Perfect Conductivity   Lecture 2

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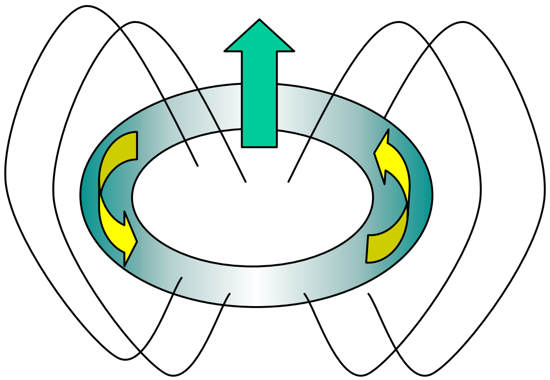
Outline

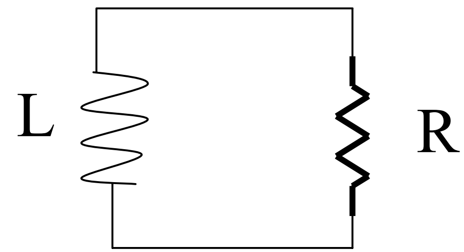
1. **Persistent Currents**
2. **Parts of a Physical Theory**
3. **Circuits and Time Constants**
4. **Distributive Systems and Time constants**

   A.Quasistatics

   B.MagnetoQuasiStatics (MQS)

Persistent Currents





If the field is turned off, then

$$I(t) = I|_{t=0}\, e^{-t/\tau_{LR}}$$

The time constant $\tau_{LR} = L/R$

If the loop is made out of a superconductor,

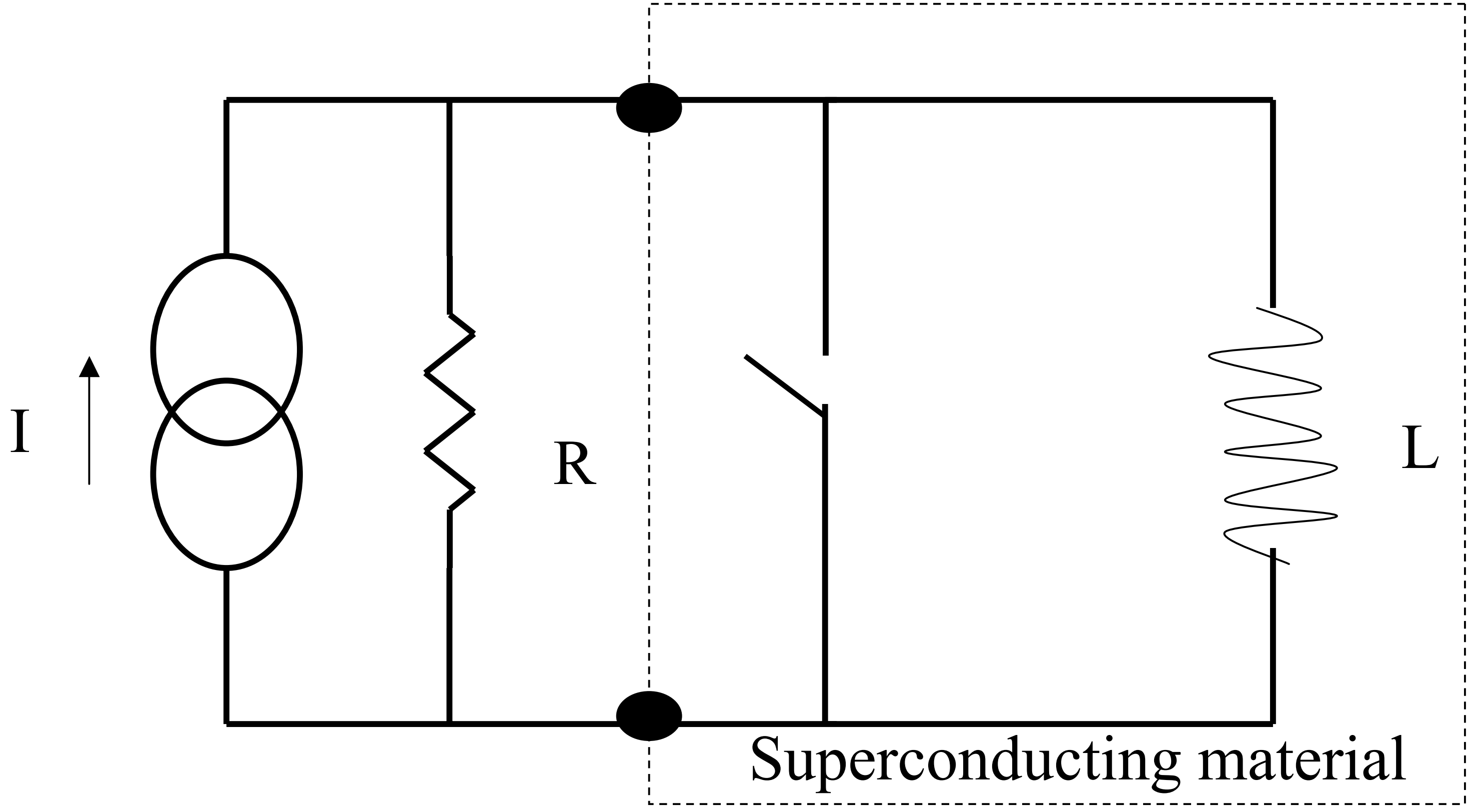
$$\lim_{R\to 0} \tau_{LR} \to \infty \qquad I(t) = I|_{t=0} \qquad \text{for } t \geq 0.$$

Experimentally the dc resistivity of a superconductor is at least as small as $10^{-25}\,\Omega\text{-m}$. The superconducting state is "truly" zero dc resistance.

Perfect Conductivity: $t \ll \tau_{RL} = L/R$, sytem looks like R is zero

Superconductivity: for all time, R is zero

Charging up a superconducting loop



This Persistent Mode is the basis of MRI magnets, SMES, flux memory….

Parts of a Physical Theory

1. Governing Laws:

   Maxwell's Equations, Newton's equations,

2. Constitutive Laws:

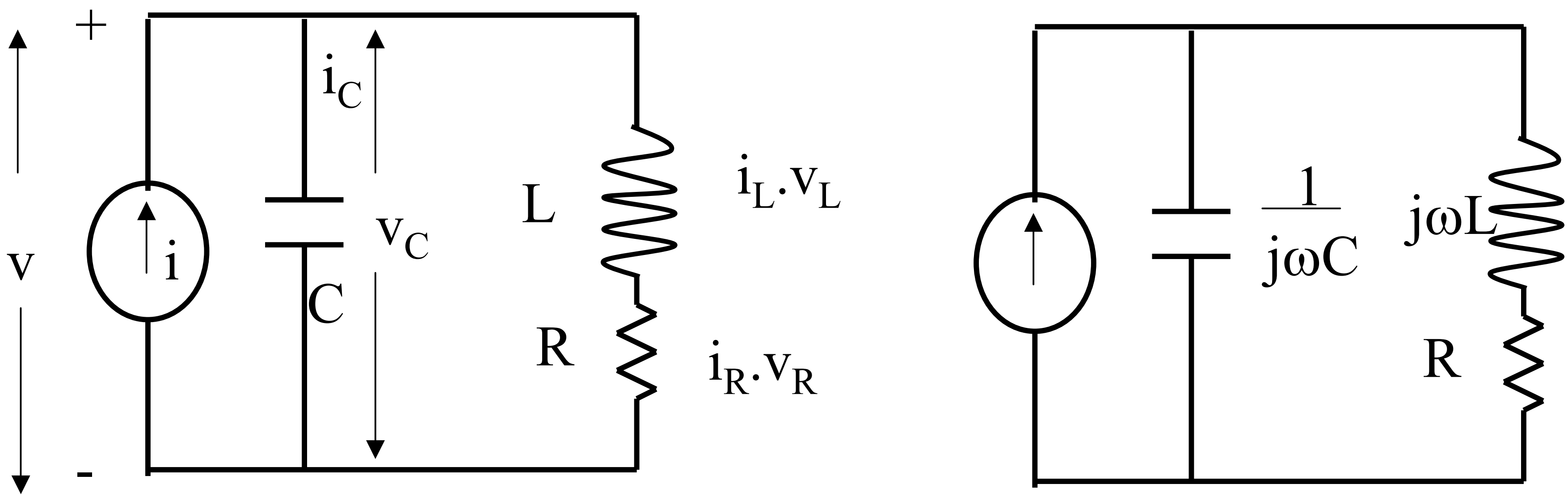
   Models of the system

   like ohm's law,

3. Summary Relations:

   Transfer functions, Dispersion relations

LRC Circuit



1. Governing Equations

Current conservation: $i = i_C + i_L \qquad i_L = i_R$

Energy Conservation $v = v_C = v_R + v_L$

2. Constitutive Relations

For the resistor

$$v_R = i_R \, R \, ,$$

for the inductor as

$$v_L = L \, \frac{d}{dt} \, i_L \, ,$$

and for the capacitor as

$$i_C = C \, \frac{d}{dt} \, v_C \, ,$$

$$\widehat{\imath} \equiv \; |i| \, e^{j\phi}$$

and

$$v = \text{Re} \left\{ \widehat{v} \, e^{j\omega t} \right\}$$

so that

$$v = \text{Re} \left\{ \widehat{v} \, e^{j\omega t} \right\}$$

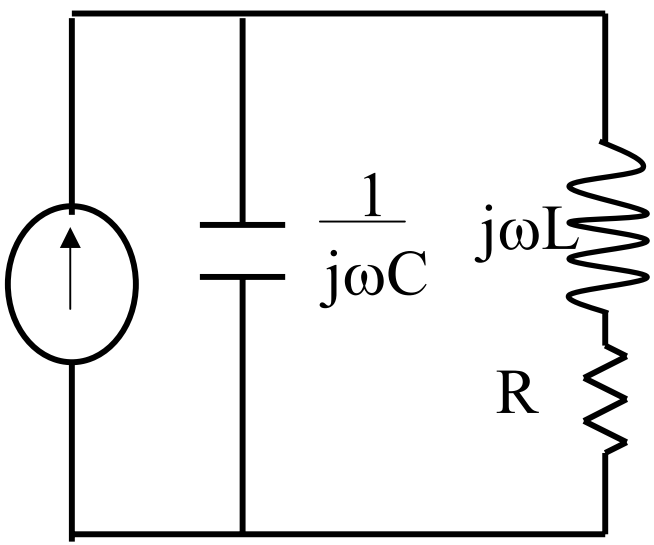
$$\widehat{v}_R = \widehat{\imath}_R \, ,$$

$$\widehat{v}_L = j\omega L \, \widehat{\imath}_L$$

$$\widehat{\imath}_C = j\omega C \, \widehat{v}_C$$

3. Summary Relation

$$Z(\omega) = R\left(\frac{1 + j\omega\tau_{RL}}{\left(1 - (\omega\tau_{LC})^2\right) + j\omega\tau_{RC}}\right) .$$



the *inductive time constant:*

$$\tau_{RL} \equiv \frac{L}{R} , \qquad (1)$$

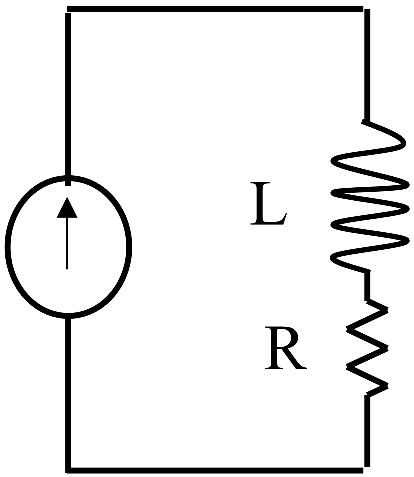
the *capacitive time constant:*

$$\tau_{RC} \equiv RC , \qquad (2)$$

and the *coupling time constant*:

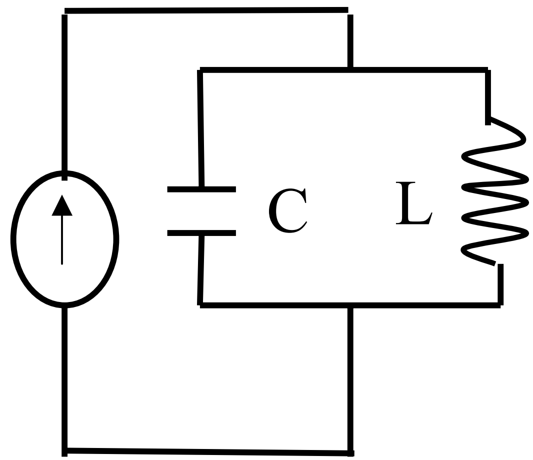
$$\tau_{LC} \equiv \sqrt{LC} = \sqrt{\tau_{RL}\,\tau_{RC}} . \qquad (3)$$

Simpler Circuits and Time Constants



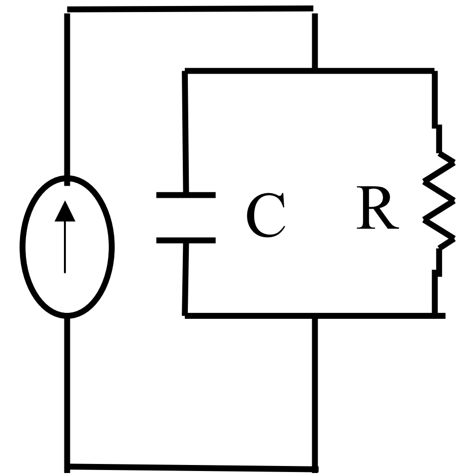
$$\tau_{RL} \equiv \frac{L}{R}$$

Energy stored in inductor



$$\tau_{LC} \equiv \sqrt{LC}$$

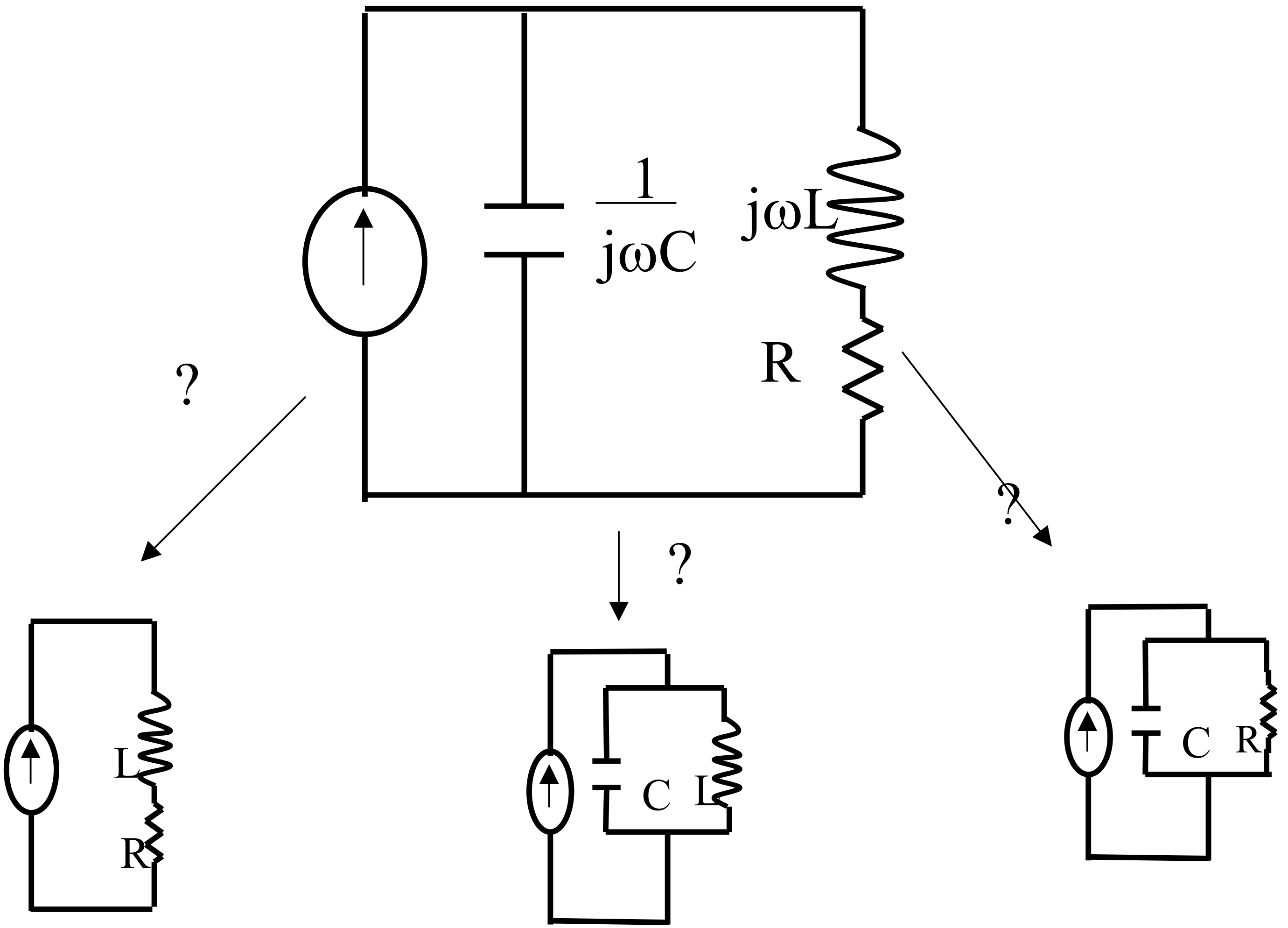
Resonant transfer of energy between L and C



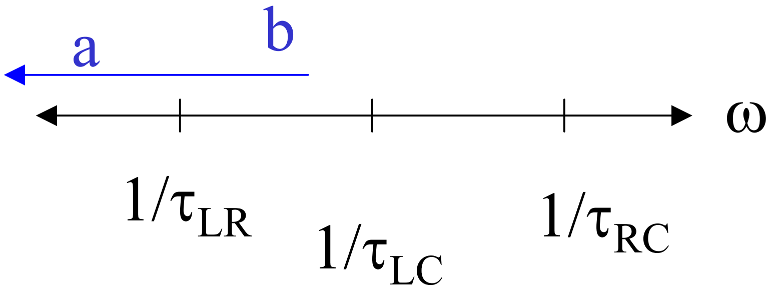
$$\tau_{RC} \equiv RC$$

Energy stored in capacitor

Reducing the Circuit to a simpler form ?



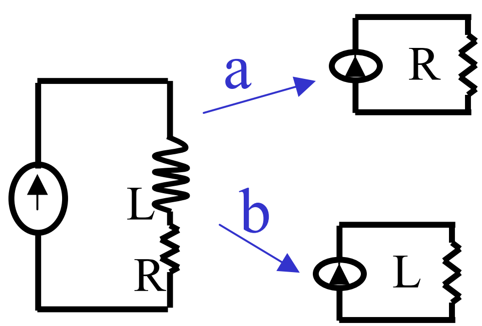
Order of time constants

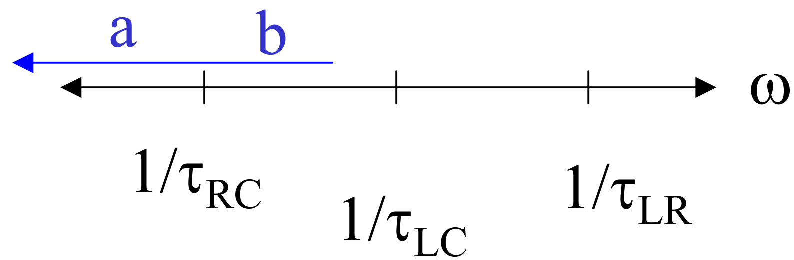


$\tau_{LR} > \tau_{RC}$ Low R $R < \sqrt{\dfrac{L}{C}}$

$$\lim_{\substack{\omega\tau_{LC} \ll 1 \\ \tau_{RC} < \tau_{RL}}} Z(\omega) \approx R\,(1 + j\omega\tau_{RL})$$

Low frequency circuit

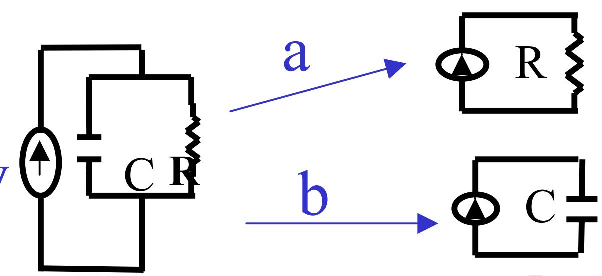




$\tau_{LR} < \tau_{RC}$ High R $R > \sqrt{\dfrac{L}{C}}$

$$\lim_{\substack{\omega\tau_{LC} \ll 1 \\ \tau_{RL} < \tau_{RC}}} Z(\omega) \approx R\left(\frac{1}{1 + j\omega\tau_{RC}}\right)$$

Low frequency circuit



Moral of time constants

If you know what frequency range you want to study or what physics dominates the problem,

then you can solve a simpler problem.

Useful, especially in more complex situations.

Distributed Systems

1. Governing Equations: Maxwell's Equations

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$ Faraday's Law

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J}$$ Ampere's Law

$$\nabla \cdot \mathbf{D} = \rho$$ Gauss Law

$$\nabla \cdot \mathbf{B} = 0$$ Gauss' Magnetic Law

Conservations laws

$$\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = 0$$ Charge conservation

Also Poynting's

Distributied systems con't

2. Constitutive Relations

$$\mathbf{B}(\mathbf{r},\omega) = \mu(\omega)\,\mathbf{H}(\mathbf{r},\omega)$$

$$\mathbf{D}(\mathbf{r},\omega) = \epsilon(\omega)\,\mathbf{E}(\mathbf{r},\omega)$$

Local in space,
linear time invariant

$$\mathbf{J}(\mathbf{r},\omega) = \sigma(\omega)\,\mathbf{E}(\mathbf{r},\omega) \longrightarrow$$ Ohm's Law

3. Summary relations

Complex: Search first for first order in time approximation

Quasistatic Limit

$$\ell \ll \lambda_{em} = \frac{2\pi c}{\omega}$$

Length scale of system

Wavelength of E&M wave

Speed of light

Frequency (angular)

If the dimensions of a structure are much less than the wavelength of an electromagnetic field interacting with it, the coupling between the associated electric and magnetic fields is weak and a quasistatic approximation is appropriate.

Time Constants

$$\tau_{em} \equiv \frac{\ell}{c} = \ell\sqrt{\mu\epsilon}$$

Electromagnetic coupling time

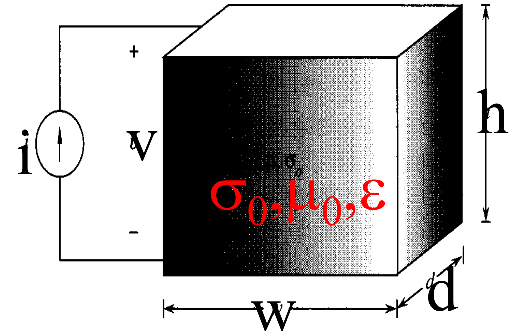
$$\tau_e \equiv \frac{\epsilon}{\sigma_o}$$

Charge relaxation time

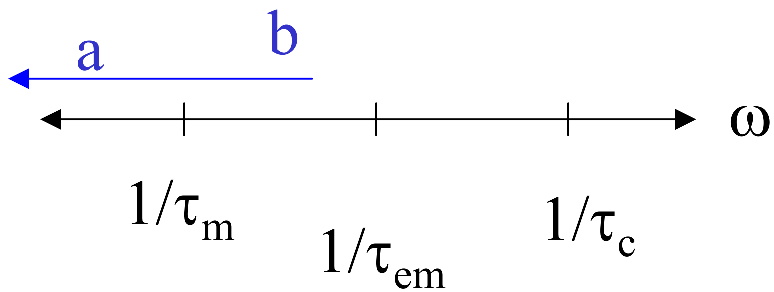
$$\tau_m \equiv \mu\sigma_o\ell^2$$

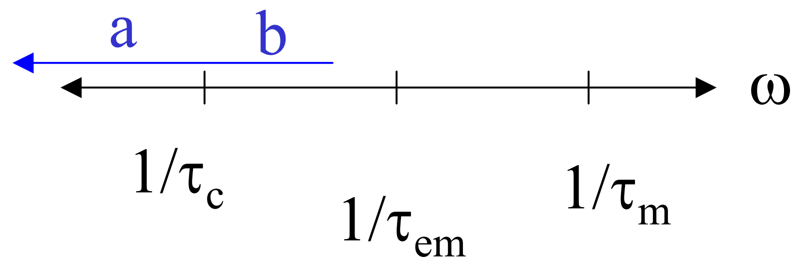
Magnetic diffusion time

$$\tau_{em} = \sqrt{\tau_e\tau_m}$$



Order of time constants





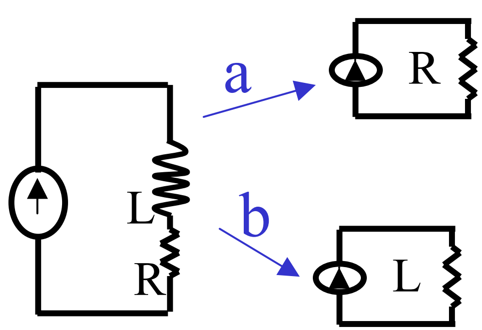
$\tau_m > \tau_c$ High conductivity $\sigma_o > \frac{1}{\ell}\sqrt{\frac{\epsilon}{\mu}}$

MQS

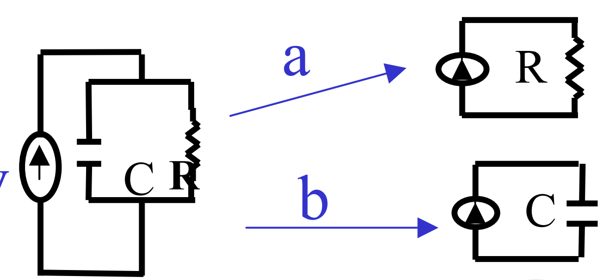
$\tau_m < \tau_e$ Low conductivity $\sigma_o < \frac{1}{\ell}\sqrt{\frac{\epsilon}{\mu}}$

EQS

Low frequency circuit



Low frequency circuit



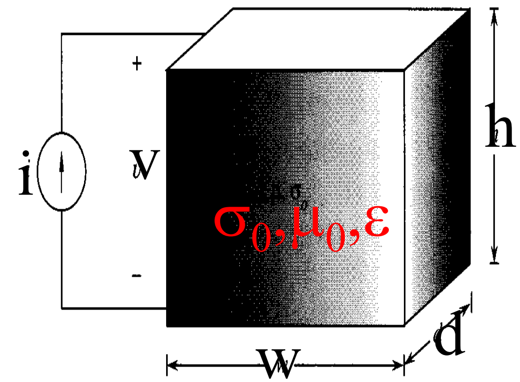
MagnetoQuasiStatics

$$\nabla \times \mathbf{H} = \mathbf{J}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \cdot \mathbf{J} = 0$$

Solve first



$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

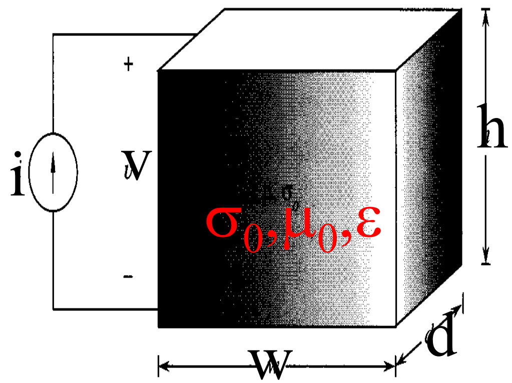
Solve for **E** once **B** is found

Boundary conditions:

$$n \times (H_2 - H_1) = K \qquad n \cdot (B_2 - B_1) = 0 \qquad n \cdot (J_2 - J_1) = 0$$

MQS: Magnetic Diffusion Equation

For a metal **B** = μ_0 **H**, **D** = ε_0 **E** and **J** = σ_0**E**, so that



$$\left(\mu\sigma_o \frac{\partial}{\partial t} - \nabla^2\right)\mathbf{H} = 0$$

Magnetic Diffusion Equation