Josephson Circuits I.

Outline

1. RCSJ Model Review
2. Response to DC and AC Drives
   • Voltage standard
3. The DC SQUID
4. Tunable Josephson Junction

October 21, 2003
JJ RCSJ Model as Circuit Element

\[ i = I_c \sin \varphi + \frac{v}{R} + C \frac{dv}{dt} \]  

and  

\[ v = \frac{\Phi_o}{2\pi} \frac{d\varphi}{dt} \]

Therefore,

\[ i = I_c \sin \varphi + \frac{\Phi_o}{2\pi R} \frac{d\varphi}{dt} + C \left( \frac{\Phi_o}{2\pi} \right)^2 \frac{d^2\varphi}{dt^2} \]
DC Current Drive

A. Static Solution: \[ \varphi = \sin^{-1} \frac{i}{I_c} \quad \text{for} \quad i \leq I_c \]

B. Dynamical Solution

\[ \beta_c = \frac{\tau_{RC}}{\tau_J} = \frac{R^2 C}{L_J} = \frac{2\pi I_c R^2 C}{\Phi_o} \]
The voltage source is DC with \( v=V_0 \), so that

\[
\varphi(t) = \varphi(0) + \frac{2\pi}{\Phi_0} V_0 t
\]

The resulting current across the JJ is ac

\[
i_J = I_c \sin \left( \frac{2\pi}{\Phi_0} V_0 t + \varphi(0) \right)
\]

The current across the resistor is dc

\[
i_R = \frac{V_0}{R}
\]

The total current is then

\[
i = \frac{V_0}{R} + I_c \sin \left( \frac{2\pi}{\Phi_0} V_0 t + \varphi(0) \right)
\]

\[
\langle i \rangle = \frac{V_0}{R}
\]
AC Voltage Drive

The voltage source $v(t) = V_o + V_s \cos \omega_s t$

Then the gauge-invariant phase is

$$\phi(t) = \phi(0) + \frac{2\pi}{\Phi_o} V_o t + \frac{2\pi V_s}{\Phi_o \omega_s} \sin \omega_s t$$

The current across the resistor is

$$i_R(t) = \frac{V_o}{R} + \frac{V_s}{R} \cos \omega_s t$$

The resulting current across the JJ is

$$i_J = I_c \sin \left( \phi(0) + \frac{2\pi}{\Phi_o} V_o t + \frac{2\pi V_s}{\Phi_o \omega_s} \sin \omega_s t \right)$$

The current across the capacitor is

$$i_C(t) = -C V_s \omega_s \sin \omega_s t$$

The total current is then

$$i(t) = i_R + i_J + i_C$$
AC Voltage Drive

\[ \langle i \rangle = \frac{V_o}{R} + I_c \sum_{n=-\infty}^{+\infty} (-1)^n J_n \left( \frac{2\pi V_o}{\Phi \omega_s} \right) \sin \varphi(0) \delta_{2\pi f_j, n\omega_s} \]
AC Voltage vs Current Drives

\[ V_s \cos \omega_s t \]

\[ V_o \]

\[ I_c \sin \phi \]

\[ I_R \]

\[ I_{cap} \]

\[ I_s \cos \omega_s t \]

\[ I_c \sin \phi \]

\[ I_J \]

\[ I_R \]

\[ I_{cap} \]

\[ <i> = \frac{V_o}{R} \]

\[ 2\pi V_o \]

\[ \Phi_s \omega_s \]

\[ i(mA) \]

\[ <i>(mV) \]
10 V conventional Josephson voltage standard chip. The chip is 1 cm x 2 cm and contains 20,208 series connected Nb-AlOx-Nb junctions.

HYPRES is the only commercial manufacturer of the superconducting integrated circuit used in Primary Voltage Standard Systems. HYPRES chips are used in the primary voltage standards in national laboratories around the world including Italy, France, United Kingdom, Australia, China, Malaysia, Japan, England, Canada, Norway, United States, Netherlands and Mexico. The HYPRES Josephson Junction Array Voltage Standard circuits provide the ultimate accuracy for realizing and maintaining the SI Volt.

**Features/Specifications**

- Niobium/Aluminum Oxide/Niobium, SiO2 dielectric, Niobium wiring technology.
- All Niobium technology. Refractory. Impervious to moisture and thermal cycling.
- 20,208 Josephson junctions (10 V chip) 3,660 Josephson junctions (1 V chip)
- 18 x 38 micrometers junction area.
- Installed in a FR-4 epoxy glass mount.
- RF input WR-12 waveguide flange.
- RF Distribution - 16 way parallel x 1263 cells in series (10 V) - 4 way parallel x 915 cells in series (1 V)
- Designed for a frequency range of 72-78 GHz
- Operating temperature of 4.2 K
- Common DC terminal resistance is < 1 Ohm - typical
- Approximately 10 mW operating power at the input flange for 10 V chip (2 mW for 1 V)
- -11V to +11 V range for 10 V chip.
- -2.5 V to + 2.5 V range for 1 V chip.
- Stability time is typically 1 hour for the 10 V chip, 5 hours for the 1V chip
- 0.005 PPM accuracy at 10 V (10 V chip)
- 0.05 PPM accuracy at 1V (1V chip)
- Calibration certificate supplied with each chip.
- Two (2) year warranty

http://www.hypres.com/
Parametric Inductor

\[ i(t) = I_c \sin \varphi(t) + \frac{v(t)}{R} + C \frac{d}{dt} v(t) \]

Take the time derivative of the currents, and for the Josephson term:

\[ \frac{d}{dt} i_J(t) = \left[ \frac{2\pi I_c}{\Phi_0} \cos \varphi(t) \right] v(t) \]

The parameteric (time-dependent) inductance can be defined as

\[ L_J = \frac{\Phi_0}{2\pi I_c \cos \varphi(t)} \]

*On the zero-voltage branch, for \( I_s \ll I_0 \) and \( I_0 + I_s \ll I_c \), then \( I_0 \approx I_c \sin \phi \), so that

\[ L_J \approx \frac{\Phi_0}{2\pi I_c \sqrt{1 - \left(\frac{I_0}{I_c}\right)^2}} \]
Inductance along $V=0$ branch

$$L_J \approx \frac{\Phi_o}{2\pi I_c \sqrt{1 - (I_o/I_c)^2}}$$

$$L\left[\frac{\Phi_o}{2\pi I_c}\right]$$
Our goal is to show that the DC SQUID circuit in (a) is equivalent to the circuit for a single junction with an effective $I_c$ and an effective resistance. The inductance of the SQUID loop in (a) is considered negligible.

In fact, we will also show later that there is an equivalent statement even for the underdamped case.
DC SQUID Equivalent Circuit

\[ i = i J_1 + i J_2 + i R_1 + i R_2 \]
\[ = I_{c1} \sin \varphi_1 + I_{c1} \sin \varphi_2 + \frac{v}{R_1} + \frac{v}{R_2} \]

Use flux quantization, \( \varphi_2 - \varphi_1 = \frac{2\pi \Phi}{\Phi_o} \) and the fact that the junctions are identical

\[ i = 2I_{c1} \cos \left( \frac{\pi \Phi_{\text{ext}}}{\Phi_o} \right) \sin \left( \varphi + \frac{\pi \Phi_{\text{ext}}}{\Phi_o} \right) + \left( \frac{1}{R_1} + \frac{1}{R_2} \right)v(t) \]

\[ I = I_c \sin \varphi + \frac{\Phi_0}{2\pi} \frac{d}{dt} \varphi + \frac{1}{R} \]
DC SQUID Equivalent Circuit

\[ i = I_c \sin \varphi + \frac{1}{R} \frac{\Phi_o}{2\pi} \frac{d\varphi}{dt} \quad \text{with} \quad I_c = 2I_{c1} \cos \left( \frac{\pi \Phi_{ext}}{\Phi_o} \right) \]

Therefore, for this overdamped equivalent circuit, for \( i > I_c \)

\[ \langle v(t) \rangle = iR \sqrt{1 - \left[ \frac{2I_{c1}}{i} \cos \left( \frac{\pi \Phi_{ext}}{\Phi_o} \right) \right]^2} \]

\[ \langle v(t) \rangle = iR \sqrt{1 - \left( \frac{i_{\text{max}}}{i} \right)^2} \]
DC SQUID Voltage Modulation

From Van Duzer and Turner, Figure 5.11a, page 272
DC SQUID Voltage Modulation

RC-Shunted SQUID

I_{bias} kept slightly above I_C
Φ_{bias} = 0.75Φ_o for sensitivity

Goal is to maximize: \( \frac{dV}{d\Phi} \approx \frac{R}{L} \)
Assume that $L_s > L_J = \Phi_0/2 \pi I_c(\Phi_{\text{ext}})$, then the range of modulation of the current is about

$$\delta i = i^+_{\text{max}} - i^-_{\text{max}} \approx \Phi_0/L$$

The sensitivity of the output voltage to the input flux is

$$\frac{\delta \langle v \rangle}{\delta \Phi_{\text{ext}}} \approx \frac{R_D \delta i}{\frac{1}{2} \Phi_0} = \frac{2R_D}{L}$$

With $R_D \sim R = 1 \text{ Ohm}$ and $L \sim 1 \text{ nH}$, then the sensitivity is about one microvolt per flux quantum, so that small fractions of a flux quantum can be measured.
DC SQUID and Thermal Noise

Assume again that \( L_s > L_J = \Phi_0/2 \pi I_c(\Phi_{\text{ext}}) \), then to prevent thermal noise from effecting the SQUID’s performance, one needs

Energy stored in the SQUID >> Thermal Energy

\[
\frac{1}{2} \frac{\Phi_0^2}{L} \gg \frac{1}{2} k_B T
\]

Therefore, \( L_s < 1 \) nH at 4 K, and 0.1 nH at 100 K.
Equivalent Tunable Junction

\[ I_c = 2I_{c1} \cos \left( \frac{\pi \Phi_{\text{ext}}}{\Phi_0} \right) \]

\[ \frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} \]

\[ C = C_1 + C_2 \]

\[ \beta_c = \frac{\tau_{RC}}{\tau_J} = \frac{R^2C}{L_J} = \frac{2\pi I_c R^2 C}{\Phi_0} \]

with \( L_{\text{loop}} \ll L_J \)
Tunable Inductance along $V=0$ branch

$$L_J \approx \frac{\Phi_0}{2\pi I_c \sqrt{1 - (I_0/I_c)^2}}$$

$$L \left[ \frac{\Phi_0}{2\pi I_c} \right]$$

$$\Phi [\text{in } \Phi_0]$$
Three-Junction Loop Measurements

Three-junction Loop
Jct. Size ~ 0.45\(\mu\)m, 0.55\(\mu\)m
Loop size ~16x16\(\mu\)m\(^2\)
L\(_{3\text{-junction}}\) ~ 30pH
I\(_c\) ~ 1 & 2\(\mu\)A
E\(_J/\)E\(_c\) ~ 350 & 550

DC SQUID
Shunt capacitors ~ 1pF
Jct. Size ~ 1.1\(\mu\)m
Loop size ~20x20\(\mu\)m\(^2\)
L\(_{\text{SQUID}}\) ~ 50pH
I\(_c\) ~ 10 & 20\(\mu\)A
M ~ 35pH
J\(_c\) ~ 350 & 730A/cm\(^2\)

- Measure switching current of DC SQUID
- Vary external flux, temperature and SQUID ramp rate
Thermal Activation of Nb Persistent Current Qubit

Device A:
- $J_c = 365 \text{ A/cm}^2$
- $E_J = 2400 \mu\text{eV}$
- $E_J/E_c = 380$
- Jct. width = 0.563 $\mu$m
- $\alpha = 0.613$
- $Q = 1.0 \times 10^6$

Device B:
- $J_c = 730 \text{ A/cm}^2$
- $E_J = 4000 \mu\text{eV}$
- $E_J/E_c = 560$
- Jct. width = 0.529 $\mu$m
- $\alpha = 0.589$
- $Q = 1.2 \times 10^6$

Highest Q of any submicron Nb junction
Coupling between qubits

Equivalent, tunable 3rd junction