Problem 5.1

We make use of the following equations for the SHO eigenfunctions

\[ \hat{a}^\dagger |\phi_n\rangle = \sqrt{n+1} |\phi_{n+1}\rangle \]  

(1)

\[ \hat{a} |\phi_n\rangle = \sqrt{n} |\phi_{n-1}\rangle \]  

(2)

and the representation of \( x \) in terms of the creation and annihilation operators

\[ x = \sqrt{\frac{\hbar}{2m\omega_0}} (\hat{a}^\dagger + \hat{a}) \]  

(3)

to give

\[ \langle \phi_n | x^4 | \phi_n \rangle = \left( \frac{\hbar}{2m\omega_0} \right)^2 \langle | (\hat{a}^\dagger + \hat{a})^3 (\sqrt{n+1} |\phi_{n+1}\rangle + \sqrt{n} |\phi_{n-1}\rangle) \rangle \]

\[ = \left( \frac{\hbar}{2m\omega_0} \right)^2 \langle \phi_n | (\hat{a}^\dagger + \hat{a}) \left[ \sqrt{(n+2)(n+1)} |\phi_{n+2}\rangle + n |\phi_n\rangle + (n+1) |\phi_{n+1}\rangle + \sqrt{n(n-1)} |\phi_{n-2}\rangle \right] \]

\[ = \left( \frac{\hbar}{2m\omega_0} \right)^2 \langle \phi_n | (\hat{a}^\dagger + \hat{a}) \left[ \sqrt{(n+3)(n+2)(n+1)} |\phi_{n+3}\rangle + n\sqrt{n+1} |\phi_{n+1}\rangle + (n+1) \sqrt{n+1} |\phi_{n+1}\rangle \right] \]

\[ + \left( \frac{\hbar}{2m\omega_0} \right)^2 \langle \phi_n | (\hat{a}^\dagger + \hat{a}) \left[ (n-1) \sqrt{n} |\phi_{n-1}\rangle + (n+1) \sqrt{n+1} |\phi_{n+1}\rangle \right] \]

\[ + \left( \frac{\hbar}{2m\omega_0} \right)^2 \langle \phi_n | (\hat{a}^\dagger + \hat{a}) \left[ (n-1) \sqrt{n} |\phi_{n-1}\rangle + (n+1) \sqrt{n} |\phi_{n-1}\rangle + \sqrt{n(n-1)(n-2)} |\phi_{n-3}\rangle \right] \]

Because eigenfunctions of the SHO are orthogonal to each other, we can simplify the expression to be

\[ \left( \frac{\hbar}{2m\omega_0} \right)^2 \langle \phi_n | \left[ n(n-1) + n^2 + n(n+1) + n(n+1) + (n+1)^2 + (n+1)(n+2) \right] |\phi_n\rangle \]
\[
\hat{H} = \left( \frac{\hbar}{2m\omega_0} \right)^2 (6n^2 + 6n + 3)
\]

**Problem 5.2**

(a) Making use of Eq. (10.14) in HSO we have

\[
\hat{H}_a = \hat{a}^\dagger \hat{a} + \hat{a} \hat{a}^\dagger = 2 \left( y^2 - \frac{d^2}{dy^2} \right).
\]

For the calculation of \(\hat{H}_b\):

\[
\hat{b}^\dagger \hat{b} f = \left( y^3 - \frac{d}{dy} \right) \left( y^3 + \frac{d}{dy} \right) f = \left( y^3 + \frac{d}{dy} \right) \left( y^3 f + \frac{df}{dy} \right) = y^6 f + y^3 \frac{d}{dy} f - 3y^2 f - y^2 \frac{d^2 f}{dy^2} - \frac{d^2}{dy^2} f
\]

(4)

\[
\hat{b} \hat{b}^\dagger f = \left( y^3 + \frac{d}{dy} \right) \left( y^3 - \frac{d}{dy} \right) f = \left( y^3 + \frac{d}{dy} \right) \left( y^3 f - \frac{df}{dy} \right) = y^6 f - y^3 \frac{d}{dy} f + 3y^2 f + y^2 \frac{df}{dy} - \frac{d^2}{dy^2} f
\]

(5)

Summing the above two equations gives

\[
\hat{H}_b = \hat{b}^\dagger \hat{b} + \hat{b} \hat{b}^\dagger = 2 \left[ y^6 - \frac{d^2}{dy^2} \right]
\]

(b) From Eq. (10.56) in HSO we have

\[
\left[ \hat{a}, \hat{a}^\dagger \right] = \left[ y + \frac{d}{dy}, y - \frac{d}{dy} \right] = 2
\]

Subtracting Eq.(4) and Eq.(5) we obtain

\[
\left[ \hat{b}, \hat{b}^\dagger \right] = \hat{b} \hat{b}^\dagger - \hat{b}^\dagger \hat{b} = 6y^2
\]

(c)

\[
\hat{H}_b \left[ \hat{b} \psi_n \right] = \left[ \hat{n}^\dagger \hat{b} + \hat{b} \hat{n} \right] \psi_n = \left[ \left( \hat{b} \hat{b}^\dagger - 6y^2 \right) \hat{b} + \hat{b} \hat{b}^\dagger \right] \psi_n
\]

\[
= \left[ \hat{b} \hat{b}^\dagger \hat{b} + \hat{b} \left( \hat{b} \hat{b}^\dagger - 6y^2 \right) - 6y^2 \hat{b} \right] \psi_n = \left[ \hat{b} \hat{b}^\dagger \hat{b} - 6y^2 \hat{b} \right] \psi_n - 6\hat{b} y^2 \psi_n
\]

\[
= \left[ \epsilon_n^{(b)} - 6y^2 \right] \hat{b} \psi_n - 6\hat{b} y^2 \psi_n \neq \text{constant} \left[ \hat{b} \psi_n (y) \right]
\]
Therefore, $\hat{b}$ is not an annihilation operator.

(d) $\left[ \hat{b}, \hat{H} \right] = \hat{b}\hat{H} - \hat{H}\hat{b} = \hat{b}\hat{b}^\dagger \hat{b} + \hat{b}\hat{b}^\dagger - \hat{b}\hat{b}^\dagger \hat{b} = \hat{b}\hat{b}^\dagger - \hat{b}\hat{b} = \hat{b} \left( \hat{b}^\dagger \hat{b} + 6y^2 \right) - \hat{b}\hat{b} = \hat{b}^\dagger \hat{b} - 6b^2 = \left( \hat{b}^\dagger \hat{b} + 6y^2 \right) \hat{b} - \hat{b}^\dagger \hat{b} \hat{b} - 6b^2 = 6y^2 \hat{b} + 6b^2$

From Ehrenfest’s theorem we have

$$\frac{d}{dt} \langle \hat{b} \rangle = \frac{6}{i\hbar} \left[ \langle y^2 \hat{b} \rangle + \langle \hat{y}^2 \rangle \right]$$

The above equation is not readily solvable, but a similar computation gives

$$\frac{d}{dt} \langle \hat{a} \rangle = \frac{4}{i\hbar} \langle \hat{a} \rangle$$

which can be used to solve for the time evolution of $\langle \hat{a} \rangle$.

**Problem 5.3**

(a) Since the wavefunction is even, we know $\langle x \rangle = 0$, and hence the variance is equal to (making use of Eq.(3))

$$\Delta x^2 = \langle \psi | x^2 | \psi \rangle = \frac{\hbar}{2m\omega_0} \left( \langle \hat{a}^\dagger + \hat{a} \rangle \right)^2 = \frac{\hbar}{2m\omega_0} \langle \psi | \left( \hat{a}^\dagger + \hat{a} \right)^2 \left[ c_0 | \psi_0 \rangle + c_2 | \psi_2 \rangle \right]$$

$$= \frac{\hbar}{2m\omega_0} \langle \psi | \left( \hat{a}^\dagger + \hat{a} \right) \left[ c_0 | \psi_1 \rangle + c_2 \sqrt{3} | \psi_3 \rangle + c_2 \sqrt{2} | \psi_1 \rangle \right]$$

$$= \frac{\hbar}{2m\omega_0} \langle \psi \left[ | c_0 \sqrt{2} | \psi_2 \rangle + 2 \sqrt{3} c_2 | \psi_4 \rangle + 2 c_2 | \psi_2 \rangle + c_0 | \psi_0 \rangle + 3 c_2 | \psi_2 \rangle + c_2 \sqrt{2} | \psi_0 \rangle \right]$$

$$= \frac{\hbar}{2m\omega_0} \left[ | c_0 \rangle^2 + c_0^* c_2 \sqrt{2} + c_2^* c_0 \sqrt{2} + 5 | c_2 \rangle^2 \right] = \frac{\hbar}{2m\omega_0} \left[ 1 + 4 | c_2 \rangle^2 + \sqrt{2} (c_0^* c_2 + c_2^* c_0) \right]$$

To minimize the above expression, we can make the following assumption:

$$c_2 = \sqrt{r}$$

$$c_0 = -\sqrt{1 - r}$$

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where \(0 \leq r \leq 1\). Essentially we need to minimize the following expression
\[
f(r) = 4r - 2\sqrt{2}\sqrt{r(1-r)}
\]

Note that \(f(r)\) is negative when
\[
16r^2 < 8r(1-r) \iff 24r^2 < 8r \iff r < \frac{1}{3}
\]

Taking the derivative of \(f(r)\) gives
\[
f'(r) = 4 - \frac{\sqrt{2}(1-2r)}{\sqrt{r-r^2}}
\]

Solving \(f'(r) = 0\) gives
\[
r = \frac{3 \pm \sqrt{6}}{6}
\]

However, we know only the solution that’s smaller than \(\frac{1}{3}\) is the one we are looking for. Thus
\[
r = \frac{3 - \sqrt{6}}{6} = 0.0918
\]

\[
c_2 = \sqrt{r} = 0.3030
\]

\[
c_0 = -\sqrt{1-|c_2|^2} = -0.9530
\]

\[
\Delta x = \sqrt{\frac{\hbar}{2m\omega_0} \left[ 1 + 4r - 2\sqrt{2}\sqrt{r-r^2} \right]} = 0.7420 \times \sqrt{\frac{\hbar}{2m\omega_0}}
\]

(b) The time evolution of the wavefunction is
\[
\psi(x, t) = c_0 e^{-i\frac{E_0}{\hbar}t} \psi_0(x) + c_2 e^{-i\frac{E_2}{\hbar}t} \psi_2(x) = e^{-i\frac{E_0}{\hbar}t} \left[ c_0 \psi(x) + c_2 e^{-i\frac{(E_2-E_0)}{\hbar}t} \psi_2(x) \right]
\]

Note that
\[
E_2 = \hbar\omega_0 \left( 2 + \frac{1}{2} \right)
\]
\[
E_0 = \frac{1}{2}\hbar\omega_0
\]
\[
\frac{E_2 - E_0}{\hbar} = 2\omega_0
\]
Since the overall phase factor does not matter, the time dependence of $\Delta x$ can be obtained from replacing $c_2$ with $c_2 e^{-i2\omega_0 t}$ in Eq.(6)

$$\Delta x = \sqrt{\frac{\hbar}{2m\omega_0}} \left[ 1 + 4|c_2|^2 + \sqrt{2}c_0 c_2 \cos(2\omega_0 t) \right]$$

The width of the wavefunction oscillates at a frequency $2\omega_0$.

(c) A strong uniform force makes this state swing back and forth at the SHO characteristic frequency $\omega_0$, which is separate from the with breathing motion. $\Delta x$ is not always minimized as the breathing motion suggests.

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Problem 5.4

(a) The inverted harmonic oscillator is mathematically equivalent to a simple harmonic oscillator with parameter $i\omega_0$ instead of $\omega_0$. Replacing $\omega_0$ with $i\omega_0$ in Eq.(11.3) and Eq.(11.4) gives

$$\frac{d}{dt} \langle \hat{x} \rangle = \frac{\langle \hat{p} \rangle}{m}$$

$$\frac{d}{dt} \langle \hat{p} \rangle = m\omega_0^2 \langle x \rangle$$

(b) The wave packet will spread because the potential in this case does not have the right curvature to hold the wave packet.

(c) Again, this problem is mathematically equivalent to the Gaussian squeezed state solutions introduced in HSO. Replacing $\omega_0$ with $i\omega_0$ in equations (11.78) through (11.81) in HSO gives

$$\frac{d}{dt} X(t) = \frac{P(t)}{m}$$

$$\frac{d}{dt} P(t) = m\omega_0^2 (t)$$

$$\hbar \frac{d}{dt} \Theta (t) = \left[ \frac{\hbar^2 \alpha(t)}{2m} - \frac{m\omega_0^2}{8\alpha(t)} \right] - \frac{P^2(t)}{2m} - \frac{1}{2} m\omega_0^2 X^2(t)$$

$$i\hbar \frac{d}{dt} \alpha(t) = \frac{2\hbar^2 \alpha^2(t)}{m} + \frac{1}{2} m\omega_0^2$$

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5
Problem 5.5

(a) We make an analogy between SHO and the LC circuit:

\[ x(t) \longleftrightarrow i(t) = \zeta \]

\[ p(t) \longleftrightarrow \frac{v(t)}{B} \]

\[ \dot{\zeta} = -i\hbar \frac{\partial}{\partial x} \leftrightarrow \frac{v(t)}{B} = -i\hbar \frac{\partial}{\partial \zeta} \tag{7} \]

The classical energy within this analogy is

\[ E = \frac{\left[ \frac{v(t)}{B} \right]^2}{2m} + \frac{1}{2} \tilde{m} \omega_0^2 [v(t)]^2 = \frac{1}{2} L i^2(t) + \frac{1}{2} C v^2(t) \]

Matching terms proportional to \( i^2(t) \) gives

\[ \tilde{m} = L^2 C \]

Matching terms proportional to \( v^2(t) \) gives

\[ \frac{1}{\tilde{m} B^2} = C \Rightarrow B = \frac{1}{LC} \]

Substituting back into Eq.(7) we obtain

\[ \hat{v} = -i \frac{\hbar}{LC} \frac{\partial}{\partial \zeta} \]

(b)

\[ \hat{E}\psi = i\hbar \frac{\partial}{\partial t} \psi = \frac{1}{2} L i^2 \psi + \frac{1}{2} C v^2 \psi = \frac{1}{2} L \zeta^2 \psi - \frac{1}{2} \frac{\hbar^2}{2L^2 C} \frac{\partial^2}{\partial \zeta^2} \psi \]

(c) The Schrödinger equation obtained above is the same as that obtained from substituting \( m \) as \( \tilde{m} = L^2 C \), \( \omega_0^2 = \frac{1}{LC} \), and \( x \) as \( \zeta \) into Eq.(10.46) in HSO. Making the same substitutions into Eq.(10.47) and Eq.(10.48) gives the creation and annihilation operators

\[ \hat{a} = \sqrt{\frac{L}{2\hbar \omega_0}} \zeta + \sqrt{\frac{\hbar \omega_0}{2L}} \frac{\partial}{\partial \zeta} \]

\[ \hat{a}^\dagger = \sqrt{\frac{L}{2\hbar \omega_0}} \zeta - \sqrt{\frac{\hbar \omega_0}{2L}} \frac{\partial}{\partial \zeta} \]