Trigonometric and Hyperbolic Identities

$$\sin x = \frac{1}{2i}\left(e^{ix} - e^{-ix}\right) = -i\sinh(ix) \qquad \sinh x = \frac{1}{2}\left(e^{x} - e^{-x}\right) = -i\sin(ix)$$

$$\cos x = \frac{1}{2}\left(e^{ix} + e^{-ix}\right) = \cosh(ix) \qquad \cosh x = \frac{1}{2}\left(e^{x} + e^{-x}\right) = \cos(ix)$$

$$\sin(x \pm y) = \sin x\cos y \pm \cos x \sin y \qquad \sinh(x \pm y) = \sinh x\cosh y \pm \cosh x \sinh y$$

$$\cos(x \pm y) = \cos x\cos y \mp \sin x \sin y \qquad \cosh(x \pm y) = \cosh x\cosh y \pm \sinh x \sinh y$$

$$\sin x \pm \sin y = 2\sin\frac{x \pm y}{2}\cos\frac{x \mp y}{2} \qquad \sinh x \pm \sinh y = 2\sinh\frac{x \pm y}{2}\cosh\frac{x \mp y}{2}$$

$$\cos x + \cos y = 2\cos\frac{x+y}{2}\cos\frac{x-y}{2} \qquad \cosh x + \cosh y = 2\cosh\frac{x+y}{2}\cosh\frac{x-y}{2}$$

$$\cos x - \cos y = -2\sin\frac{x+y}{2}\sin\frac{x-y}{2} \qquad \cosh x - \cosh y = 2\sinh\frac{x+y}{2}\sinh\frac{x-y}{2}$$

$$\sin x\cos y = \frac{1}{2}\left[\sin(x+y) + \sin(x-y)\right] \qquad \sinh x\cosh y = \frac{1}{2}\left[\sinh(x+y) + \sinh(x-y)\right]$$

$$\cos x\cos y = \frac{1}{2}\left[\cos(x+y) + \cos(x-y)\right] \qquad \cosh x\cosh y = \frac{1}{2}\left[\cosh(x+y) + \cosh(x-y)\right]$$

$$\sin x\sin y = -\frac{1}{2}\left[\cos(x+y) - \cos(x-y)\right] \qquad \sinh x\sinh y = \frac{1}{2}\left[\cosh(x+y) - \cosh(x-y)\right]$$

$$\sin(2x) = 2\sin x\cos x \qquad \sinh(2x) = 2\sinh x\cosh x$$

$$\cos(2x) = \cos^2 x - \sin^2 x \qquad \cosh(2x) = \cosh^2 x + \sinh^2 x$$

$$\sin^2 x = \frac{1}{2}\left[1 - \cos(2x)\right] \qquad \sinh^2 x = \frac{1}{2}\left[\cosh(2x) - 1\right]$$

$$\cos^2 x = \frac{1}{2}\left[1 + \cos(2x)\right] \qquad \cosh^2 x = \frac{1}{2}\left[\cosh(2x) + 1\right]$$

$$\sin\left(\frac{\pi}{2} - x\right) = \cos x \qquad \sin(\pi - x) = \sin x \qquad \cos(\pi - x) = -\cos x$$

Useful Integrals

$$\int x\cos x dx = \cos x + x\sin x$$

$$\int x\sin x dx = \sin x - x\cos x$$

$$\int_{-\infty}^{\infty}\frac{\sin x}{x}dx = \pi$$

$$\int_0^L \begin{matrix}\cos^2\\ \sin^2\end{matrix}\left(\frac{n\pi}{L}x\right)dx = \frac{L}{2} \qquad n = 1, 2, 3, \ldots$$

Gaussians

$$\phi(x) = (\pi L^2)^{-1/4} e^{-x^2/2L^2}$$

$$\int_{-\infty}^{\infty}\phi^2(x)dx = 1$$

$$\int_0^{\infty} x\phi^2(x)dx = \frac{L}{2\sqrt{\pi}}$$

$$\int_{-\infty}^{\infty} x^2\phi^2(x)dx = \frac{L^2}{2}$$

Fourier Transforms

$\psi(x) = \int_{-\infty}^{\infty} A(k) e^{ikx} \frac{dk}{2\pi}$	$A(k) = \int_{-\infty}^{\infty} \psi(x) e^{-ikx} dx$
$\psi(x)$	$A(k)$
$\phi(x)$	$B(k)$
$a\psi(x) + b\phi(x)$	$aA(k) + bB(k)$
$\psi(x - x_0)$	$e^{-ikx_0} A(k)$
$e^{ik_0 x}\psi(x)$	$A(k - k_0)$
$\psi^*(x)$	$A^*(-k)$
$\psi(-x)$	$A(-k)$
$\psi(ax)$	$\frac{1}{\lvert a\rvert} A\left(\frac{k}{a}\right)$
$\psi(x) * \phi(x)$	$A(k)B(k)$
$\psi(x)\phi(x)$	$A(k) * B(k)$ [See below]
$\frac{d}{dx}\psi(x)$	$ikA(k)$
$\int_{-\infty}^{x} \psi(x')dx'$	$\frac{1}{ik}A(k) + \pi A(0)\delta(k)$
$x\psi(x)$	$i\frac{d}{dk}A(k)$
$A(x)$	$2\pi\psi(-k)$

$\delta(x)$	1
$u(x)$	$\frac{1}{ik} + \pi\delta(k)$
$\delta(x - x_0)$	e^{-ikx_0}
$e^{-x^2/2L^2}$	$\sqrt{2\pi L^2}\, e^{-L^2 x^2/2}$
$\psi(x) = \begin{cases} 1\,, \lvert x\rvert < L \\ 0\,, \lvert x\rvert > L \end{cases}$	$2\,\frac{\sin kL}{k}$
$e^{-\alpha\lvert x\rvert}$	$\frac{2\alpha}{\alpha^2 + k^2}$
sech x	$\sqrt{\frac{\pi}{2}}\,\mathrm{sech}\left(\frac{\pi}{2}k\right)$

$$\int_{-\infty}^{\infty} \psi^*(x)\phi(x)dx = \int_{-\infty}^{\infty} A^*(k)B(k)\frac{dk}{2\pi}$$ Parseval's Theorem

$$A(k) * B(k) = \int_{-\infty}^{\infty} A(k')B(k - k')\frac{dk}{2\pi}$$ Convolution in k-domain

Fundamental Constants

Speed of light	$c = 2.998 \times 10^8$ m/s
Electron charge	$e = 1.602 \times 10^{-19}$ C
Electron mass	$m_e = 9.109 \times 10^{-31}$ kg
Proton mass	$m_p = 1.672 \times 10^{-27}$ kg
Planck's constant	$\hbar = 1.055 \times 10^{-34}$ Js
Boltzmann's constant	$k_B = 1.381 \times 10^{-23}$ J/K
Bohr rad. $a_0 = \dfrac{4\pi\epsilon_0\hbar^2}{m_e e^2}$	$a_0 = 0.529$ Å
Rydberg $I_H = \dfrac{\hbar^2}{2m_e a_0^2}$	$I_H = 13.61$ eV

Particles and Waves

$$E = \sqrt{(pc)^2 + (m_0 c^2)^2}$$

$$E = \hbar\omega$$

$$p = \hbar k$$

Photon: $E = pc$

Electron: $E = \dfrac{p^2}{2m}$

Group velocity: $v(k) = \dfrac{d\omega}{dk}$

Compton scattering: $\Delta\lambda = \dfrac{h}{mc}(1 - \cos\theta)$

Heisenberg Uncertainty

$$\Delta x \Delta p \geq \frac{\hbar}{2} \qquad \Delta E \Delta t \geq \frac{\hbar}{2} \qquad \text{etc.}$$

Operators

$$\hat{x} = x = i\frac{\partial}{\partial k}$$

$$\hat{p} = \hbar k = -i\hbar\frac{\partial}{\partial x}$$

$$\hat{E} = i\hbar\frac{\partial}{\partial t}$$

$$\hat{H} = -\frac{\hbar^2}{2m}\cdot\frac{\partial^2}{\partial x^2} + V(\hat{x})$$

More Operators

$$[\hat{A}, \hat{B}] \equiv \hat{A}\hat{B} - \hat{B}\hat{A}$$

$$[\hat{x}, \hat{p}] = i\hbar$$

$$[\hat{x}, \hat{H}] = \frac{\hbar^2}{m}\cdot\frac{\partial}{\partial x}$$

$$[\hat{p}, \hat{H}] = -i\hbar\frac{\partial\hat{V}}{\partial x}$$

$$|\phi(x)\rangle \equiv \phi(x)$$

$$\langle\phi(x)| \equiv \int_{-\infty}^{\infty} dx\,\phi^*(x)\,\text{ᗧ} \cdots$$

$$\langle\phi(x)|\,\hat{O}\,|\psi(x)\rangle \equiv \int_{-\infty}^{\infty} dx\,\phi^*(x)\hat{O}\psi(x)$$

Ehrenfest Theorem

$$\frac{d}{dt}\left\langle\hat{Q}\right\rangle = \left\langle\frac{\partial\hat{Q}}{\partial t}\right\rangle + \frac{1}{i\hbar}\left\langle[\hat{Q}, \hat{H}]\right\rangle$$

Schrödinger Equation

$$\hat{E}\psi = \hat{H}\psi$$

$$i\hbar\frac{\partial}{\partial t}\psi(x,t) = -\frac{\hbar^2}{2m}\cdot\frac{\partial^2}{\partial x^2}\psi(x,t) + V(x)\psi(x,t)$$

Time independent:

$$E\psi(x) = -\frac{\hbar^2}{2m}\cdot\frac{\partial^2}{\partial x^2}\psi(x) + V(x)\psi(x)$$

Time evolution of eigenfunctions of $\hat{H}$:

$$\phi_n(x,t) = \phi_n(x,0)e^{-(E_n/\hbar)t}$$

Eigenfunction Expansion

[Assiming $\phi_n(x)$ constitute a complete orthonormal set in Hilbert space, which they would, if they are eigenfunctions of a Hermitian operator.]

$$\psi(x) = \sum\Big/\!\!\int a_n\phi_n(x)$$

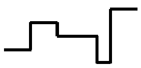
$$a_n = \langle\phi_n(x)|\psi(x)\rangle$$

Special Case Solutions to the Schrödinger Equation

Free Space $[V(x) = 0]$

$$\phi_k^+(x) = e^{ikx} \quad \phi_k^-(x) = e^{-ikx}$$

Piece-wise Linear Potentials

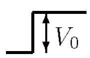
$V(x) =$ 

In every region:

$$k = \sqrt{\frac{2m(E-V)}{\hbar^2}} \qquad \psi(x) = c_1 e^{ikx} + c_2 e^{-ikx}$$

At region boundaries:

$$\psi_1(x_n) = \psi_2(x_n) \qquad \left.\frac{\partial \psi_1}{\partial x}\right|_{x_n} = \left.\frac{\partial \psi_2}{\partial x}\right|_{x_n}$$

Single step: 

$$R = \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2} \qquad T = \frac{4k_1 k_2}{(k_1 + k_2)^2}$$

Infinite Square Well

$$V(x) = \begin{cases} 0 & , \; 0 < x < L \\ \infty & , \; x < 0, x > L \end{cases}$$

$$\phi_n(x) = \sqrt{\frac{2}{L}} \sin \frac{n\pi}{L} x \qquad E_n = \frac{\hbar^2 n^2 \pi^2}{2mL^2}$$

Simple Harmonic Oscillator

$$V(x) = \frac{1}{2} m \omega_0^2 x^2$$

Energy: $E_n = \hbar\omega_0 \left(n + \frac{1}{2}\right)$

Ground state:

$$\Delta x = \sqrt{\frac{\hbar}{2m\omega_0}} \qquad \Delta p = \sqrt{\frac{\hbar m \omega_0}{2}}$$

SHO Eigenstates:

$$\psi_n(x) = \left(\frac{m\omega_0}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{2^n n!}} \exp\left(-\frac{m\omega_0}{2\hbar} x^2\right) H_n\left(\sqrt{\frac{m\omega_0}{\hbar}}\, x\right)$$

$H_0(y) = 1, \quad H_1(y) = y, \quad H_2(y) = 1 - 2y^2, \quad$ etc.

SHO Operators

$$\hat{a} \equiv \left(\sqrt{\frac{m\omega_0}{2\hbar}}\, x + \sqrt{\frac{\hbar}{2m\omega_0}} \cdot \frac{\partial}{\partial x}\right)$$

$$\hat{a}^\dagger \equiv \left(\sqrt{\frac{m\omega_0}{2\hbar}}\, x - \sqrt{\frac{\hbar}{2m\omega_0}} \cdot \frac{\partial}{\partial x}\right)$$

$$\hat{a}\psi_n(x) = \sqrt{n}\, \psi_{n-1}(x) \qquad \hat{a}^\dagger \psi_n(x) = \sqrt{n+1}\, \psi_{n+1}(x)$$

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega_0}} (\hat{a}^\dagger + \hat{a}) \qquad \hat{p} = i\sqrt{\frac{\hbar m \omega_0}{2}} (\hat{a}^\dagger - \hat{a})$$

$$\hat{H} = \hbar\omega_0 \left(\hat{a}^\dagger \hat{a} + \frac{1}{2}\right) \qquad [\hat{a}, \hat{a}^\dagger] = 1$$

Coupled Systems

Let $\quad \psi(x,t) = \sum_n c_n(t) \phi_n(x) \quad$ Then:

$$i\hbar \frac{d}{dt} c_n(t) = \sum_m c_m(t) \langle \phi_n(x) | \hat{H} | \phi_m(x) \rangle$$

LC Circuit Quantization

$$i\hbar \frac{\partial}{\partial t} \psi(v,t) = -\frac{\hbar^2 \omega_0^2}{2C} \cdot \frac{\partial^2}{\partial v^2} \psi(v,t) + \frac{1}{2} C v^2 \psi(v,t)$$

$$\hat{v} = v \qquad \hat{\imath} = -i\hbar \frac{\omega_0}{\sqrt{LC}} \cdot \frac{\partial}{\partial v}$$

$$\hat{a} \equiv \left(\sqrt{\frac{C}{2\hbar\omega_0}}\, v + \sqrt{\frac{\hbar\omega_0}{2C}} \cdot \frac{\partial}{\partial v}\right)$$

$$\hat{a}^\dagger \equiv \left(\sqrt{\frac{C}{2\hbar\omega_0}}\, v - \sqrt{\frac{\hbar\omega_0}{2C}} \cdot \frac{\partial}{\partial v}\right)$$

$$\hat{v} = \sqrt{\frac{\hbar\omega_0}{2C}} (\hat{a}^\dagger + \hat{a}) \qquad \hat{\imath} = i\sqrt{\frac{\hbar\omega_0}{2L}} (\hat{a}^\dagger - \hat{a})$$

EM Field Quantization

Cubic box $L \times L \times L$. Use the LC circuit results after making the following substitutions:

$$\hat{v} \to \hat{e} \qquad \hat{\imath} \to \hat{h}$$

$$C \to \epsilon_0 L^3 \qquad L \to \mu_0 L^3$$