Problem 10.1

The following succinct solution is provided by Leo and Kevin. According to Virial’s theorem, the capacitor energy is half the total energy, which is given in Eq. (26.40) in HSO:

\[
\frac{1}{2} C \Delta v^2 = \frac{1}{2} \langle E \rangle = \frac{\hbar \omega_0}{2} \left( \frac{1}{e^{\hbar \omega_0/k_B T} - 1} + \frac{1}{2} \right)
\]

Therefore

\[
\Delta v = \sqrt{\frac{\hbar \omega_0}{C} \left( \frac{1}{e^{\hbar \omega_0/k_B T} - 1} + \frac{1}{2} \right)}
\]

Similarly for the current,

\[
\Delta i = \sqrt{\frac{\hbar \omega_0}{L} \left( \frac{1}{e^{\hbar \omega_0/k_B T} - 1} + \frac{1}{2} \right)}
\]

Problem 10.2

(a) The number of states for which the reservoir has one unit of excitation is

\[
\Omega_1 = C_1^3 = 3
\]

The number of states for which the reservoir has zero unit of excitation is

\[
\Omega_0 = C_0^3 = 1
\]

Therefore we obtain the excitation probabilities for the oscillator

\[
p_0 = \frac{\Omega_1}{\Omega_0 + \Omega_1} = \frac{3}{4}
\]
\[ p_1 = \frac{\Omega_0}{\Omega_0 + \Omega_1} = \frac{1}{4} \]

(b) The number of states for which the reservoir has zero unit of excitation is

\[ \Omega_0 = C_7^0 = 1 \]

The number of states for which the reservoir has one unit of excitation is

\[ \Omega_1 = C_7^1 = 7 \]

The number of states for which the reservoir has two units of excitation is

\[ \Omega_2 = C_7^2 = 21 \]

Therefore we obtain the excitation probabilities for the oscillator

\[ p_0 = \frac{\Omega_0}{\Omega_0 + \Omega_1 + \Omega_2} = \frac{21}{29} \]

\[ p_1 = \frac{\Omega_1}{\Omega_0 + \Omega_1 + \Omega_2} = \frac{7}{29} \]

\[ p_2 = \frac{\Omega_2}{\Omega_0 + \Omega_1 + \Omega_2} = \frac{1}{29} \]

(c) We use the approach outline in part (d). For the situation in part (a),

\[ \frac{1}{T} \approx \frac{k_B}{\hbar \omega_0} (\ln \Omega_1 - \ln \Omega_0) = \frac{k_B}{\hbar \omega_0} \ln \frac{3}{4} = \frac{k_B \ln 3}{\hbar \omega_0} \]

For the situation in part (b),

\[ \frac{1}{T} \approx \frac{k_B}{\hbar \omega_0} (\ln \Omega_2 - \ln \Omega_1) = \frac{k_B}{\hbar \omega_0} \ln \frac{21}{7} = \frac{k_B \ln 3}{\hbar \omega_0} \]

(d) For a reservoir of \(4m-1\) two-level systems and \(E = m\hbar \omega_0\) total energy, the number of states for which the oscillator is in the ground state is equal to the number of states for which the reservoir has \(m\) units of excitation:

\[ \Omega_m = C_m^{4m-1} = \frac{(4m-1)}{m (3m-1)!} \]

The number of states for which the oscillator is in the first excited state is equal to the number of states for which the reservoir has \(m - 1\) units of excitation:
\[ \Omega_{m-1} = C_{m-1}^{4m-1} = \frac{(4m-1)!}{(m-1)!(3m)!} \]

Therefore the temperature estimate is

\[ \frac{1}{T} \approx \frac{k_B}{\hbar \omega_0} \ln \frac{\Omega_m}{\Omega_{m-1}} = \frac{k_B}{\hbar \omega_0} \ln \frac{3m}{m} = \frac{k_B}{\hbar \omega_0} \ln 3 \]

\[ T \approx \frac{\hbar \omega_0}{k_B \ln 3} \tag{1} \]

(e) We have shown that the value of \( p_1/p_0 \) is 1/3 for the 3 two-level systems, 7 two-level systems, \( 4m-1 \) two-level systems cases. For the infinite reservoir case, we note that Eq.(26.30) in HSO gives

\[ p_n = e^{-n\hbar \omega_0/k_B T} \left( 1 - e^{-\hbar \omega_0/k_B T} \right) \]

Hence we obtain

\[ \frac{p_1}{p_0} = e^{-\hbar \omega_0/k_B T} \]

The above expression is different from that for the finite two-level systems cases, but in fact if we substitute the approximation for the temperature \( T \) eq.(1) we obtain

\[ e^{-\hbar \omega_0/k_B T} \approx \frac{1}{3} \]

which is consistent with the finite two-level systems cases. For the value of \( p_2/p_1 \), we use the results in part (a) and part (b) to get

\[ \frac{p_2}{p_1} = \begin{cases} 0 & \text{3 two-level systems} \\ \frac{1}{7} & \text{7 two-level systems} \end{cases} \]

For the \( 4m - 1 \) two-level systems case

\[ \frac{p_2}{p_1} \frac{\Omega_{m-2}}{\Omega_{m-1}} = \frac{C_{m-2}^{4m-1}}{C_{m-1}^{4m-1}} = \frac{m - 1}{3m + 1} \]

In the limit of large \( m \), the above expression reduces to 1/3 as can be seen from the following table
\[
\begin{array}{c|c}
\text{Ratio} & \text{Infinite Reservoir} \\
\hline
p_1/p_0 & e^{-\hbar \omega_0/k_B T} \\
p_2/p_1 & e^{-\hbar \omega_0/k_B T}
\end{array}
\]

**Problem 10.3**

(a) From problem 27.2 in HSO we know the bandgap of diamond is

\[ E_G = 5.5eV \]
The conduction band edge density of states and the valence band edge density of states are

\[ N_C(T), N_V(T) \sim 10^{10} \left( \frac{T}{300K} \right)^{3/2} \text{cm}^{-3} \]

There are two types of dopants considered in this problem, \( N \) and \( P \). The donor energy for \( N \) is 1.7eV below the conduction band.

\[ E_C - E_D = 1.7\text{eV} \]

The donor energy for \( P \) is 0.2eV below the conduction band.

\[ E_C - E_D = 0.2\text{eV} \]

In this problem we are assuming that the dopant concentration is

\[ N_D = 10^{15}\text{cm}^{-3} \]

The parameters for silicon are as follows. From the caption of Fig. 27.6 in HSO we know that

\[ E_G = 1.12\text{eV} \]

Michael provided the silicon conduction and the valence band edge densities of states:

\[ N_C(T) = 2.86 \times 10^{19} \left( \frac{T}{300} \right)^{3/2} \]

\[ N_V(T) = 3.1 \times 10^{19} \left( \frac{T}{300} \right)^{3/2} \]

The silicon example done in the text has \( P \) doping with donor energy 0.045eV below the conduction band

\[ E_C - E_D = 0.045\text{eV} \]

The doping level is

\[ N_D = 10^{15}\text{cm}^{-3} \]

In this part, a function script `ps10p3parta.m` is written to be called in follow-on parts of the problem.

```
% start of ps10p3parta.m
function output = ps10p3parta(mu)
```

5
global Nc
global Nv
global Eg
global ND
global kB
global T
global ED

\[ p = \frac{N_v}{\exp(\mu/k_B/T)+1}; \]
\[ n = \frac{N_c}{\exp((E_g-\mu)/k_B/T)+1}; \]

\[ NDp = \frac{ND}{2+\exp((\mu-ED)/kB/T)}; \]

\[ \text{output} = p + NDp - n; \]

\%\%\% end of ps10p3parta.m \%\%\%\%\%\%

(b) The file ps10p3partb.m calls ps10p3parta.m with parameters for silicon. The computed Fermi level as a function of temperature is as follows

![Silicon chemical potential graph](image)

The \( n \) and \( p \) carrier concentrations are plotted below:
clear;
clear global;
global Nc
global Nv
global Eg
global ND
global kB
global T
global ED

kB=8.617342e-5; %eV/K

Eg = 1.12; %eV
ED= Eg-0.045; %eV

ND=10^15; % cm^-3

% room temperature Nc for silicon
Nc_room = 2.86e19; % cm^-3
Nv\_room = 3.1e19; \% cm\(^{-3}\)

Taxis=linspace(10,700,100);

for count=1:length(Taxis)
    T=Taxis(count);
    Nc = Nc\_room*(T/300)\(^{(3/2)}\);
    Nv = Nv\_room*(T/300)\(^{(3/2)}\);
    mu(count)=fzero('ps10p3parta',[0 Eg]);
    n(count) = Nc/(exp((Eg-mu(count))/kB/T)+1);
    p(count) = Nv/(exp(mu(count)/kB/T)+1);
end

figure(1);
plot(Taxis,mu);
xlabel('T in Kelvin');
ylabel('\mu(T) in eV');
title('Silicon chemical potential');

figure(2);
semilogy(Taxis,n,Taxis,p,'--');
axis([0 700 10e12 10e15]);
xlabel('T(K)');
ylabel('carrier concentration (cm\(^{-3}\))');
title('P-doped Si characteristics');
legend('n','p');

(c) Using parameters for N-doped diamond we obtain
(d) Using parameters for P-doped diamond we obtain
(e) \( P \) doping has a higher carrier concentration and also wider operating range about 400 to 2300 Kelvin and therefore it is a better impurity.

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**Problem 10.4**

(a) The Hamiltonian is separable into an SHO in the \( x \), \( y \), and \( z \) directions. The first three eigen-states to an SHO is

\[
\phi_0(x) = \frac{1}{\pi^{1/4}} e^{-x^2/2}
\]

\[
\phi_1(x) = \frac{2}{\pi^{1/4}} xe^{-x^2/2}
\]

\[
\phi_2(x) = -\frac{1}{\sqrt{2\pi^{1/4}}} e^{-x^2/2}(1 - 2x^2)
\]
States with two quanta of excitation are states with two quanta of excitation in one direction or states with one quanta of excitation in two of the three directions. Hence the six eigen-states are

\[
\Phi_1 = \phi_2(x)\phi_0(y)\phi_0(z) = -\frac{1}{\sqrt{2\pi^{3/4}}} e^{-r^2/2}(1 - 2x^2)
\]

\[
\Phi_2 = \phi_0(x)\phi_2(y)\phi_0(z) = -\frac{1}{\sqrt{2\pi^{3/4}}} e^{-r^2/2}(1 - 2y^2)
\]

\[
\Phi_3 = \phi_0(x)\phi_0(y)\phi_2(z) = -\frac{1}{\sqrt{2\pi^{3/4}}} e^{-r^2/2}(1 - 2z^2)
\]

\[
\Phi_4 = \phi_1(x)\phi_1(y)\phi_0(z) = \frac{4}{\pi^{3/4}} e^{-r^2/2}xy
\]

\[
\Phi_5 = \phi_1(x)\phi_0(y)\phi_1(z) = \frac{4}{\pi^{3/4}} e^{-r^2/2}xz
\]

\[
\Phi_6 = \phi_0(x)\phi_1(y)\phi_1(z) = \frac{4}{\pi^{3/4}} e^{-r^2/2}yz
\]

(b) We need to come up with six eigen-states of the form \(R_{nl}Y_{lm}(\theta, \phi)\). We construct these states via linear superposition of the states in part (a). See Table 29.2 in HSO for the spherical harmonics involved:

\[
Y_{0,0} = \frac{1}{\sqrt{4\pi}}
\]

\[
Y_{2,0} = \sqrt{\frac{5}{16\pi}} \frac{2z^2 - x^2 - y^2}{r^2}
\]

\[
Y_{2,\pm 1} = \mp \sqrt{\frac{15}{8\pi}} \frac{z(x \pm iy)}{r^2}
\]

\[
Y_{2,\pm 2} = \sqrt{\frac{15}{32\pi}} \frac{(x \pm iy)^2}{r^2} = \sqrt{\frac{15}{32\pi}} \frac{x^2 - y^2 \pm 2ixy}{r^2}
\]

The constructed states are

\[
\psi^1 = \frac{1}{\sqrt{3}} (\Phi_1 + \Phi^2 + \Phi^3) = -\sqrt{\frac{2}{3\pi^{1/4}}} e^{-r^2/2}(3 - 2r^2)Y_{0,0}
\]

\[
\psi^2 = \frac{1}{\sqrt{6}} (\Phi_1 + \Phi^2 - 2\Phi^3) = -\sqrt{\frac{16}{15\pi^{1/4}}} r^2 e^{-r^2/2}Y_{2,0}
\]
\[ \psi^{3,4} = \frac{1}{\sqrt{2}} (\Phi^5 \pm i \Phi^2) = \pm \frac{8}{\pi^{1/4}} \sqrt{15} r^2 e^{-r^2/2} Y_{2,\pm1} \]

\[ \psi^{5,6} = \frac{1}{\sqrt{3}} (\sqrt{2}(\Phi^2 - \Phi^1) \pm i \Phi^4) = -\frac{2}{\pi^{1/4}} \sqrt{\frac{32}{45}} r^2 e^{-r^2/2} Y_{2,\pm2} \]

(c) We’ve answered this part by the way we construct eigen-states in part (b).