Lecture 5: Classical Model of a Superconductor

Outline

1. **First and Second London Equations**
2. **Examples**
   - Superconducting Slab
   - Bulk Sphere
3. **Non-simply connected superconductors**
   - Hollow cylinder
   - Superconducting circuits
     - o DC flux transformer
     - o Superconducting memory loop
     - o Magnetic monopole detector
4. **Two Fluid Model**

Superconductor: Classical Model

$$\mathbf{E} = \frac{\partial}{\partial t}(\Lambda \mathbf{J})$$ first London Equation

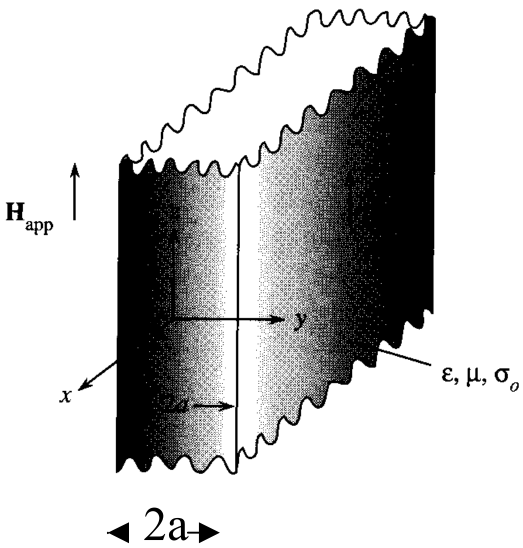
$$\nabla \times (\Lambda \mathbf{J}) = -\mathbf{B}$$ second London Equation

$$\Lambda \equiv \frac{m^\star}{n^\star (q^\star)^2} \qquad \lambda \equiv \sqrt{\frac{\Lambda}{\mu_o}}$$ penetration depth

When combined with Maxwell's equation in the MQS limit

$$\left(\frac{1}{\lambda^2} - \nabla^2\right)\mathbf{H} = 0$$

Superconducting Infinite Slab



$$\mathbf{H}_{\mathsf{app}} = \mathsf{Re}\left\{\hat{H}_o\, e^{j\omega t}\right\}\mathbf{i}_z$$

$$\left(\frac{1}{\lambda^2} - \nabla^2\right)\mathbf{H} = 0$$

Let $\mathbf{H}(\mathbf{r}, t) = \mathsf{Re}\left\{\hat{H}(y)\, e^{j\omega t}\right\}\mathbf{i}_z$

Therefore,

$$\left(\frac{1}{\lambda^2} - \frac{d^2}{dy^2}\right)\hat{H}(y) = 0$$

and

$$\hat{H}(y) = C\cosh(y/\lambda)$$

Boundary Conditions demand

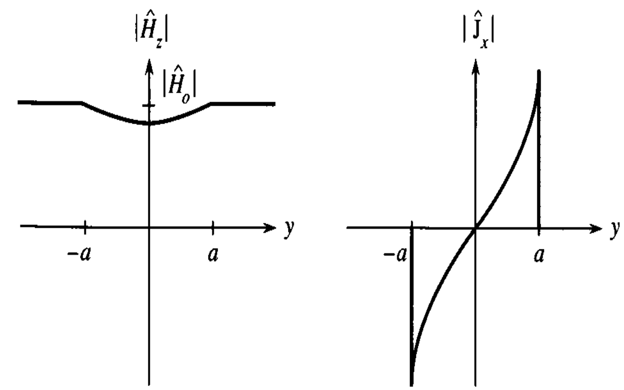
$$H_z(a) = H_z(-a) = C\cosh(a/\lambda) = \hat{H}_o$$

Fields and Currents for |y|< a

$$\mathbf{H} = \mathrm{Re}\left\{\hat{H}_o \frac{\cosh(y/a)}{\cosh(a/\lambda)}\, e^{j\omega t}\right\}\mathbf{i}_z \quad \mathbf{J} = \mathrm{Re}\left\{\frac{\hat{H}_o}{\lambda}\, \frac{\sinh(y/\lambda)}{\cosh(a/\lambda)}\, e^{j\omega t}\right\}\mathbf{i}_x$$

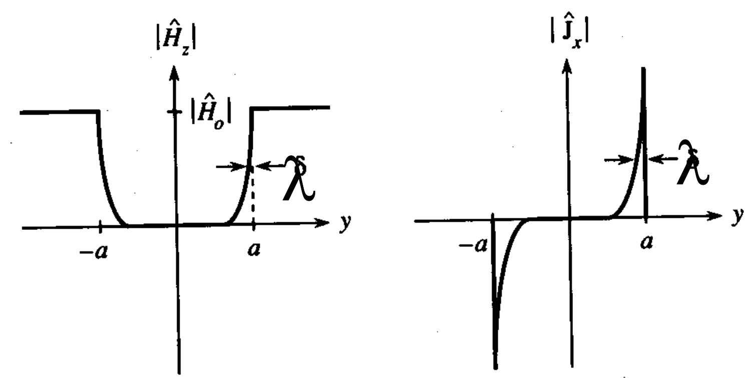
Thin film limit

$$a \ll \lambda$$

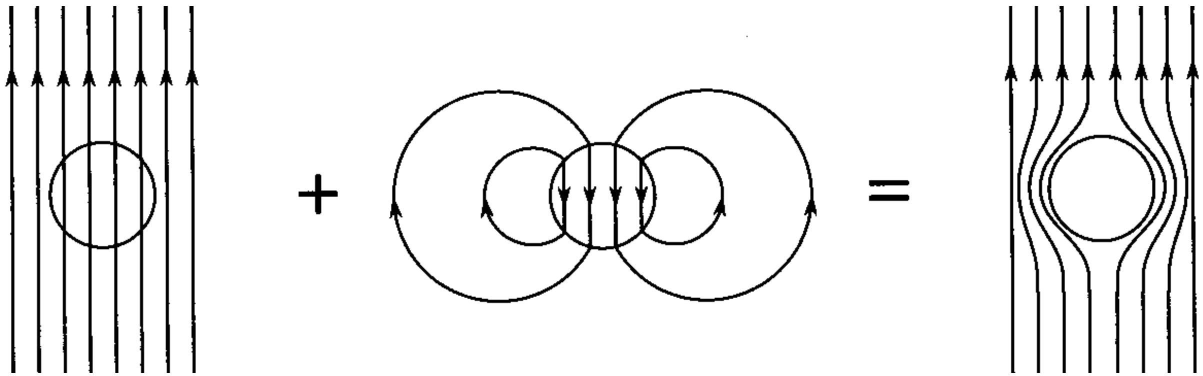


Bulk limit

$$a \gg \lambda$$



Superconducting Sphere: Bulk Approximation R >> λ

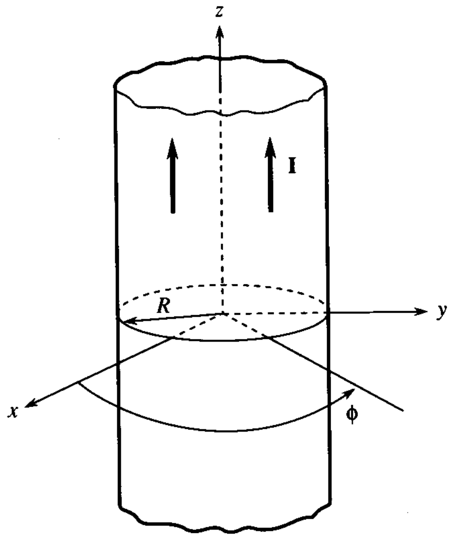


$$\mathbf{H}(r \le R) = 0$$

$$\mathbf{H}(r \ge R) = \mathrm{Re}\left\{\hat{H}_o\left(1 - \left(\frac{R}{r}\right)^3\right)\cos\theta\, e^{j\omega t}\right\}\mathbf{i}_r - \mathrm{Re}\left\{\hat{H}_o\left(1 + \frac{1}{2}\left(\frac{R}{r}\right)^3\right)\sin\theta\, e^{j\omega t}\right\}\mathbf{i}_\theta\,.$$

$$\mathbf{K}(r = R) = -\mathrm{Re}\left\{\frac{3}{2}\hat{H}_o \sin\theta\, e^{j\omega t}\right\}\mathbf{i}_\phi$$

Current along a cylinder: bulk superconductor



$$\mathbf{I} = \mathrm{Re}\left\{\hat{I}_o\, e^{j\omega t}\right\}\mathbf{i}_z$$

$$\mathbf{J}(r \leq R) = \frac{\mathbf{I}}{\pi R^2}$$

The fields from Ampere's law

$$\oint_C \mathbf{H}\cdot d\mathbf{l} = \int_S \mathbf{J}\cdot d\mathbf{s}$$

Inside: $H\, 2\pi r = 0$

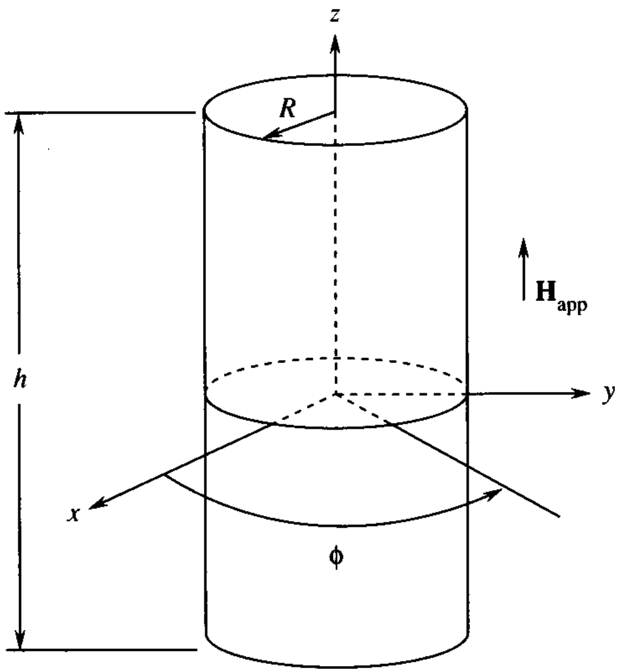
$$\mathbf{H}(r \leq R) = 0$$

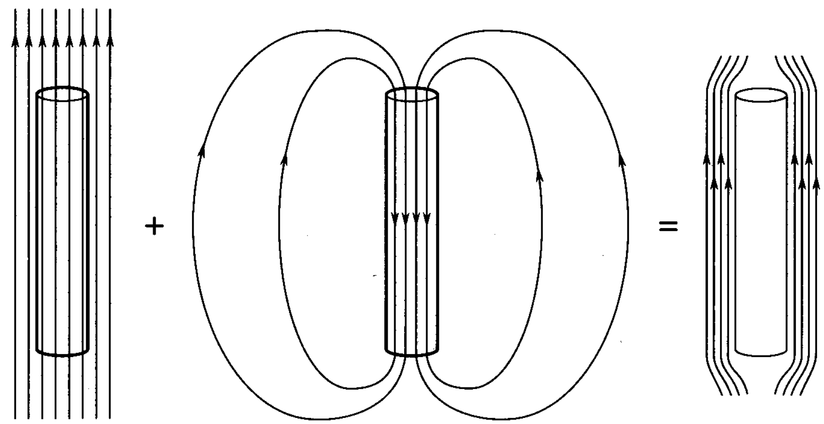
Outside: $H\, 2\pi r = I$

$$\mathbf{H}(r \geq R) = \mathrm{Re}\left\{\frac{\hat{I}_o}{2\pi r}\, e^{j\omega t}\right\}\mathbf{i}_\phi$$

Therefore, $\mathbf{K}(r = R) = \dfrac{\mathbf{I}}{2\pi R}$

Field along a cylinder: bulk superconductor





$$\mathbf{H}_{\mathsf{app}} = \mathsf{Re}\left\{\hat{H}_o\, e^{j\omega t}\right\}\mathbf{i}_z$$

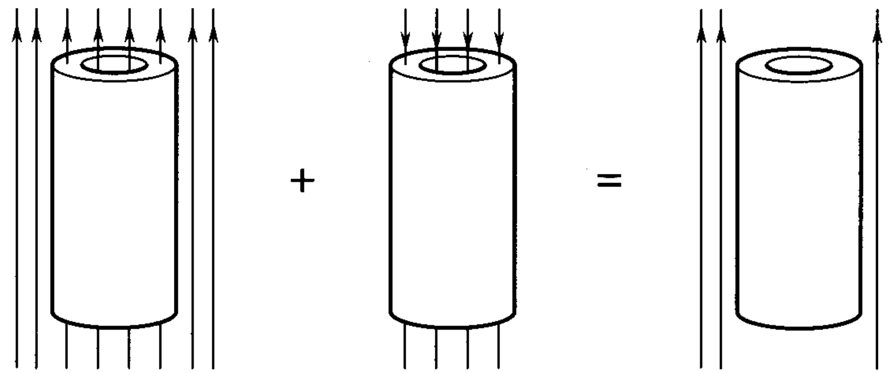
$$\mathbf{H}_{\mathsf{out}} = \mathsf{Re}\left\{\hat{H}_o\, e^{j\omega t}\right\}\mathbf{i}_z$$

$$\mathbf{H}_{\mathsf{in}} = 0$$

$$\mathbf{K}(r = R) = -H_o\,\mathbf{i}_\phi$$

Field along a hollow cylinder

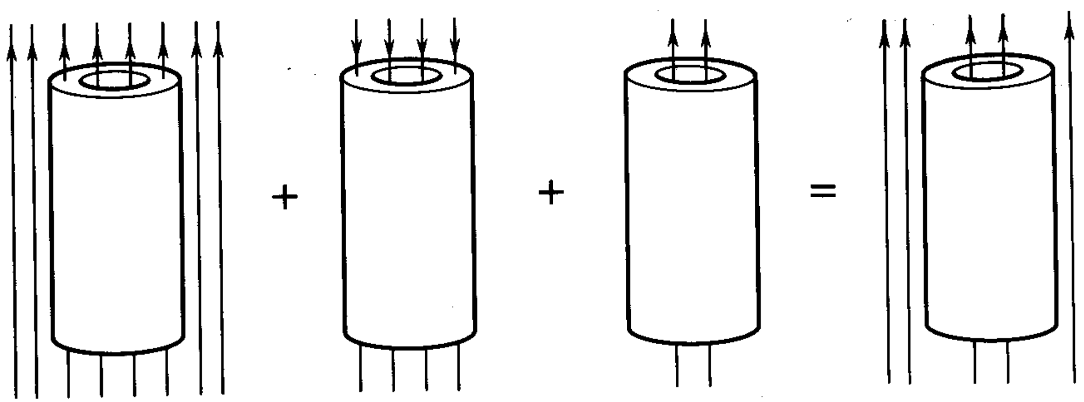
Solution 1



or

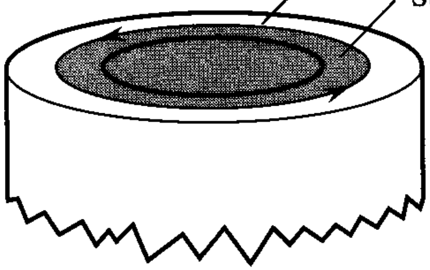
Solution 2

?



$$\mathbf{H}_{\mathrm{app}} = \mathrm{Re}\left\{ \hat{H}_o \, e^{j\omega t} \right\} \mathbf{i}_z$$

Multiply Connected Superconductor



Maxwell $$\oint_C \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt} \int_S \mathbf{B} \cdot d\mathbf{s}$$

First London $$\oint_C \mathbf{E} \cdot d\mathbf{l} = \frac{d}{dt} \oint_c \Lambda \mathbf{J} \cdot d\mathbf{l}$$

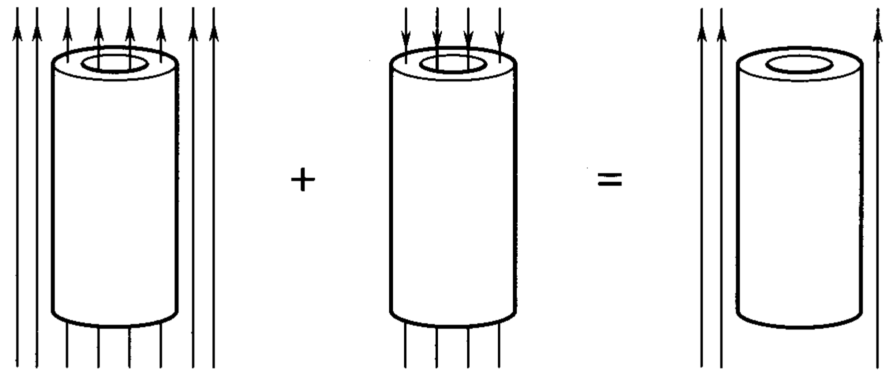
Therefore $$\frac{d}{dt}\left[\Phi + \oint_c \Lambda \mathbf{J} \cdot d\mathbf{s}\right] = 0$$

and $$\Phi + \oint_c \Lambda \mathbf{J} \cdot d\mathbf{s} = \Phi_C = \text{constant}$$

For a contour within the bulk where $\mathbf{J} = 0$, flux remains constant

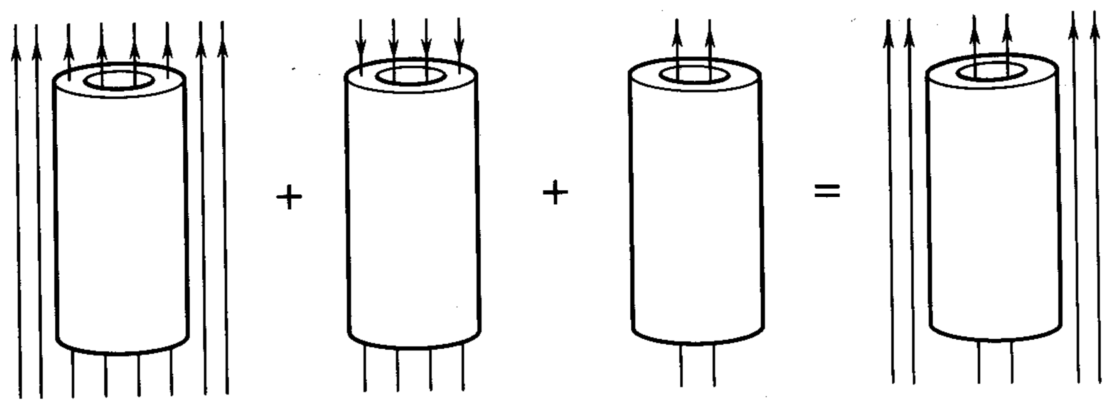
Field along a hollow cylinder

Zero Field
Initially Solution



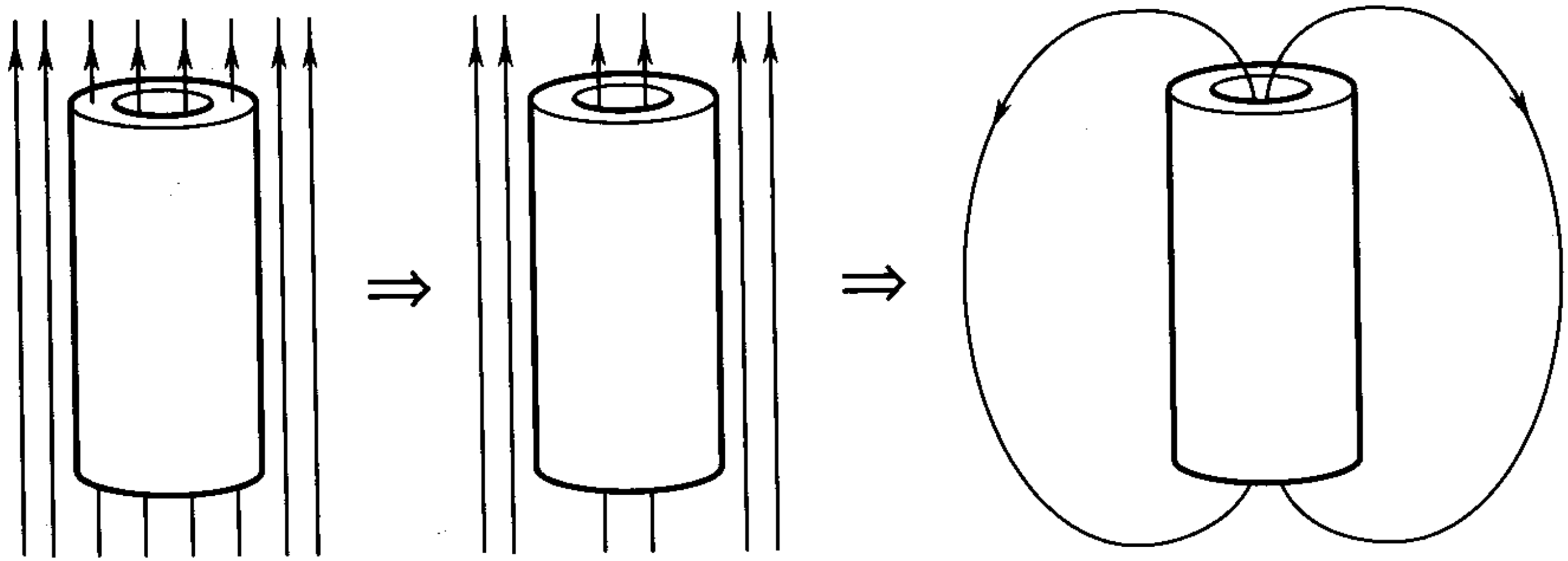
Finite Field

Initially Solution



$$\mathbf{H}_{\mathrm{app}} = \mathrm{Re}\left\{\widehat{H}_o\, e^{j\omega t}\right\}\mathbf{i}_z$$

Flux trapped in a hollow cylinder



$T > T_c$
$\mathbf{H}_{app} \neq 0$

$T < T_c$
$\mathbf{H}_{app} \neq 0$

$T < T_c$
$\mathbf{H}_{app} = 0$

Superconducting Circuits

A generalization to any closed superconducting circuit is that the total flux linkage in a circuit remains constant.

Then if a circuit has N elements that can contain flux,

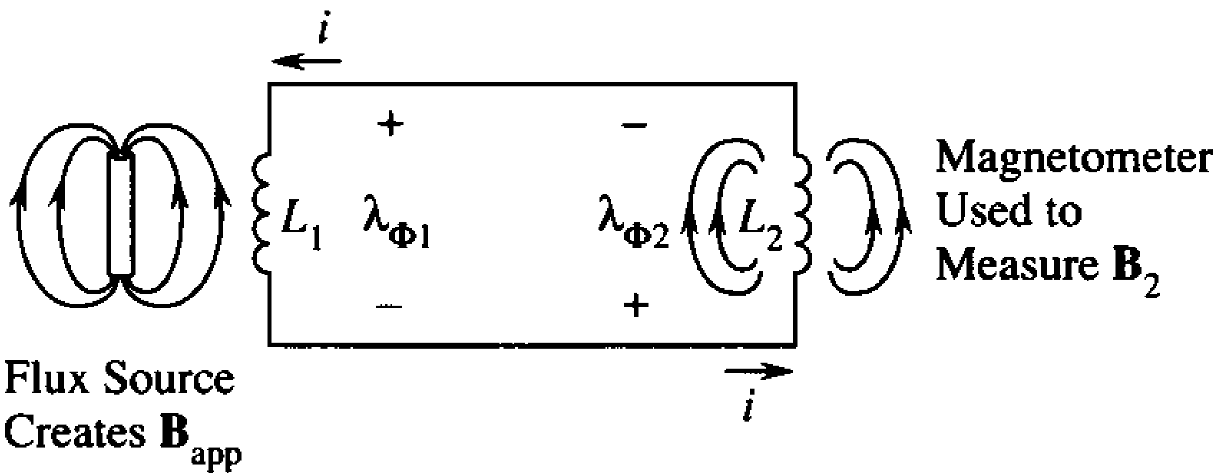
$$\lambda_{\Phi 1} + \lambda_{\Phi 2} + \ldots = \text{constant}$$

Sources of Flux linkage

$$\lambda_{\Phi a} = L_a i_a + M_{ab} i_b + \ldots + \lambda_{\Phi_{\text{ext}}}$$

Self-inductance   Mutual inductance   External flux

DC Flux Transformer



$$\lambda_{\Phi 1} + \lambda_{\Phi 2} = 0$$

$$\lambda_{\Phi 1} = \lambda_{\Phi o} + L_1 i$$

$$\lambda_{\Phi 2} = L_2 i = -\lambda_{\Phi o} \frac{L_2}{L_1 + L_2}$$

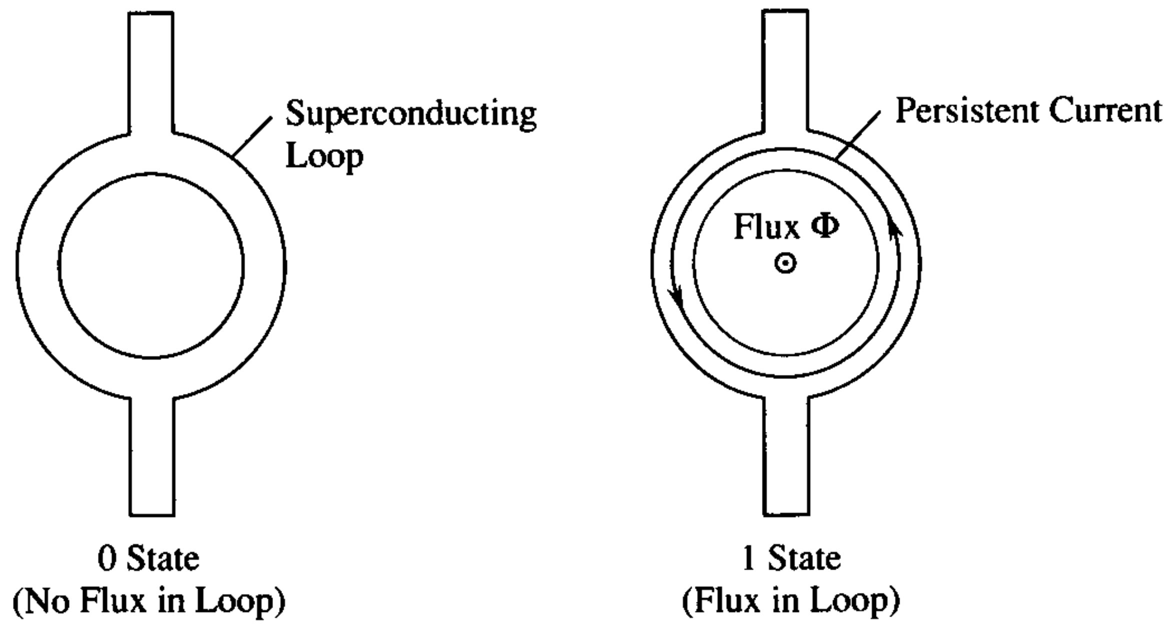
Flux can be transported

If the **B** field is measured of the transported flux

$$\beta \equiv \frac{|\mathrm{B}_2|}{|\mathrm{B}_{\mathrm{app}}|} = \frac{N_1 A_1}{N_2 A_2} \frac{L_2}{L_1 + L_2}$$

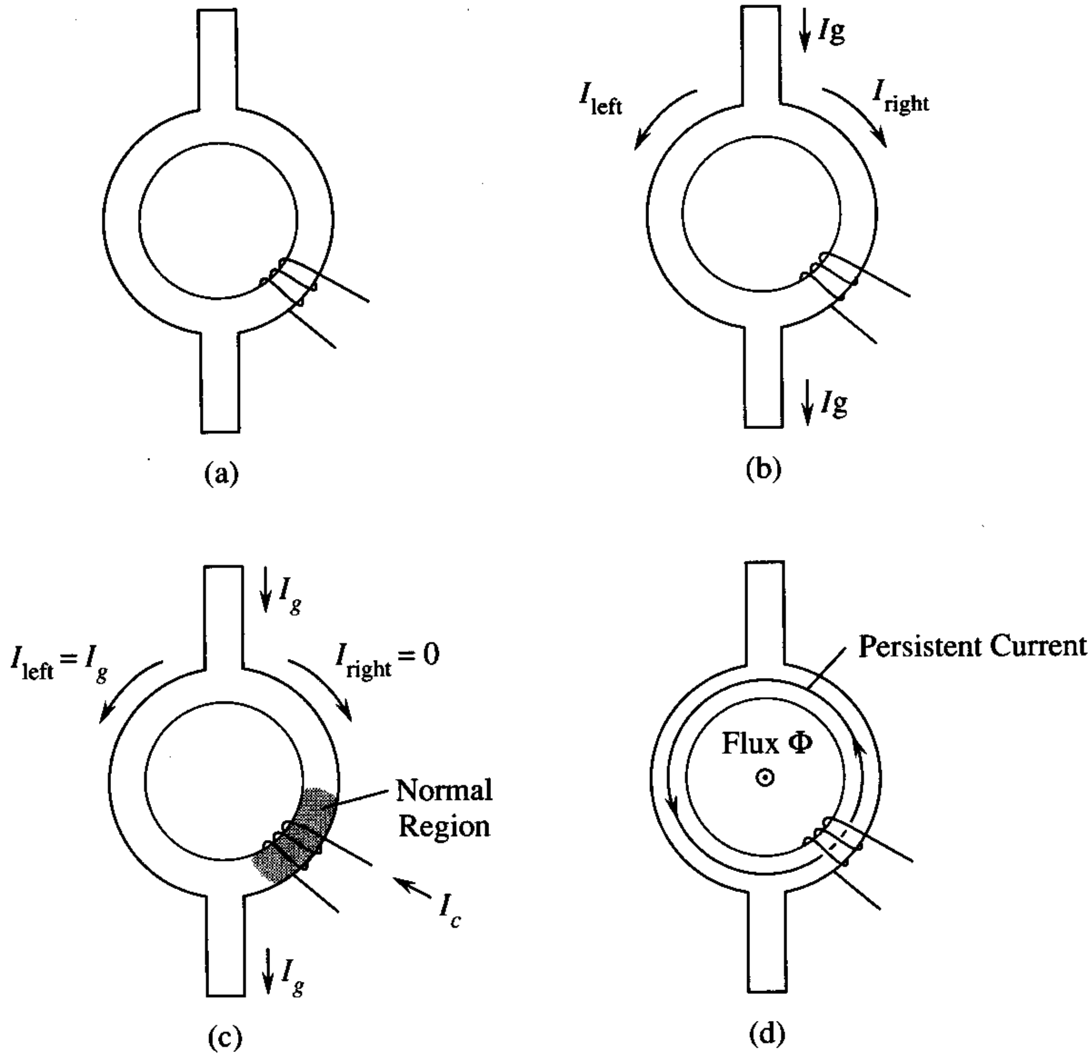
B can be amplified

Superconducting Memory



0 to 1 Storage

0



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Magnetic Monopole Detector

Maxwell's Equations with Monopole density ρ_m

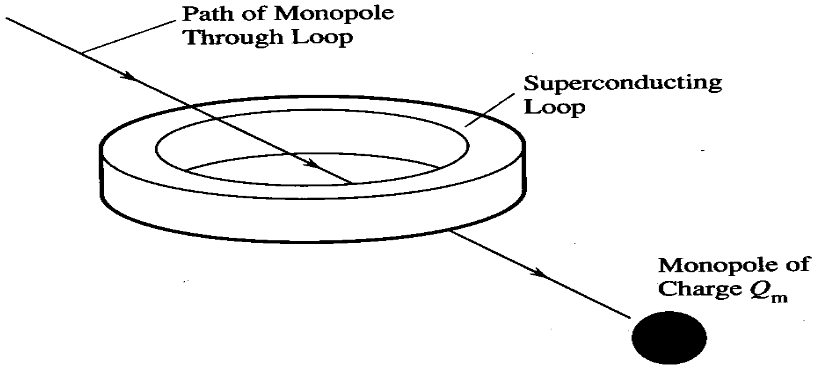
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} - \mathbf{J}_\mathsf{m} \qquad \nabla \cdot \mathbf{D} = \rho_\mathsf{e}$$

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J}_\mathsf{e} \qquad \nabla \cdot \mathbf{B} = \rho_\mathsf{m}$$

The signs insure electric and magnetic charge conservation.

$$\nabla \cdot \mathbf{J}_\mathsf{e} + \frac{\partial}{\partial t}\rho_\mathsf{e} = 0 \qquad \nabla \cdot \mathbf{J}_\mathsf{m} + \frac{\partial}{\partial t}\rho_\mathsf{m} = 0$$

Magnetic Monopole Detector



Take the line integral $$\nabla \times \mathbf{E} = -\frac{\partial \mathrm{B}}{\partial t} - \mathbf{J}_{\mathsf{m}}$$

$$-\oint_{\mathsf{C}} \mathbf{E} \cdot d\mathbf{l} = \frac{d}{dt} \int_{\mathsf{S}} \mathbf{B} \cdot d\mathbf{s} + \int_{\mathsf{S}} \mathbf{J}_{\mathsf{m}} \cdot d\mathbf{s}$$

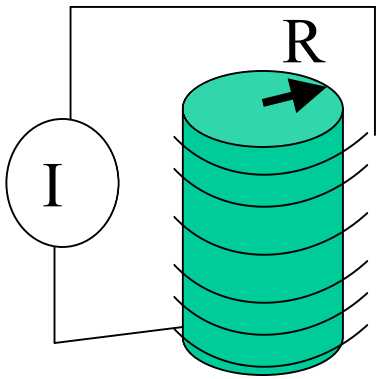
$$0 = \frac{d\Phi}{dt} + I_{\mathsf{m}}$$

$$\frac{d}{dt}\left(\Phi + Q_{\mathsf{m}}\right) = 0$$

Total of Flux and magnetic charge is conserved.

Inductance measurement

From the measurement of the inductance, the penetration depth can determined.



For a normal metal

$$\Phi = \frac{N}{L} I \, N \pi R^2$$

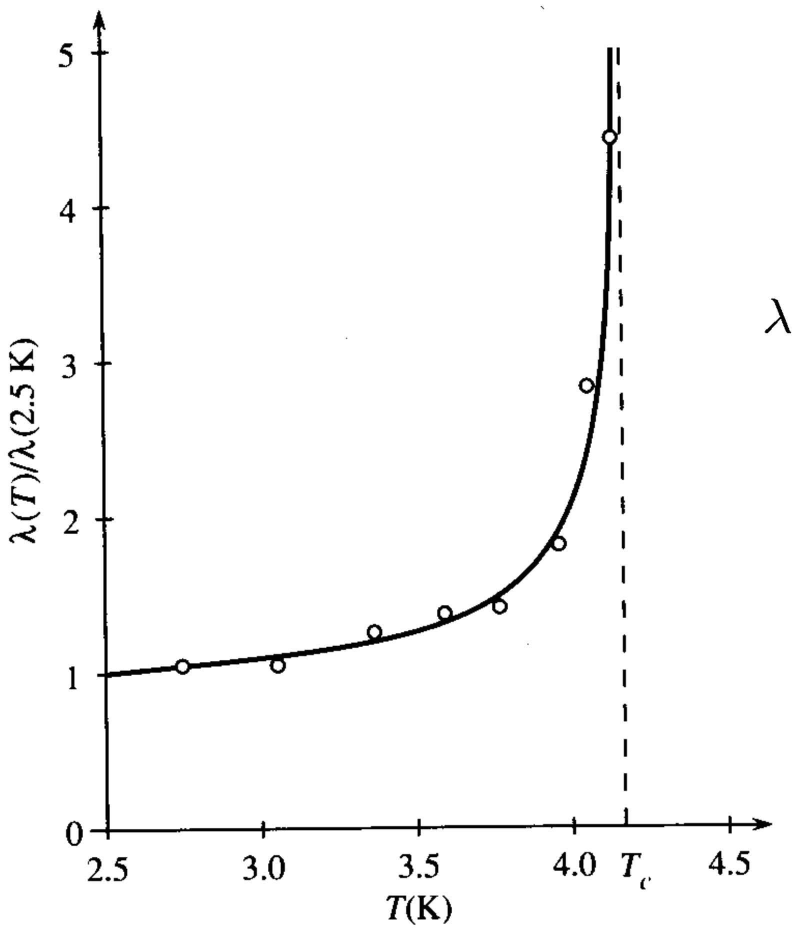
And $L = \frac{N^2}{L} \pi R^2$

For a superconductor,

$$\Phi = \frac{N}{L} I \, N 2 \pi R \lambda$$

and $L = \frac{N^2}{L} 2 \pi R \lambda$

Experiment



$$\lambda(T) = \frac{\lambda_o}{\sqrt{1-(T/T_c)^4}} \qquad \text{for } T \leq T_c.$$

The penetration depth λ is temperature dependent !

Temperature dependent λ

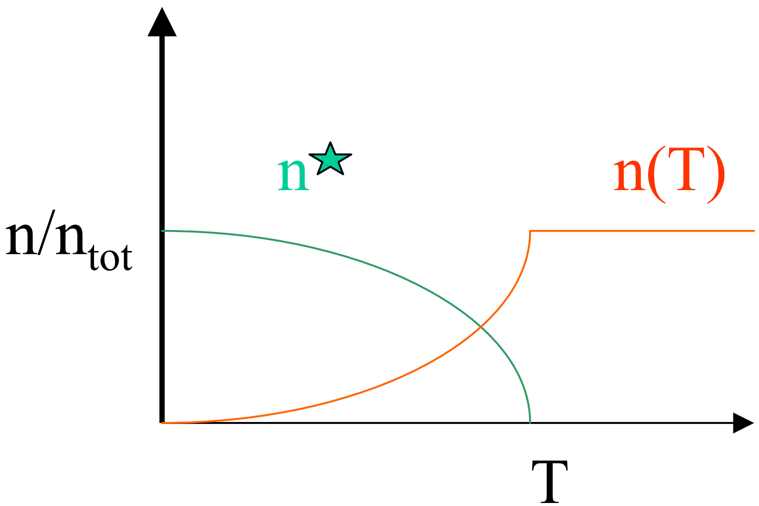
$$\lambda(T) = \sqrt{\frac{\Lambda}{\mu_o}} = \sqrt{\frac{m^\star}{n^\star (q^\star)^2 \mu_o}} = \frac{\lambda_o}{\sqrt{1-(T/T_c)^4}} \qquad \text{for } T \leq T_c.$$

A good guess to let $n^\star$ depend on temperature for $T < T_c$

$$n^\star(T) = \frac{1}{2} n_{\text{tot}} \left(1 - \left(\frac{T}{T_c}\right)^4\right)$$

$$n_{\text{tot}} = n(T) + 2n^\star(T)$$

$$n(T) = n_{\text{tot}} \left(\frac{T}{T_c}\right)^4$$

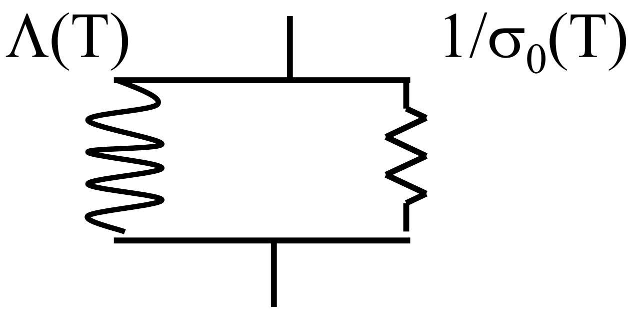


Two Fluid Model for $\omega\tau_{tr} << 1$, $T < T_c$

$$\mathbf{J}_{tot} = \mathbf{J}_s(T) + \mathbf{J}_n(T)$$

$$\mathbf{E} = \frac{\partial}{\partial t}\left(\Lambda(T)\,\mathbf{J}_s\right) \qquad \mathbf{E} = \frac{1}{\tilde{\sigma}_o(T)}\mathbf{J}_n$$

$$\Lambda(T) = \frac{m}{n_{tot}e^2}\left(\frac{1}{1-(T/T_c)^4}\right) \qquad \tilde{\sigma}_o(T) = \frac{n_{tot}e^2\tau_{tr}}{m}\left(\frac{T}{T_c}\right)^4$$



$$\mathbf{J} = \mathbf{J}_n + \mathbf{J}_s = \left(\tilde{\sigma}_o(T) + \frac{1}{j\omega\mu_o(\lambda(T))^2}\right)\mathbf{E}$$

Two Fluid Model

Constitutive relations for two fluid model

$$\mathbf{E} = \frac{\partial}{\partial t}\left(\Lambda(T)\,\mathbf{J}_\mathsf{s}\right) \qquad \mathbf{E} = \frac{1}{\tilde{\sigma}_o(T)}\mathbf{J}_\mathsf{n}$$

$$\nabla \times \left(\Lambda(T)\,\mathbf{J}_\mathsf{s}\right) = -\mathbf{B}$$

Maxwell

$$\nabla \times \mathbf{H} \approx \mathbf{J} = \mathbf{J}_\mathsf{n} + \mathbf{J}_\mathsf{s} \qquad \nabla \times \mathbf{E} = -\frac{\partial}{\partial t}\mathbf{B}$$

Gives

$$\left(1 - \lambda^2\nabla^2 + \mu_o\tilde{\sigma}_o\lambda^2\frac{\partial}{\partial t}\right)\mathbf{B}$$

Complex wavenumber

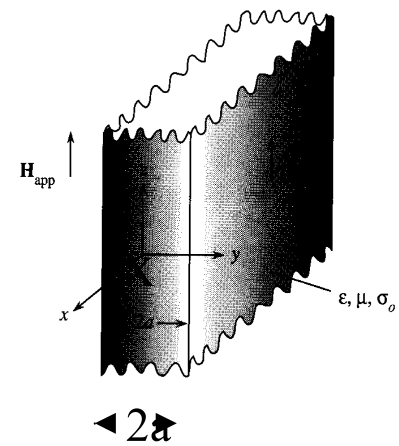
For a sinusoidal drive,

$$\left(1 - \lambda^2(T)\nabla^2 + j2\left(\frac{\lambda(T)}{\delta(T)}\right)^2\right)\hat{\mathrm{B}} = 0$$

For a slab in a uniform field

$$\hat{\mathrm{B}} = \mu_o \hat{H}_o \frac{\cosh ky}{\cosh ka}\,\mathbf{i}_z$$

$$(k(T))^2 = \frac{1}{(\lambda(T))^2} + j\frac{2}{(\delta(T))^2}$$



The smaller length determines the length scale