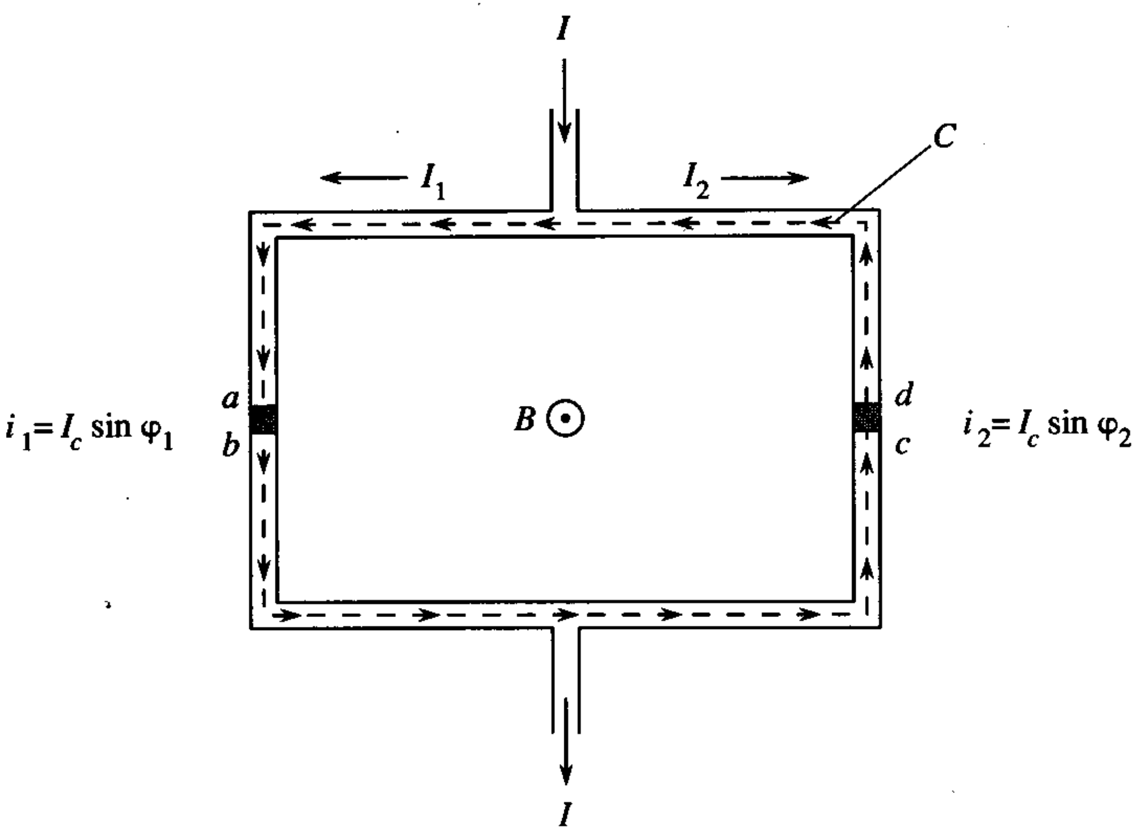
Lecture 12: Superconducting Quantum Interference Devices

OUTLINE

1. Superconducting Quantum Interference
2. SQUIDs
    - SQUID Equations
    - SQUID Magnetometers
    - Josephson loop vs SQUID Loop
3. Distributed Josephson Junctions
    - Short Josephson Junctions
        - Josephson Phasors (pendula)
    - Long Josephson Junctions

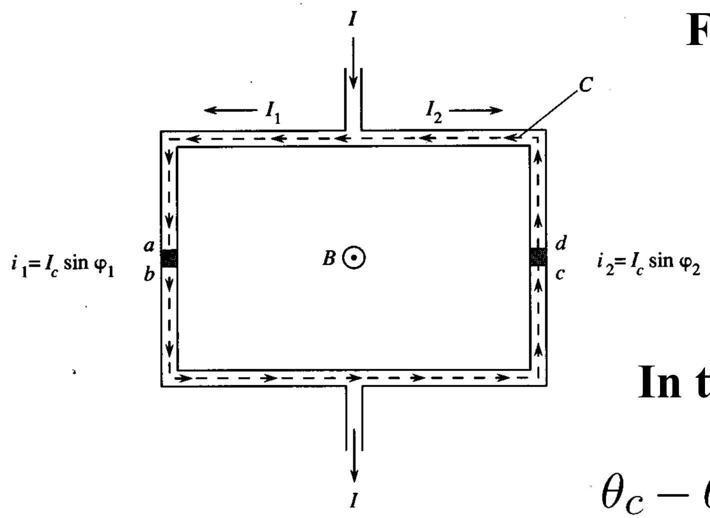
Superconducting Quantum Interference



$$
\begin{aligned}
i &= i_1 + i_2 = I_c \sin \varphi_1 + I_c \sin \varphi_2 \\
&= 2 I_c \cos \left(\frac{\varphi_1 - \varphi_2}{2} \right) \sin \left(\frac{\varphi_1 + \varphi_2}{2} \right)
\end{aligned}
$$

Phase difference around the loop

$$\oint_C \nabla\theta\cdot d\mathbf{l} = 2\pi n = (\theta_b-\theta_a)+(\theta_c-\theta_b)+(\theta_d-\theta_c)+(\theta_a-\theta_d)$$



From the definition of the gauge invariant phase

$$\theta_b-\theta_a = -\varphi_1 - \frac{2\pi}{\Phi_o}\int_a^b \mathbf{A}\cdot d\mathbf{l}$$

$$\theta_d-\theta_c = \varphi_2 - \frac{2\pi}{\Phi_o}\int_c^d \mathbf{A}\cdot d\mathbf{l}$$

In the superconductor the supercurrent equation gives

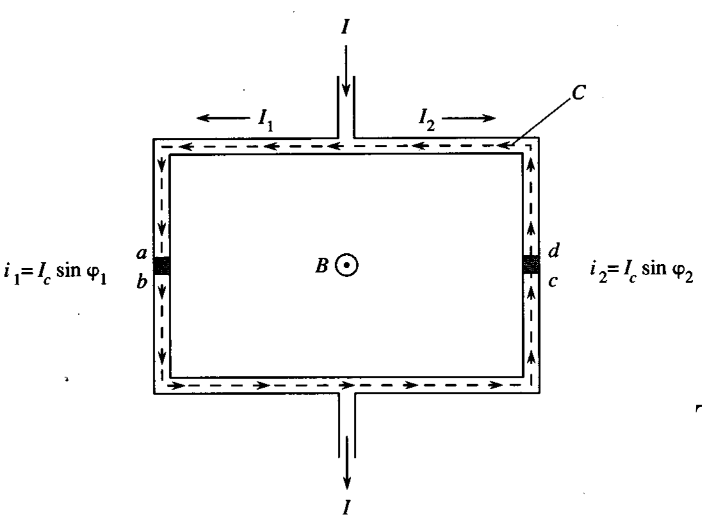
$$\theta_c-\theta_b = \int_b^c \nabla\theta\cdot d\mathbf{l} = -\Lambda\int_b^c \mathbf{J}\cdot d\mathbf{l} - \frac{2\pi}{\Phi_o}\int_b^c \mathbf{A}\cdot d\mathbf{l}$$

$$\theta_a-\theta_d = \int_d^a \nabla\theta\cdot d\mathbf{l} = -\Lambda\int_d^a \mathbf{J}\cdot d\mathbf{l} - \frac{2\pi}{\Phi_o}\int_d^a \mathbf{A}\cdot d\mathbf{l}$$

Adding them together gives

$$\varphi_2-\varphi_1 = 2\pi n + \frac{2\pi}{\Phi_o}\oint_C \mathbf{A}\cdot d\mathbf{l} + \Lambda\int_b^c \mathbf{J}\cdot d\mathbf{l} + \Lambda\int_d^a \mathbf{J}\cdot d\mathbf{l}$$

SQUID Equations



$$\varphi_2 - \varphi_1 = 2\pi n + \frac{2\pi\Phi}{\Phi_o} + \Lambda \int_{C'} \mathbf{J} \cdot d\mathbf{l}$$

Often the contour can be taken where **J** = 0, in this case

$$\varphi_2 - \varphi_1 = 2\pi n + \frac{2\pi\Phi}{\Phi_o}$$

The total current can be written then as

$$i = 2I_c \cos\left(\frac{\pi\Phi}{\Phi_o}\right) \sin\left(\varphi_1 + \frac{\pi\Phi}{\Phi_o}\right)$$

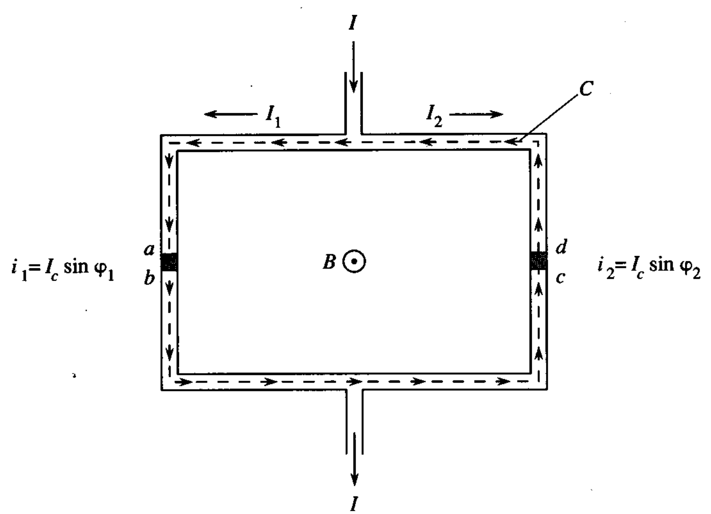
The flux in the contour is $\Phi = \Phi_{\text{ext}} + LI_{\text{cir}}$

The circulating current is given by $I_{\text{cir}} = (i_1 - i_2)/2$

The total flux can then be written as

$$\Phi = \Phi_{\text{ext}} + \frac{LI_c}{2} \sin\left(\frac{\pi\Phi}{\Phi_o}\right) \cos\left(\varphi_1 + \frac{\pi\Phi}{\Phi_o}\right)$$

Method of Solution



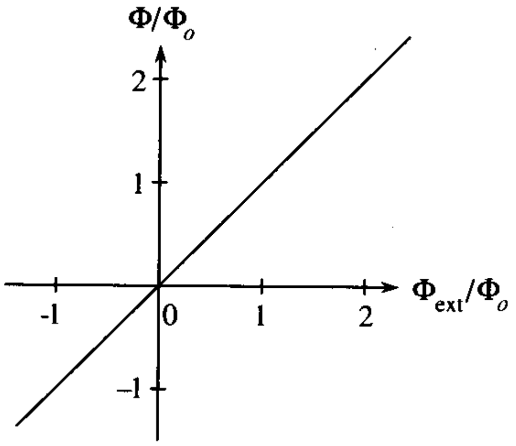
$$i = 2I_c \cos\left(\frac{\pi\Phi}{\Phi_o}\right)\sin\left(\varphi_1 + \frac{\pi\Phi}{\Phi_o}\right)$$

$$\Phi = \Phi_{\text{ext}} + \frac{LI_c}{2}\sin\left(\frac{\pi\Phi}{\Phi_o}\right)\cos\left(\varphi_1 + \frac{\pi\Phi}{\Phi_o}\right)$$

For a given external flux, there is a range of i and Φ that satisfy these equations. One wants to determine the maximum i that can be put through the SQUID and still have zero voltage. (For larger I the current will be shown to create a voltage).

SQUID without self inductance

In this case $\Phi = \Phi_{\text{ext}}$ And, therefore,



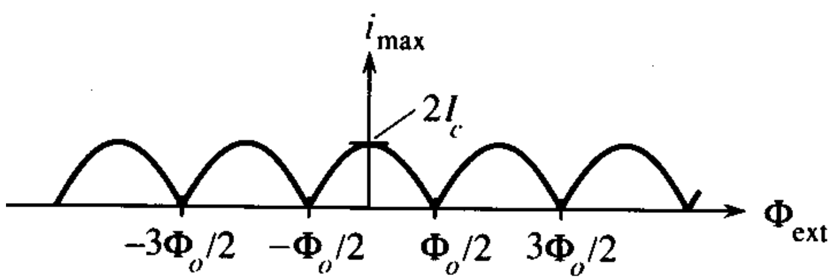
$$i = 2I_c \cos\left(\frac{\pi\Phi_{\text{ext}}}{\Phi_o}\right) \sin\left(\varphi_1 + \frac{\pi\Phi_{\text{ext}}}{\Phi_o}\right)$$

The extremum occurs when $di/d\varphi_i = 0$, that is, when

$$\cos\left(\varphi_1 + \pi\Phi_{\text{ext}}/\Phi_o\right) = 0 \quad \Rightarrow \quad \sin\left(\varphi_1 + \pi\Phi_{\text{ext}}/\Phi_o\right) = \pm 1$$

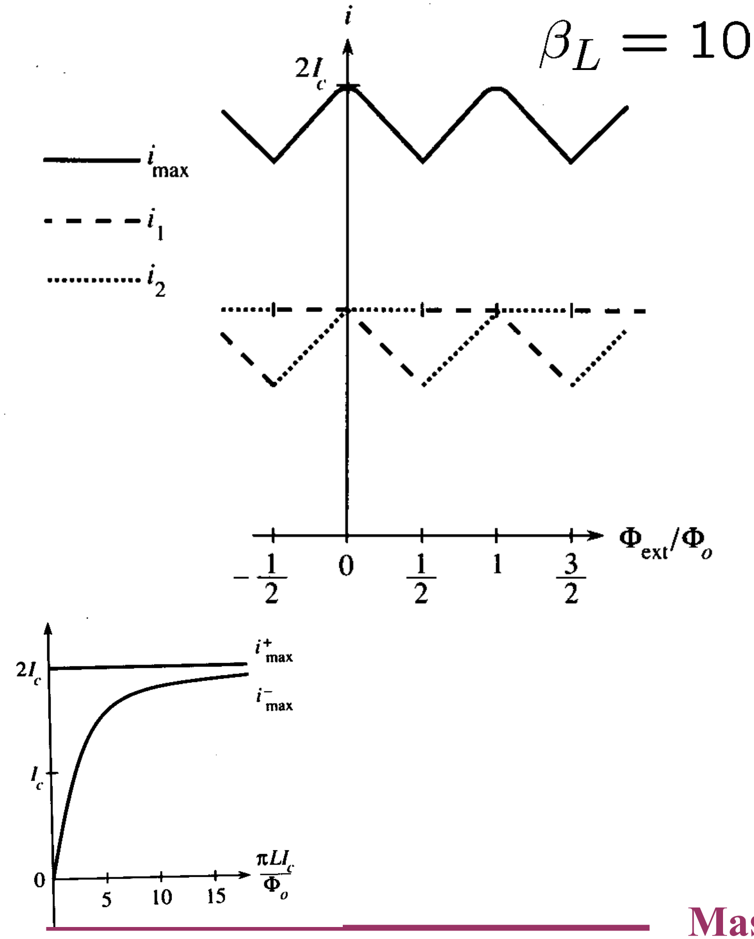
Therefore,

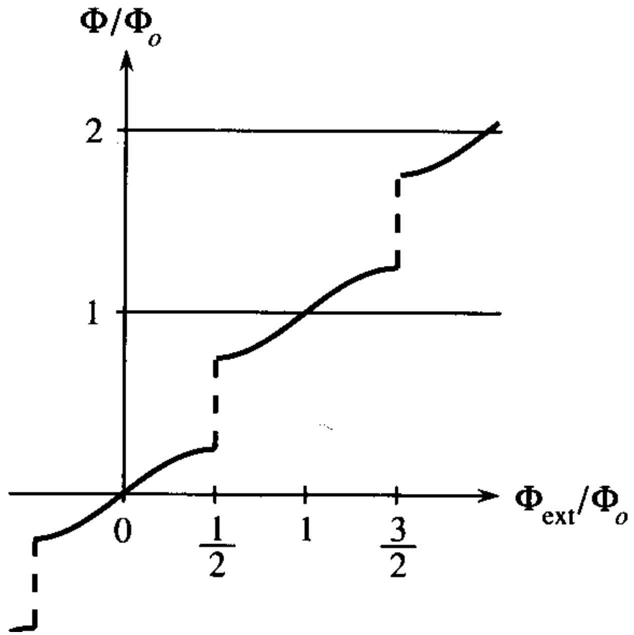
$$i_{\max} = 2I_c \left|\cos\left(\frac{\pi\Phi_{\text{ext}}}{\Phi_o}\right)\right|$$



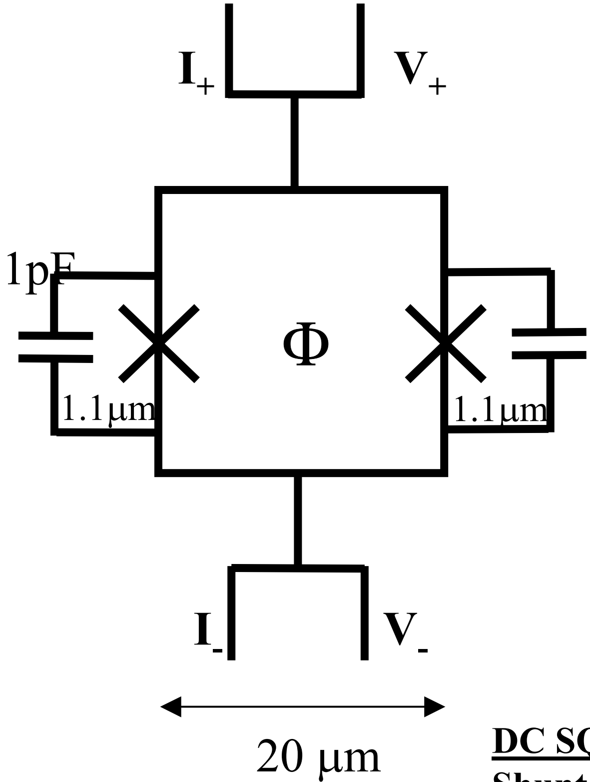
SQUID with self inductance

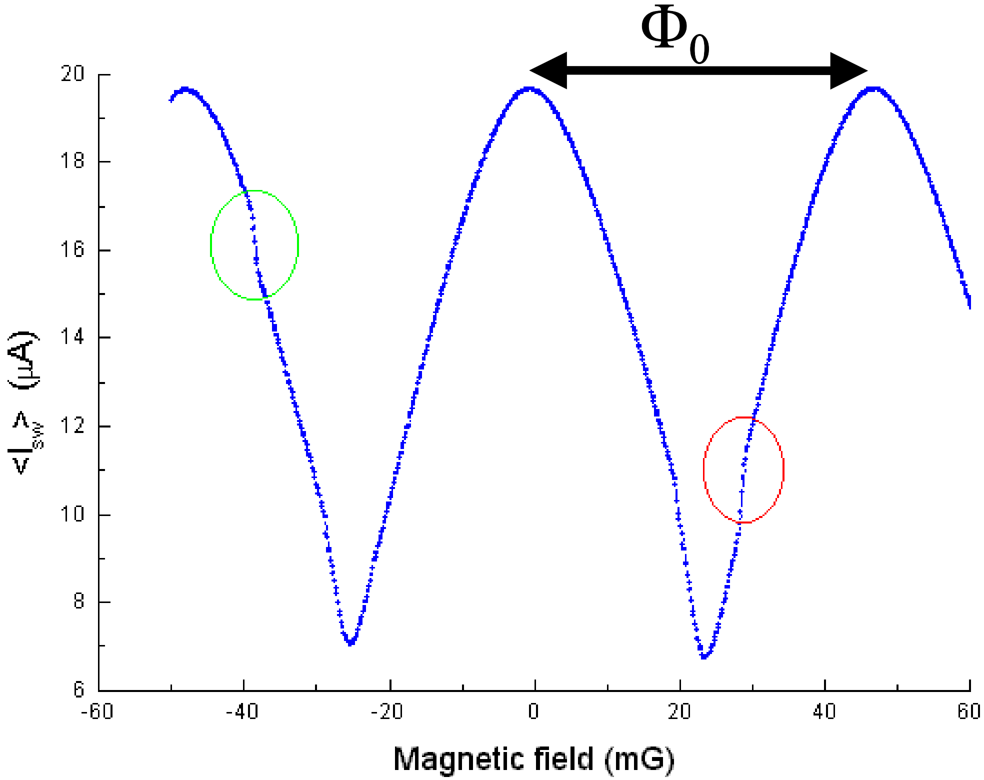
$$\beta_L = 2\pi L I_c/\Phi_o = L/L_J$$





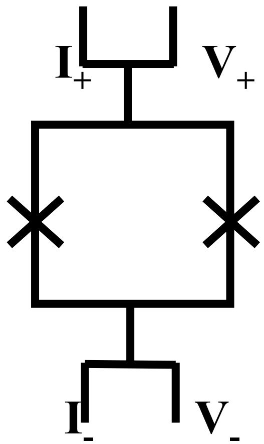
SQUID Magnetometers

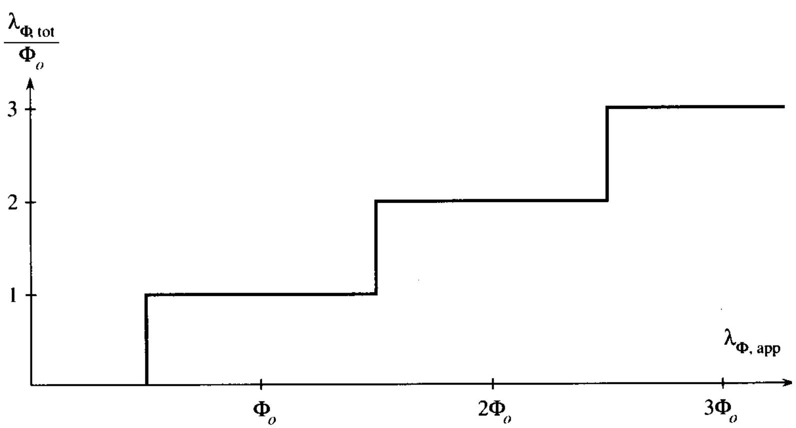


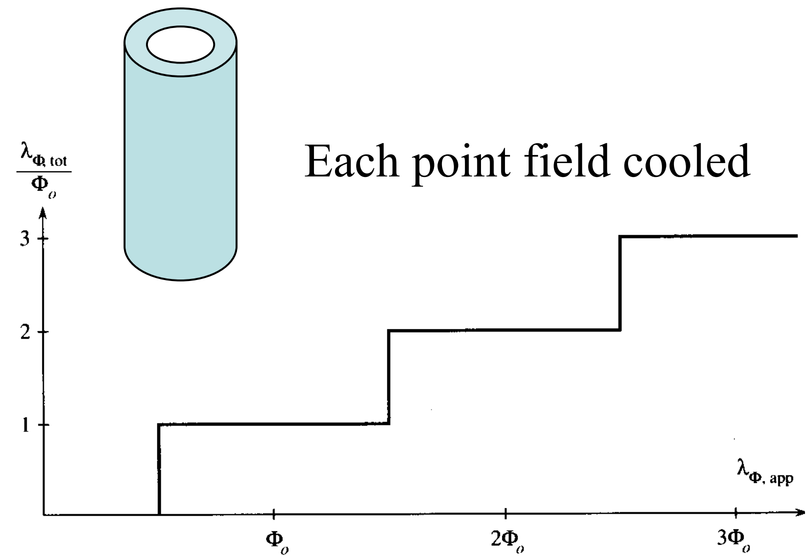


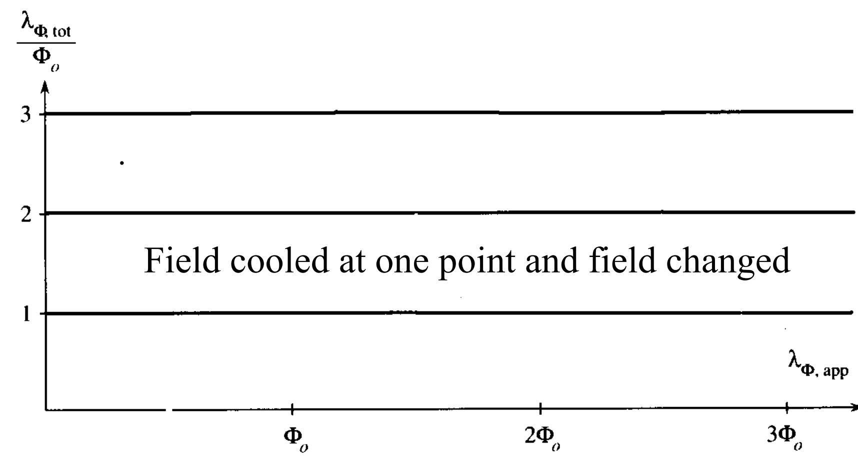
DC SQUID
Shunt capacitors ~ 1pF
Jct. Size ~ 1.1μm
Loop size ~$20x20\mu m^2$
L_{SQUID} ~ 50pH
I_c~10 & 20μA

Josephson Loop vs. Superconducting Loop

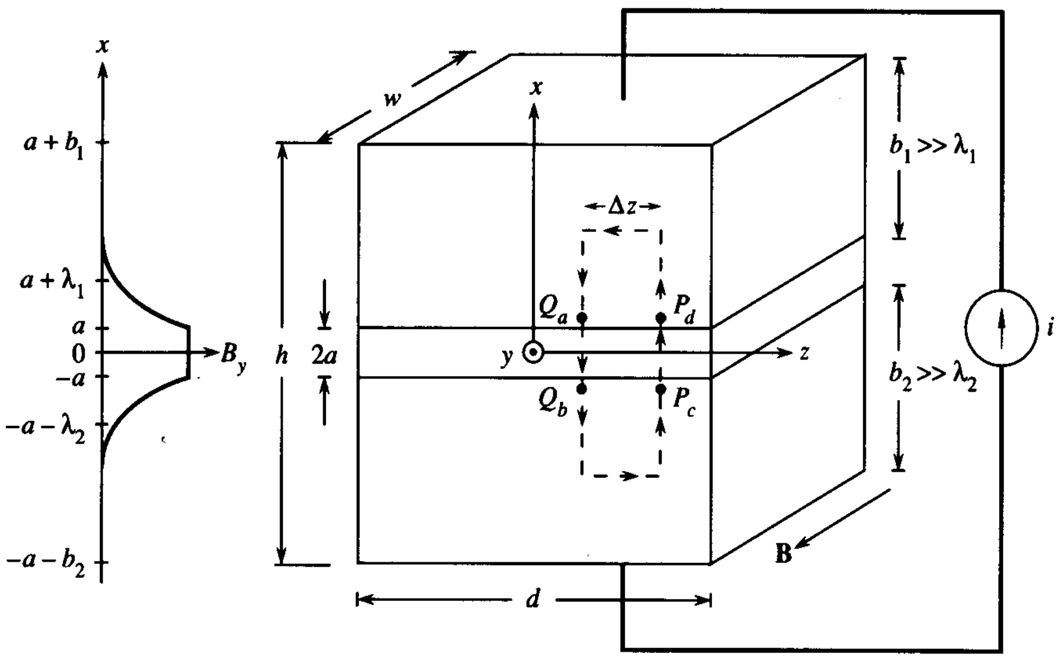








Distributed Josephson Junction



$$\varphi(Q) - \varphi(P) = \pi n + \frac{2\pi\Phi}{\Phi_o} + \underbrace{\Lambda \int_{C'} \mathbf{J}\cdot d\mathbf{l}}_{0 \text{ as } \Delta z \to 0}$$

$$\Phi = B_y\underbrace{(\lambda_1 + \lambda_2 + 2a)}_{h_{\text{eff}}}\Delta z$$

Therefore,

$$\frac{\partial\varphi}{\partial z} = \frac{2\pi}{\Phi_o} B_y h_{\text{eff}}$$

Likewise, for the y-directions

$$\frac{\partial\varphi}{\partial y} = -\frac{2\pi}{\Phi_o} B_z h_{\text{eff}}$$

Also,

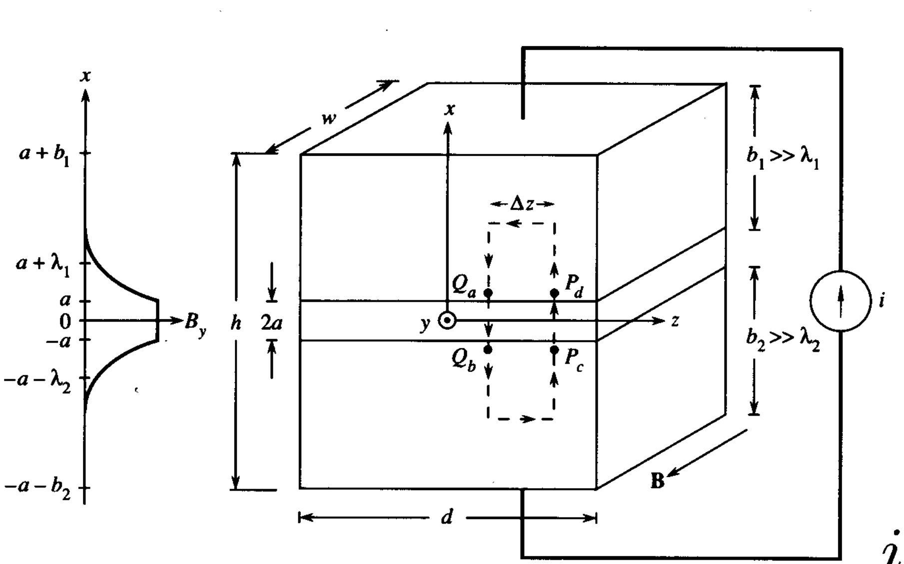
$$J_s(y,z,t) = J_c(y,z)\sin\varphi(z,t)$$

In general this must be solved self-consistently with Ampere's Law

$$\nabla \times \mathbf{B} = \mu_o \mathbf{J} + \epsilon\mu_o \frac{\partial \mathbf{E}}{\partial t}$$

Short Josephson Junction

Let the self fields generated by the currents be negligible, and $\mathbf{B} = B_o \, \mathbf{i}_y$, then



$$\frac{\partial \varphi}{\partial z} = \frac{2\pi}{\Phi_o} B_o h_{\text{eff}}$$

That can be integrated directly,

$$\varphi(z) = \frac{2\pi}{\Phi_o} B_o h_{\text{eff}} z + \varphi(0)$$

The total current with constant J_c is

$$i = \int_{-d/2}^{d/2} \int_{-w/2}^{w/2} J_c \sin \varphi(z) \, dy \, dz$$

Flux through the junction

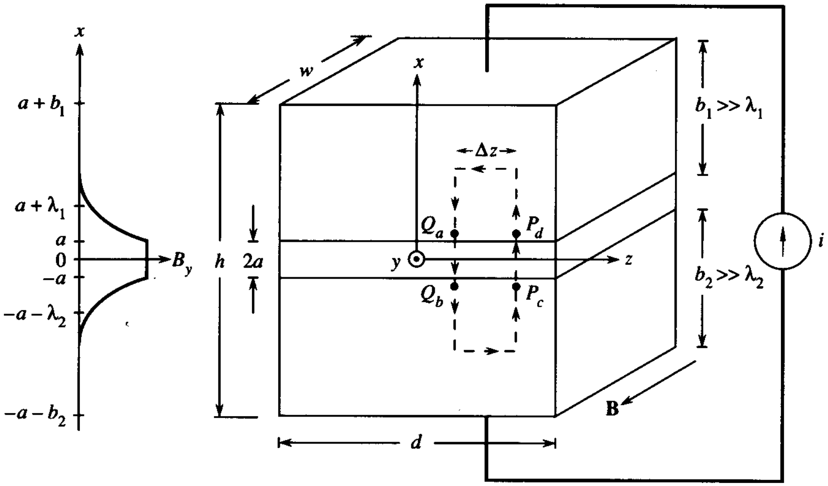
$$\Phi_J = B_o h_{\text{eff}} d$$

Critical current of the junction

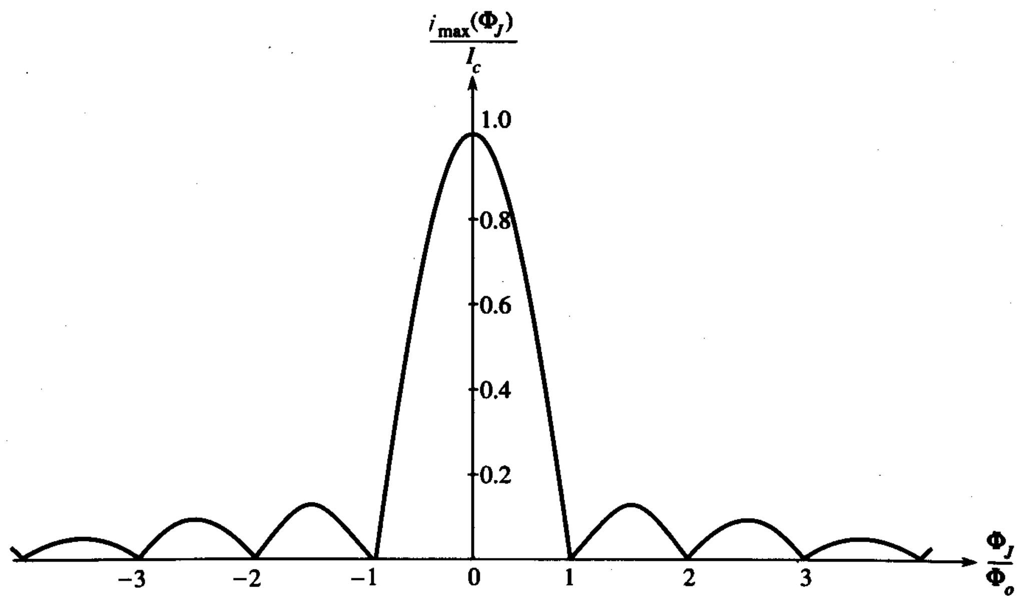
$$I_c = J_c w d$$

$$i\left(\Phi_J, \varphi(0)\right) = I_c \frac{\sin \frac{\pi \Phi_J}{\Phi_o}}{\frac{\pi \Phi_J}{\Phi_o}} \sin(\varphi(0))$$

Short Josephson Junction: "single-slit interference"



$$i_{\mathsf{max}}(\Phi_J) = I_c \left| \frac{\sin \frac{\pi \Phi_J}{\Phi_o}}{\frac{\pi \Phi_J}{\Phi_o}} \right|$$



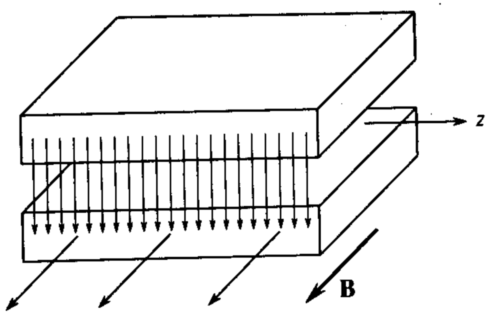
Flux through the junction

$$\Phi_J = B_o h_{\mathsf{eff}} d$$

Critical current of the junction

$$I_c = J_c w d$$

Vortices in Short Junctions

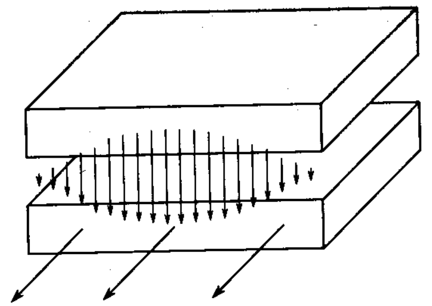


$\Phi_J = 0$

$\varphi = \varphi(0)$

$\varphi\left(\frac{d}{2}\right) - \varphi\left(-\frac{d}{2}\right) = 0$

$\varphi(0) = \frac{\pi}{2}$

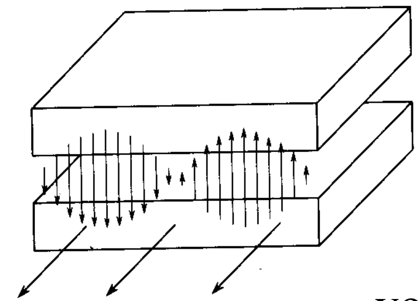


$\Phi_J = 1/2\ \Phi_o$

$\varphi = \pi \frac{z}{d} + \varphi(0)$

$\varphi\left(\frac{d}{2}\right) - \varphi\left(-\frac{d}{2}\right) = \pi$

$\varphi(0) = \frac{\pi}{2}$



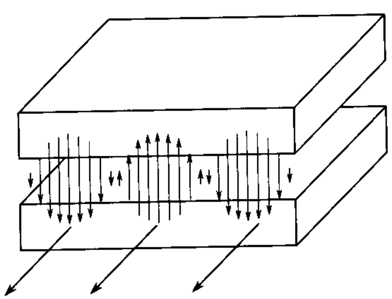
vortex

$\Phi_J = \Phi_o$

$\varphi = 2\pi \frac{z}{d} + \varphi(0)$

$\varphi\left(\frac{d}{2}\right) - \varphi\left(-\frac{d}{2}\right) = 2\pi$

$\varphi(0) = 0$

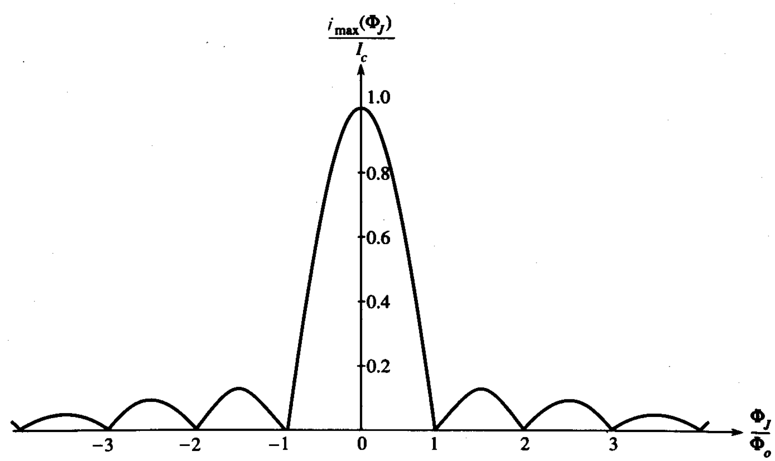


$\Phi_J = 3/2\ \Phi_o$

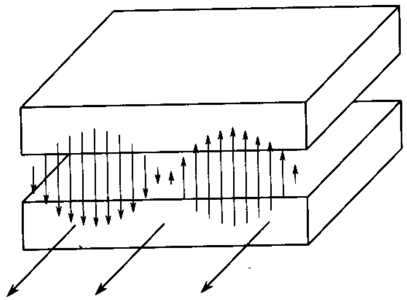
$\varphi = 3\pi \frac{z}{d} + \varphi(0)$

$\varphi\left(\frac{d}{2}\right) - \varphi\left(-\frac{d}{2}\right) = 3\pi$

$\varphi(0) = -\frac{\pi}{2}$



Vortices in Short Junctions



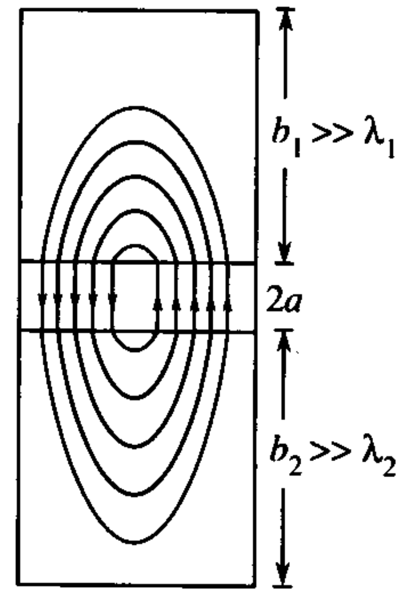
$\Phi_J = \Phi_o$

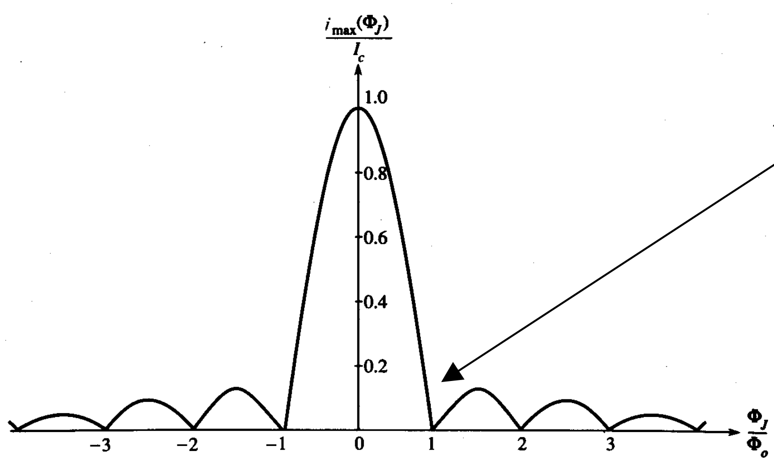
$\varphi = 2\pi \frac{z}{d} + \varphi(0)$

$\varphi\left(\frac{d}{2}\right) - \varphi\left(-\frac{d}{2}\right) = 2\pi$

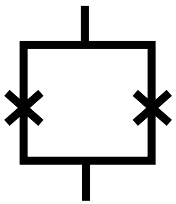
$\varphi(0) = 0$

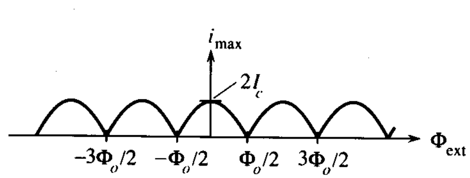
A vortex is a structure that has a 2π phase



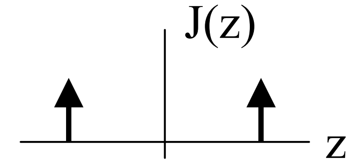


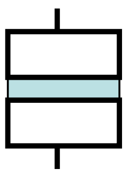
Interference Revisited (no self fields)

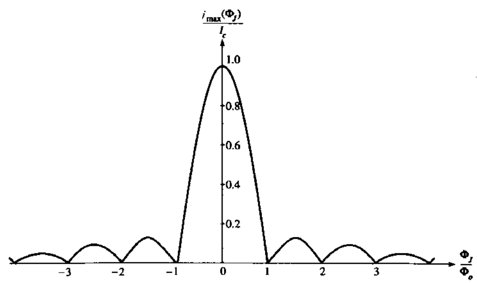




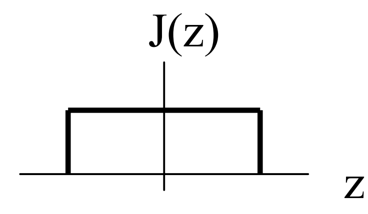
$$i_{\max} = 2I_c \left|\cos\left(\frac{\pi\Phi_{\text{ext}}}{\Phi_o}\right)\right|$$







$$i_{\max}(\Phi_J) = I_c \left|\frac{\sin\frac{\pi\Phi_J}{\Phi_o}}{\frac{\pi\Phi_J}{\Phi_o}}\right|$$



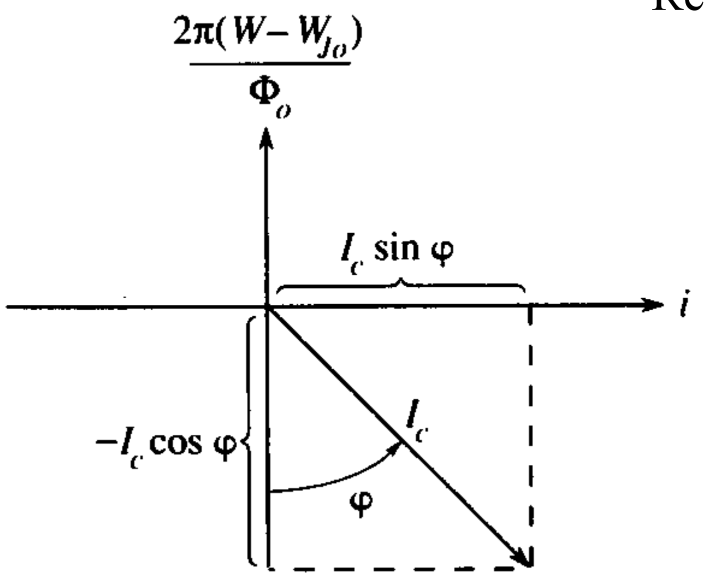
Josephson current is the Fourier transform of the current distribution.
Also in 2D. Like Fourier optics.

$$k = \frac{2\pi}{\Phi_o} B_o h_{\text{eff}}$$

$$i = w\,\text{Im}\left\{e^{j\varphi(0)} \int_{-\infty}^{\infty} J_c(z)\, e^{jkz}\, dz\right\}$$

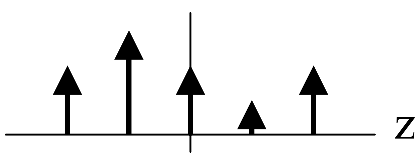
$$i_{\max} = w \left|\int_{-\infty}^{\infty} J_c(z)\, e^{jkz}\, dz\right|$$

Josephson Phasors (no self fields)



Rewrite the current-phase relation as

$$i(t) = w\,\mathrm{Im}\left\{\int_{-\infty}^{\infty} J_c(z)\, e^{j\varphi(z,t)}\, dz\right\}$$



So each junction can be considered a phasor (pendulum) whose projection is the current, and whose spatial and temporal dependences are given by

$$\frac{\partial\varphi}{\partial z} = \frac{2\pi}{\Phi_o} B_o h_{\mathrm{eff}}$$

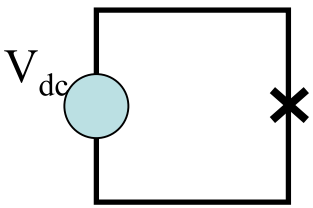
$$V(z) = \frac{\Phi_o}{2\pi}\frac{d\varphi(z)}{dt}$$

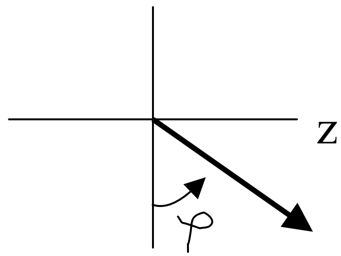
or discretely

$$\varphi_{n+1} - \varphi_n = \frac{2\pi\Phi[n]}{\Phi_o}$$

$$V[n] = \frac{\Phi_o}{2\pi}\frac{d\varphi[n]}{dt}$$

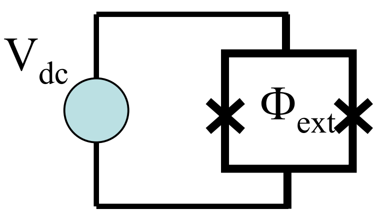
Josephson circuits (no self fields, no capacitance)

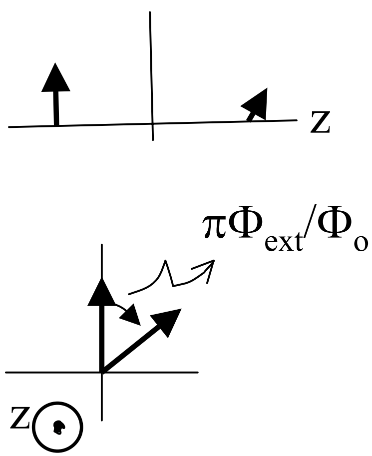




$$\varphi(t) = \varphi(0) + \frac{2\pi}{\Phi_o} V_o t$$

Each pendulum rotates

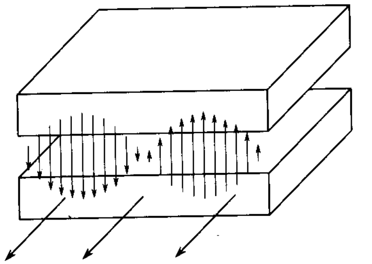




$$\varphi(t) = \varphi(0) + \frac{2\pi}{\Phi_o} V_o t + \frac{\pi \Phi_{ext}}{\Phi_o}$$

Each pendulum rotates, keeping the phase difference the same.

Josephson circuits (no self fields, no capacitance)



$\Phi_J = \Phi_o$

$\varphi = 2\pi \frac{z}{d} + \varphi(0)$

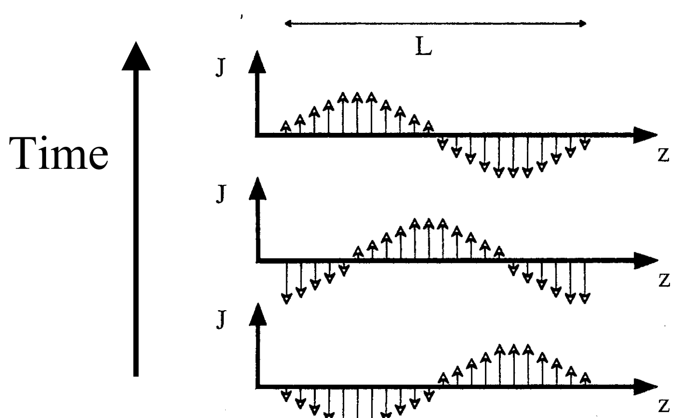
$\varphi\left(\frac{d}{2}\right) - \varphi\left(-\frac{d}{2}\right) = 2\pi$

$\varphi(0) = 0$

$$\varphi(z) = \frac{2\pi}{\Phi_o} B_o h_{\text{eff}} z + \frac{2\pi}{\Phi_o} V_o t + \varphi(0)$$

Therefore, the pattern moves with velocity $u = \dfrac{V_o}{B_o h_{\text{eff}}}$

The voltage can be written as



$$V_o = B_o h_{\text{eff}} u = \frac{d}{dt}\Phi_v = \Phi_o \frac{d}{dt} n_v$$

The number of "vortices", that is, structures that have a 2π phase

Long Josephson Junction (self fields included)

As before,

$$\frac{\partial \varphi}{\partial z} = \frac{2\pi}{\Phi_o} B_y h_{\text{eff}}$$

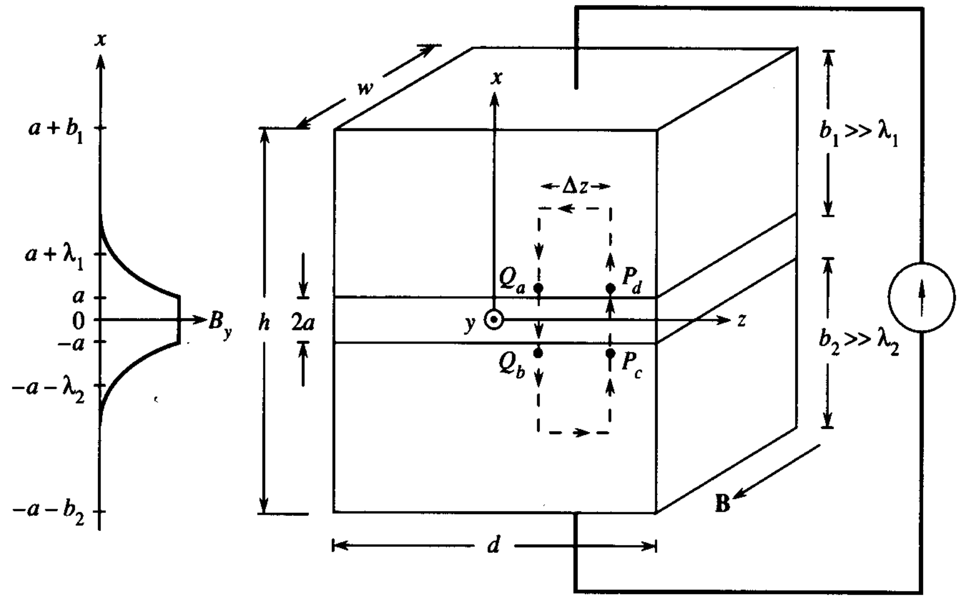
$$J_s(y,z,t) = J_c(y,z)\sin\varphi(z,t)$$

$$\nabla \times \mathbf{B} = \mu_o \mathbf{J} + \epsilon\mu_o \frac{\partial \mathbf{E}}{\partial t}$$

Assume static situation, then

$$\frac{\partial B_y(z)}{\partial z} = \mu_o J_x(z)$$

$$J_x(z) = -J_c \sin\varphi(z)$$



Therefore,

$$\frac{\partial^2 \varphi(z)}{\partial z^2} = \frac{\sin\varphi(z)}{\lambda_J^2}$$

With the *Josephson penetration depth*

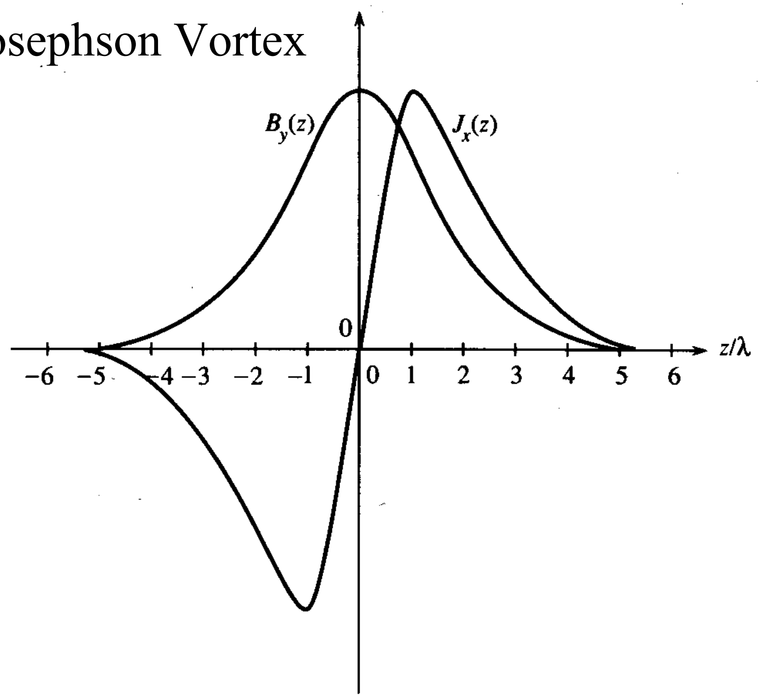
$$\lambda_J = \sqrt{\frac{\Phi_o}{2\pi\mu_o J_c h_{\text{eff}}}}$$

Josephson Penetration Depth

$$\lambda_J = \sqrt{\frac{\Phi_o}{2\pi\mu_o J_c h_{\text{eff}}}}$$

For $h_{\text{eff}} \approx 5000\,\text{Å}$, then $\lambda_J \approx 2\,\mu\text{m}$ for $J_c = 10^8\,\text{A/m}^2$, and $\lambda_J \approx 20\,\mu\text{m}$ for $J_c = 10^6\,\text{A/m}^2$.

Josephson Vortex



Energy per unit length of the vortex

$$\mathcal{E}_V = \frac{4\Phi_o J_c \lambda_J}{\pi}$$

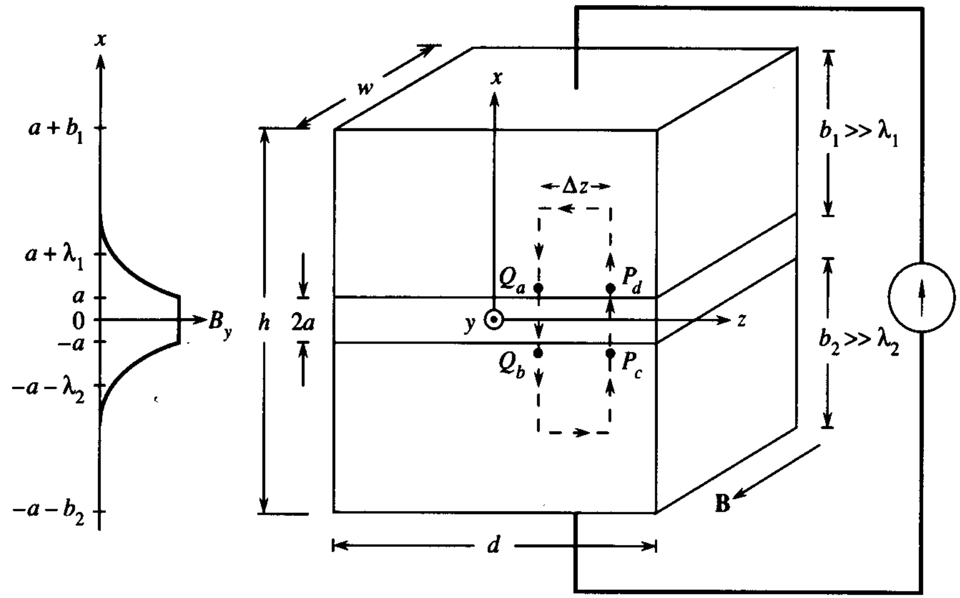
Long Josephson Junction (self fields included)

As before,

$$\frac{\partial \varphi}{\partial z} = \frac{2\pi}{\Phi_o} B_y h_{\text{eff}}$$

$$J_s(y,z,t) = J_c(y,z) \sin \varphi(z,t)$$

$$\nabla \times \mathbf{B} = \mu_o \mathbf{J} + \epsilon \mu_o \frac{\partial \mathbf{E}}{\partial t}$$



In the general time-dependent case, the sine-Gordon equation governs the phase:

$$\frac{\partial^2 \varphi}{\partial z^2} - \frac{1}{u_p^2} \frac{\partial^2 \varphi}{\partial t^2} = \frac{1}{\lambda_J^2} \sin \varphi \quad \text{where} \quad u_p = \frac{1}{\sqrt{\mu_o \epsilon}} \sqrt{\frac{a}{\lambda + a}}$$