6.730 Physics for Solid State Applications

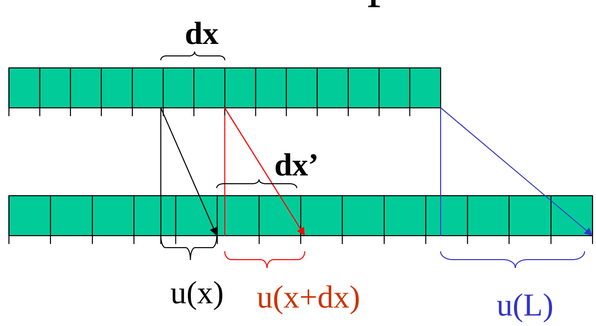
Lecture 4: Vibrations in Solids

February 11, 2004

Outline

- 1-D Elastic Continuum
- 1-D Lattice Waves
- 3-D Elastic Continuum
- 3-D Lattice Waves

Strain E and Displacement u(x)

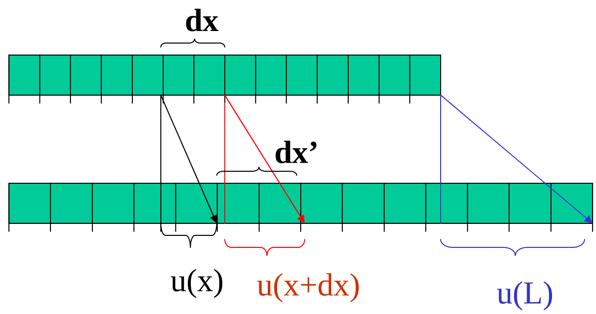


$\delta(dx) = dx' - dx$ how much does a differential length change

$\delta(dx) = u(x+dx) - u(x)$ difference in displacements

Strain:
$$E = \frac{\delta(dx)}{dx} = \frac{\partial u}{\partial x}$$

Uniform Strain & Linear Displacement u(x)

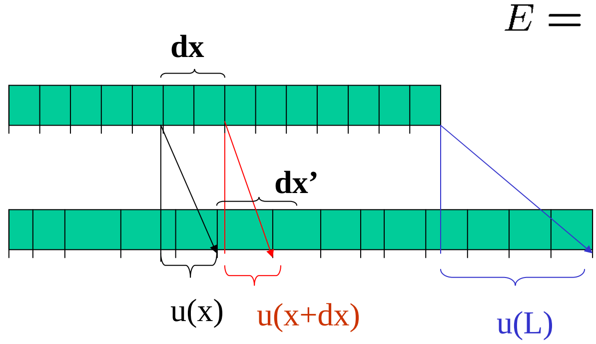


Linear displacement: $u(x) = E^0 x$

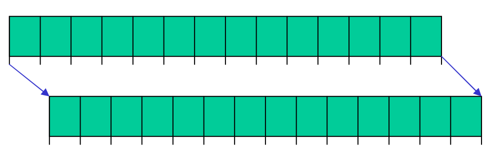
Constant Strain: $$E = \frac{\delta(dx)}{dx} = \frac{\partial u}{\partial x} = E^0$$

More Types of Strain

$$E = \frac{\delta(dx)}{dx} = \frac{\partial u}{\partial x}$$



Non-uniform
$E = E(x)$

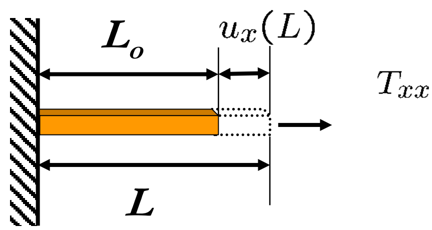


Zero Strain:
u(x) is constant
Just a Translation
We will ignore this

1-D Elastic Continuum

Stress and Strain

uniaxial loading



Stress:

$$T_{xx} = \frac{F_x}{A}\left[N/m^2\right]$$

Elongation:

$$\delta(dx) = u_x(x+dx) - u_x(x)$$

Normal Strain:

$$E_{xx} = \frac{\delta(dx)}{dx} = \frac{\partial u_x}{\partial x}$$

If u_x is uniform there is no strain, just rigid body motion.

1-D Elastic Continuum

Young's Modulus

$$T_{xx} = E_Y E_{xx}$$

Young's Modulus For Various Materials (GPa)
from Christina Ortiz

CERAMICS GLASSES AND SEMICONDUCTORS

Diamond (C)	1000
Tungsten Carbide (WC)	450 -650
Silicon Carbide (SiC)	450
Aluminum Oxide (Al_2O_3)	390
Berylium Oxide (BeO)	380
Magnesium Oxide (MgO)	250
Zirconium Oxide (ZrO)	160 - 241
Mullite ($Al_6Si_2O_{13}$)	145
Silicon (Si)	107
Silica glass (SiO_2)	94
Soda-lime glass (Na_2O - SiO_2)	69

METALS :

Tungsten (W)	406
Chromium (Cr)	289
Berylium (Be)	200 - 289
Nickel (Ni)	214
Iron (Fe)	196
Low Alloy Steels	200 - 207
Stainless Steels	190 - 200
Cast Irons	170 - 190
Copper (Cu)	124
Titanium (Ti)	116
Brasses and Bronzes	103 - 124
Aluminum (Al)	69

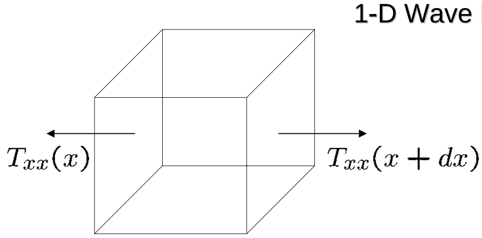
PINE WOOD (along grain): 10

POLYMERS :

Polyimides	3 - 5
Polyesters	1 - 5
Nylon	2 - 4
Polystryene	3 - 3.4
Polyethylene	0.2 -0.7
Rubbers / Biological Tissues	0.01-0.1

Dynamics of 1-D Continuum

1-D Wave Equation



Net force on incremental volume element:

$$f_x = [T_{xx}(x+dx) - T_{xx}(x)]\; dy\, dz$$

$$m\frac{\partial^2 u_x}{\partial t^2} = [T_{xx}(x+dx) - T_{xx}(x)]\; dy\, dz$$

$$\rho\frac{\partial^2 u_x}{\partial t^2}\, dx\, dy\, dz = [T_{xx}(x+dx) - T_{xx}(x)]\; dy\, dz$$

$$\boxed{\rho\frac{\partial^2 u_x}{\partial t^2} = \frac{\partial T_{xx}}{\partial x}}$$

Dynamics of 1-D Continuum

1-D Wave Equation

$$\rho\frac{\partial^2 u_x}{\partial t^2} = \frac{\partial T_{xx}}{\partial x} \qquad T_{xx} = E_Y E_{xx} \qquad E_{xx} = \frac{\partial u_x}{\partial x}$$

$$\rho\frac{\partial^2 u_x}{\partial t^2} = E_Y\frac{\partial^2 u_x}{\partial x^2}$$

$$\frac{\partial^2 u_x}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u_x}{\partial t^2} \qquad c = \sqrt{\frac{E_Y}{\rho}}$$

Velocity of sound, c, is proportional to stiffness and inverse prop. to inertia

Dynamics of 1-D Continuum
1-D Wave Equation Solutions

$$\frac{\partial^2 u_x}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u_x}{\partial t^2}$$

Clamped Bar: Standing Waves

$$u_x(x,t) = A_\pm \sin(kx)\exp(i\omega t) \qquad \omega = ck$$

$$u_{x,m,\pm}(x,t) = A_{m,\pm}\sin\left(\frac{m\pi x}{L}\right)\exp\left(\pm i\frac{m\pi c}{L}t\right)$$

$$k = \frac{m\pi}{L} \qquad \text{for} \qquad m = 1, 2, \ldots$$

Dynamics of 1-D Continuum
1-D Wave Equation Solutions

$$\frac{\partial^2 u_x}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u_x}{\partial t^2}$$

Periodic Boundary Conditions: Traveling Waves

$$u_x(x,t) = A_\pm \exp(ikx)\exp(i\omega t) \qquad \omega = ck$$

$$u_{x,n,\pm}(x,t) = B_{n,\pm}\exp\left(\pm i\frac{2n\pi x}{L}(x \pm ct)\right)$$

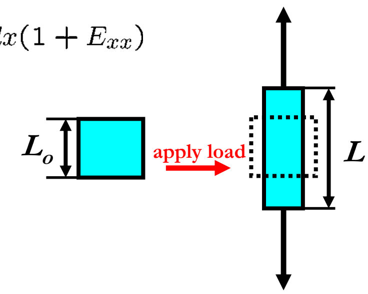
$$k = \frac{2n\pi}{L} \qquad \text{for} \qquad n = \pm 1, \pm 2, \ldots$$

3-D Elastic Continuum
Volume Dilatation

$$dx \rightarrow dx + \delta(dx) = dx + E_{xx}dx = dx(1 + E_{xx})$$

$$dy \rightarrow dy(1 + E_{yy})$$

$$dz \rightarrow dz(1 + E_{zz})$$



$$e = \frac{\delta V}{V} = \frac{dx(1 + E_{xx})dy(1 + E_{yy})dz(1 + E_{zz}) - dxdydz}{dxdydz}$$

$$e = E_{xx} + E_{yy} + E_{zz}$$

Volume change is sum of all three normal strains

3-D Elastic Continuum
Poisson's Ratio

$$E_{xx} = \frac{\partial u_x}{\partial x} \qquad E_{yy} = \frac{\partial u_y}{\partial y} \qquad E_{zz} = \frac{\partial u_z}{\partial z}$$

$$e = E_{xx} + E_{yy} + E_{zz} = \nabla \cdot \mathbf{u}(\mathbf{r})$$

ν is Poisson's Ratio – ratio of lateral strain to axial strain

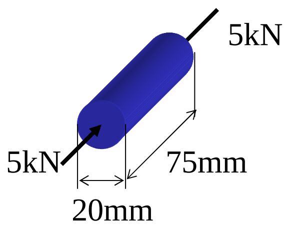
$$E_{yy} = E_{zz} = -\nu E_{xx}$$

$$e = E_{xx}(1 - 2\nu)$$

Poisson's ratio can not exceed 0.5, typically 0.3

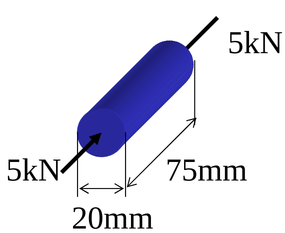
3-D Elastic Continuum
Poisson's Ratio Example

Aluminum: E_Y=68.9 GPa, ν = 0.35



3-D Elastic Continuum
Poisson's Ratio Example

Aluminum: E_Y=68.9 GPa, ν = 0.35



$$T_{xx} = \frac{F_x}{A} = \frac{5\text{x}10^3}{\pi(10\text{x}10^{-3})^2} = -15.9\text{MPa}$$

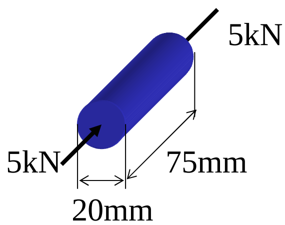
$$E_{xx} = \frac{T_{xx}}{E_Y} = \frac{-15.9\text{x}10^6}{68.9\text{x}10^9} = -0.231\text{x}10^{-3}$$

$$E_{xx} = \frac{\Delta l}{l} = -0.231\text{x}10^{-3}$$

$$\Delta l = -0.0173\text{mm}$$

3-D Elastic Continuum
Poisson's Ratio Example

Aluminum: E_Y=68.9 GPa, ν = 0.35



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$$E_{xx} = \frac{\Delta l}{l} = -0.231\text{x}10^{-3}$$

$$\Delta l = -0.0173\text{mm}$$

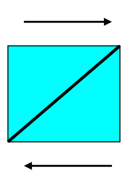
$$E_{trns} = -\nu E_{xx} = -0.35 E_{xx} = 0.081\text{x}10^{-3}$$

$$E_{trns} = \frac{\Delta d}{d}$$

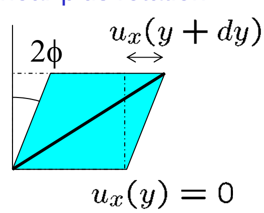
$$\Delta d = +0.001617\text{mm}$$

3-D Elastic Continuum
Shear Strain

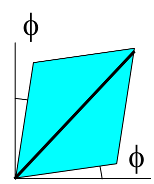
Shear loading



Shear plus rotation



Pure shear



Pure shear strain

$$\phi = E_{xy} = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right)$$

Shear stress

$$T_{xy} = G2\phi = 2GE_{xy}$$

G is shear modulus

3-D Elastic Continuum
Stress and Strain Tensors

$$E_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

$$E_{xx} = \frac{\partial u_x}{\partial x}$$

$$E_{xy} = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right)$$

$$e = \sum_{k=1}^{3} E_{kk}$$

For *most* general isotropic medium,

$$\mathbf{T} = \lambda e \mathbf{I} + 2\mu \mathbf{E}$$

Initially we had three elastic constants: E_Y, G, e

Now reduced to only two: λ, μ

3-D Elastic Continuum
Stress and Strain Tensors

$$T_{ij} = \lambda e\, \delta_{ij} + 2\mu E_{ij}$$

If we look at just the diagonal elements

$$\sum_{k=1}^{3} T_{kk} = 3\lambda e + 2\mu e$$

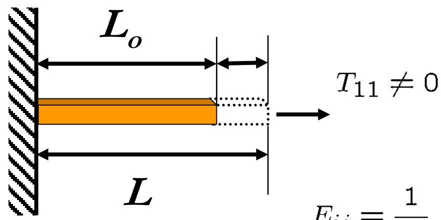
$$e = \frac{1}{3\lambda + 2\mu} \sum_{k=1}^{3} T_{kk}$$

Inversion of stress/strain relation:

$$E_{ij} = \frac{1}{2\mu}\left[T_{ij} - \frac{\lambda}{3\lambda + 2\mu}\left(\sum_k T_{kk}\right)\delta_{ij}\right]$$

3-D Elastic Continuum

Example of Uniaxial Stress



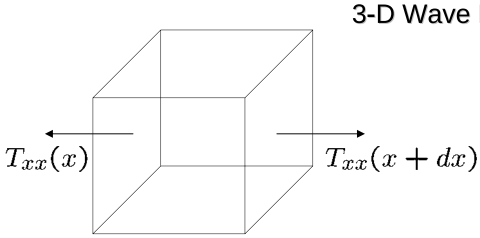
$$E_{ij} = \frac{1}{2\mu}\left[T_{ij} - \frac{\lambda}{3\lambda + 2\mu}\left(\sum_k T_{kk}\right)\delta_{ij}\right]$$

$$E_{11} = \underbrace{\frac{\lambda + \mu}{\mu(3\lambda + 2\mu)}}_{E_Y} T_{11}$$

$$E_{22} = E_{33} = -\underbrace{\frac{\lambda}{2(\lambda + \mu)}}_{\nu} E_{11}$$

Dynamics of 3-D Continuum

3-D Wave Equation



Net force on incremental volume element:

$$\mathbf{F} = \int_{\mathbf{V}} \mathbf{f} \mathbf{dxdydz}$$

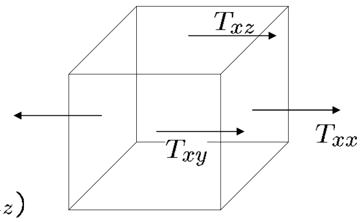
$$\mathbf{F} = \int_{\mathbf{V}} \rho \frac{\partial^2 \mathbf{u}}{\partial \mathbf{t}^2} \mathbf{dxdydz}$$

$$\mathbf{f} = \rho \frac{\partial^2 \mathbf{u}}{\partial \mathbf{t}^2}$$

Total force is the sum of the forces on *all* the surfaces

Dynamics of 3-D Continuum

3-D Wave Equation



Net force in the x-direction:

$$F_x = \sum_{\text{surfaces}} (T_{xx}\, dA_x + T_{xy}\, dA_y + T_{xz}\, dA_z)$$

$$\sum_{\text{surface}} T_{xx}\, dA_x = \frac{T_{xx}(x+dx) - T_{xx}(x)}{dx}\, dx\, dy\, dz$$

$$\sum_{\text{surface}} T_{xx}\, dA_x = \frac{\partial T_{xx}}{\partial x}\, dx\, dy\, dz$$

$$F_x = \int\int\int \left[\frac{\partial T_{xx}}{\partial x} + \frac{\partial T_{xy}}{\partial y} + \frac{\partial T_{xz}}{\partial z}\right] dx\, dy\, dz$$

Dynamics of 3-D Continuum

3-D Wave Equation

$$F_x = \int\int\int \left[\frac{\partial T_{xx}}{\partial x} + \frac{\partial T_{xy}}{\partial y} + \frac{\partial T_{xz}}{\partial z}\right] dx\, dy\, dz \qquad T_{ij} = \lambda e\, \delta_{ij} + 2\mu E_{ij}$$

$$F_x = \int_v \rho \frac{\partial^2 \mathbf{u}}{\partial t^2} dxdydz = \int\int\int \underbrace{\left[(\mu+\lambda)\frac{\partial}{\partial x}(\nabla\cdot\mathbf{u}) + \mu\nabla^2 \mathbf{u_x}\right]}_{\mathbf{f}_x} dx\, dy\, dz$$

Finally, 3-D wave equation….

$$\rho\frac{\partial^2 \mathbf{u}}{\partial t^2}(\mathbf{r}, t) = (\mu+\lambda)\nabla\left[(\nabla\cdot\mathbf{u}(\mathbf{r},\mathbf{t})\right] + \mu\nabla^2\mathbf{u}(\mathbf{r},\mathbf{t})$$

Dynamics of 3-D Continuum
Fourier Transform of 3-D Wave Equation

$$\rho\frac{\partial^2\mathbf{u}}{\partial t^2}(\mathbf{r},t) = (\mu+\lambda)\nabla\left[(\nabla\cdot\mathbf{u}(\mathbf{r},\mathbf{t})\right] + \mu\nabla^2\mathbf{u}(\mathbf{r},\mathbf{t})$$

Anticipating plane wave solutions, we Fourier Transform the equation….

$$\mathbf{u}(\mathbf{r},\mathbf{t}) = \int\frac{\mathbf{d}\omega}{\mathbf{2}\pi}\int\frac{\mathbf{d^3q}}{(\mathbf{2}\pi)^\mathbf{3}}\mathbf{U}(\mathbf{q},\omega)\mathbf{e}^{\mathbf{i}(\mathbf{q}\cdot\mathbf{r}-\omega\mathbf{t})}$$

$$\rho\omega^2\mathbf{U}(\mathbf{q},\omega) = (\lambda+\mu)\mathbf{q}\left[\mathbf{q}\cdot\mathbf{U}(\mathbf{q},\omega)\right] + \mu\mathbf{q^2}\mathbf{U}(\mathbf{q},\omega)$$

Three coupled equations for U_x, U_y, and U_z….

Dynamics of 3-D Continuum
Dynamical Matrix

$$\rho\omega^2\mathbf{U_i}(\mathbf{q},\omega) = (\lambda+\mu)\mathbf{q_i}\left[\mathbf{q}\cdot\mathbf{U}(\mathbf{q},\omega)\right]+\mu\mathbf{q^2}\mathbf{U_i}(\mathbf{q},\omega)$$

Express the system of equations as a matrix….

$$\rho\omega^2\begin{bmatrix}\mathbf{U_1}\\ \mathbf{U_2}\\ \mathbf{U_3}\end{bmatrix} = \begin{bmatrix}\mu q^2+(\lambda+\mu)q_1^2 & (\lambda+\mu)q_1q_2 & (\lambda+\mu)q_1q_3\\ (\lambda+\mu)q_2q_1 & \mu q^2+(\lambda+\mu)q_2^2 & (\lambda+\mu)q_2q_3\\ (\lambda+\mu)q_3q_1 & (\lambda+\mu)q_3q_2 & \mu q^2+(\lambda+\mu)q_3^2\end{bmatrix}\begin{bmatrix}\mathbf{U_1}\\ \mathbf{U_2}\\ \mathbf{U_3}\end{bmatrix}$$

Turns the problem into an eigenvalue problem for
the polarizations of the modes (eigenvectors) and
wavevectors **q** (eigenvalues)….

$$\rho\omega^2\mathbf{U} = \mathbf{D}\,\mathbf{U}$$

Dynamics of 3-D Continuum
Solutions to 3-D Wave Equation

$$\rho\omega^2 \mathrm{U_i}(\mathbf{q},\omega) = (\lambda+\mu)\mathbf{q_i}\,[\mathbf{q}\cdot\mathrm{U}(\mathbf{q},\omega)] + \mu\mathbf{q}^2\mathrm{U_i}(\mathbf{q},\omega)$$

Transverse polarization waves:

$$\mathbf{q}\cdot\mathbf{U}(\mathbf{q},\omega) = \mathbf{0}$$

$$\rho\omega^2 = \mu q^2 \qquad \text{for transverse waves}$$

$$\omega = c_T|\mathbf{q}| \qquad \text{where} \qquad c_T = \sqrt{\frac{\mu}{\rho}}$$

Longitudinal polarization waves:

$$\mathbf{q}\cdot\mathbf{U}(\mathbf{q},\omega) = \mathbf{q}\mathbf{U}$$

$$\rho\omega^2 U = (\lambda+2\mu)q^2 U \qquad \text{for longitudinal waves}$$

$$\omega = c_L|\mathbf{q}| \qquad \text{where} \qquad c_L = \sqrt{\frac{\lambda+2\mu}{\rho}}$$

Dynamics of 3-D Continuum
Summary

1. Dynamical Equation can be solved by inspection

$$\rho\omega^2 \mathbf{U}(\mathbf{q},\omega) = (\lambda+\mu)\mathbf{q}\,[\mathbf{q}\cdot\mathbf{U}(\mathbf{q},\omega)] + \mu\mathbf{q}^2\mathbf{U}(\mathbf{q},\omega)$$

2. There are 2 transverse and 1 longitudinal polarizations for each q

3. The dispersion relations are linear $\omega = c_i|\mathbf{q}|$

$$c_T = \sqrt{\frac{\mu}{\rho}} \qquad c_L = \sqrt{\frac{\lambda+2\mu}{\rho}}$$

4. The longitudinal sound velocity is always greater than the transverse sound velocity

$$\frac{c_L}{c_T} = \left(\frac{\lambda+2\mu}{\mu}\right)^{1/2} = \left(1+\frac{1}{1-2\nu}\right)^{1/2}$$