequency ω_0 that has a constant electric field aligned with the beam direction over a distance L. The initial electron momentum is $\hbar k_i$.



Schematic of the quantum Monotron.

The relevant coupled Hamiltonian for this problem is

$$\hat{H} \;=\; \hat{H}_{EM} \;+\; \hat{H}_e \;+\; \hat{V}_{int}$$

where the electromagnetic field Hamiltonian for a single mode is

$$\hat{H}_{EM} \;=\; \hbar\omega_0 \; (\hat{a}^\dagger \hat{a} + \frac{1}{2})$$

The Hamiltonian for the free particle is

$$\hat{H} \;=\; -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}$$

and the interaction Hamiltonian can be modeled through

$$\hat{V}_{int} \;=\; -\, qx\hat{E}$$

The resulting coupled Hamiltonian is of the form

$$\hat{H} \;=\; \hbar\omega_0 \; (\hat{a}^\dagger \hat{a} + \frac{1}{2}) \;-\; \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} \;+\; V_0(\hat{a} + \hat{a}^\dagger)x$$

valid within the cavity.

(a) What is the final electron momentum and energy if a photon is created in the cavity? Is the total energy conserved?

(b) We consider a two-level model for this problem, which we expect to be valid if the coupling is weak. The wavefunction for the coupled electron plus cavity is approximately

$$\Psi \;=\; c_1(t)|k_i, 0\rangle \;+\; c_2(t)|k_f, 1\rangle$$

Under the presumption that the electron is in the cavity, we can model the electron states as

$$|k_i\rangle \;=\; \frac{e^{ik_i x}}{\sqrt{L}} \qquad\qquad |k_f\rangle \;=\; \frac{e^{ik_f x}}{\sqrt{L}}$$

Using this information, develop a two-level model of the form

$$i\frac{d}{dt}\begin{pmatrix} c_1(t) \\ c_2(t) \end{pmatrix} \;=\; \begin{pmatrix} H_1 & V_{12} \\ V_{21} & H_2 \end{pmatrix}\begin{pmatrix} c_1(t) \\ c_2(t) \end{pmatrix} \tag{1}$$

(c) Find the energy eigenvalues for the two-level model. Note that energy conservation has led to $H_1 = H_2$, so that the result will be simple in form.

(d) Thinking classically, we know that the electron will be in the cavity for a time t consistent with

$$vt \;=\; \frac{p}{m}t \;=\; \frac{\hbar k}{m}t \;=\; d$$

Estimate the probability that the electron will have created a photon during this time.

(e) What initial electron momentum optimizes this probability?

Problem 9.3

The quantum description of the coupling between matter and the transverse part of the radiation field is given in terms of an interaction Hamiltonian of the form

$$\hat{H}_{int} \;=\; -\int \hat{\mathbf{J}}(\mathbf{r}) \cdot \hat{\mathbf{A}}(\mathbf{r}) d^3\mathbf{r}$$

where $\hat{J}$ is the current operator, and where the vector potential operator $\hat{\mathbf{A}}$ is given by

$$\hat{\mathbf{A}}(\mathbf{r}) \;=\; -i \sum_{\sigma} \sum_{\mathbf{k}} \mathbf{i}_{\sigma} \sqrt{\frac{\mu_0 \hbar c}{2\epsilon_0 |\mathbf{k}| L^3}} \left[\hat{a}_{\mathbf{k},\sigma}\, e^{i\mathbf{k}\cdot\mathbf{r}} \;-\; \hat{a}^{\dagger}_{\mathbf{k},\sigma}\, e^{-i\mathbf{k}\cdot\mathbf{r}} \right]$$

We recall that the electric field operator is given by

$$\hat{\mathbf{E}}(\mathbf{r}) \;=\; \sum_{\sigma} \sum_{\mathbf{k}} \mathbf{i}_{\sigma} \sqrt{\frac{\hbar \omega_{\mathbf{k},\sigma}}{2\epsilon_0 L^3}} \left[\hat{a}_{\mathbf{k},\sigma}\, e^{i\mathbf{k}\cdot\mathbf{r}} \;+\; \hat{a}^{\dagger}_{\mathbf{k},\sigma}\, e^{-i\mathbf{k}\cdot\mathbf{r}} \right]$$

If the matter is treated as being classical, then the resulting radiation Hamiltonian is of the form

$$\hat{H} \;=\; \int \frac{1}{2}\epsilon_0 |\hat{\mathbf{E}}(\mathbf{r})|^2 \;+\; \frac{1}{2}\mu_0 |\hat{\mathbf{H}}(\mathbf{r})|^2 \, d^3\mathbf{r} - \int \mathbf{J}(\mathbf{r},t) \cdot \hat{\mathbf{A}}(\mathbf{r})\, d^3\mathbf{r}$$

(a) Show that if the radiation field is initially in the ground state, that all radiation fields generated by a classical source are classical states.

(b) Find evolution equations for $\langle e_{\mathbf{k},\sigma} \rangle$ and $\langle h_{\mathbf{k},\sigma} \rangle$.

(c) If the radiation field is initially in the ground state, can a classical source ever generate squeezed states?