Lecture 6: Electromagnetic Power

Outline

1. **Power and energy in a circuit**
2. **Power and energy density in a distributed system**
3. **Surface Impedance**

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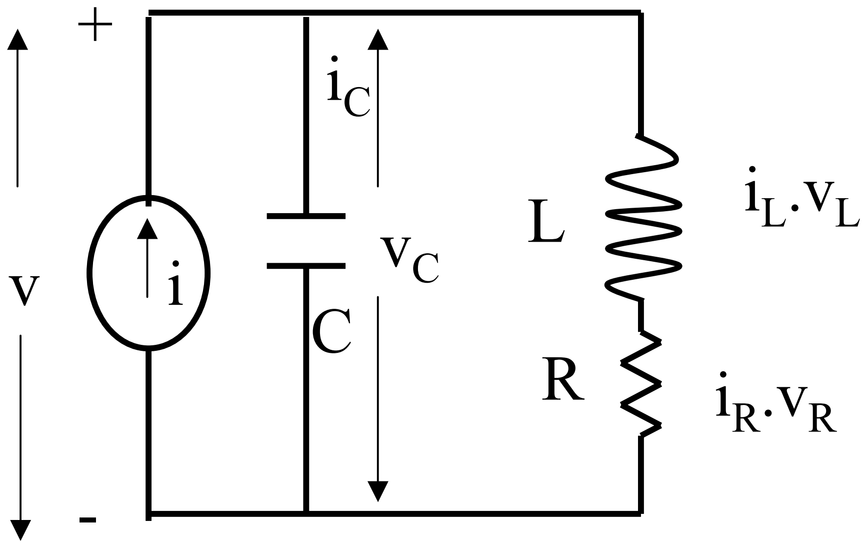
Power in a Circuit

Power: $vi = v_C i_C + v_L i_L + v_R i_R$



Constitutive relations for the resistor

$$v_R = i_R R \,,$$

the inductor

$$v_L = L \frac{d}{dt} i_L \,,$$

and the capacitor

$$i_C = C \frac{d}{dt} v_C \,,$$

$$vi = \frac{d}{dt}\left(\frac{1}{2} C v_C^2\right) + \frac{d}{dt}\left(\frac{1}{2} L i_L^2\right) + R i_R^2$$

Energy $\mathbf{W_e}$ $\mathbf{W_m}$

Average Power for a Sinusoidal Drive

The time average power is

$$\langle vi \rangle \equiv \frac{1}{T} \int_0^T vi \; dt$$

Power is a bilinear term, not a linear one, so must use real variables,

$$C(t) = \frac{1}{2} \left(\widehat{C}\, e^{j\omega t} + \widehat{C}^*\, e^{-j\omega t} \right)$$

The time average power is then

$$\langle vi \rangle = \frac{1}{4T} \int_0^T \left(\widehat{v}\widehat{\imath}^* + (\widehat{v}\widehat{\imath}^*)^* + \widehat{v}\widehat{\imath}\, e^{j2\omega t} + (\widehat{v}\widehat{\imath})^*\, e^{-j2\omega t} \right) dt$$

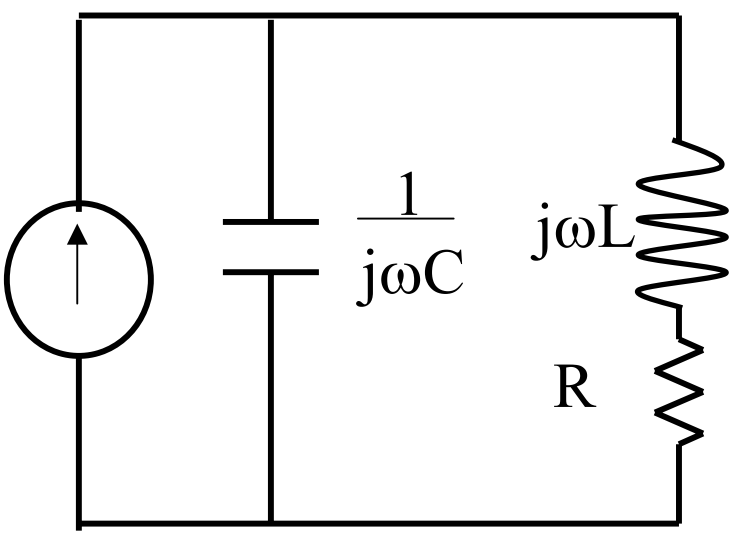
which gives

$$\langle vi \rangle = \frac{1}{2} \mathrm{Re} \left\{ \widehat{v}\widehat{\imath}^* \right\}$$

Average Power for a Sinusoidal Drive

$$\langle vi \rangle = \frac{1}{2}\,\mathrm{Re}\left\{\hat{v}\hat{i}^*\right\}$$

$$\langle vi \rangle = \frac{|\hat{i}|^2}{2}\,\mathrm{Re}\left\{Z(\omega)\right\} = \frac{|\hat{v}|^2}{2}\,\mathrm{Re}\left\{\frac{1}{Z(\omega)}\right\}$$



$$Z(\omega) = j\omega C + \frac{1}{R + j\omega L}$$

$$\langle vi \rangle = \frac{|\hat{v}|^2}{2}\,\mathrm{Re}\left\{\frac{R}{R^2 + \omega^2 L^2}\right\}$$

Power in Distributed Systems

Use the full Maxwell's Equations,

$$\mathbf{E}\cdot \left\{ \nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J} \right\}$$

$$- \quad \mathbf{H}\cdot \left\{ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \right\}$$

$$-\nabla \cdot (\mathbf{E} \times \mathbf{H}) = \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} + \mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} + \mathbf{E} \cdot \mathbf{J}$$

where we have used

$$\nabla \cdot (\mathbf{A} \times \mathbf{C}) = \mathbf{C} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{C})$$

Poynting's Theorem

Therefore, we have found Poynting's theorem, with $\mathbf{S} = \mathbf{E} \times \mathbf{H}$

$$-\oint_{\Sigma} (\mathbf{E} \times \mathbf{H}) \cdot d\mathbf{s} = \int_V \left(\mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} + \mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} + \mathbf{E} \cdot \mathbf{J} \right) dv$$

For a linear, isotropic, homogenous ohmic medium (σ_0, μ, ε)

$$-\oint_{\Sigma} \mathbf{S} \cdot d\mathbf{s} = \frac{d}{dt} \int_V \Big(\underbrace{\frac{1}{2} \epsilon \mathbf{E}^2}_{w_e} + \underbrace{\frac{1}{2} \mu \mathbf{H}^2}_{w_m} \Big) dv + \underbrace{\int_V \frac{1}{\sigma_o} \mathbf{J}^2 dv}_{\text{Joule heating}}$$

w_e w_m Joule heating

energy density

For a sinusoidal drive:

$$\langle \mathbf{S} \rangle = \frac{1}{2} \mathrm{Re} \left\{ \hat{\mathbf{E}} \times \hat{\mathbf{H}}^* \right\} \quad \text{and} \quad -\oint_{\Sigma} \langle \mathbf{S} \rangle \cdot d\mathbf{s} = \frac{1}{2} \int_V \frac{1}{\sigma_o} |\hat{\mathbf{J}}|^2 dv$$

Poynting's Theorem for a Superconductor

Maxwell's equations still give

$$-\oint_{\Sigma}(\mathbf{E}\times\mathbf{H})\cdot d\mathbf{s} = \int_V \left(\mathbf{E}\cdot\frac{\partial \mathbf{D}}{\partial t} + \mathbf{H}\cdot\frac{\partial \mathbf{B}}{\partial t} + \mathbf{E}\cdot\mathbf{J}\right) dv$$

But for a superconductor

$$\mathbf{J} = \mathbf{J}_{\mathsf{n}} + \mathbf{J}_{\mathsf{s}} \qquad \mathbf{E} = \frac{\partial}{\partial t}\left(\Lambda(T)\,\mathbf{J}_{\mathsf{s}}\right) \qquad \mathbf{E} = \frac{1}{\tilde{\sigma}_o(T)}\mathbf{J}_{\mathsf{n}}$$

Therefore,

$$-\oint_{\Sigma}\mathbf{S}\cdot d\mathbf{s} = \frac{d}{dt}\int_V \left(\frac{1}{2}\epsilon\,\mathbf{E}^2 + \frac{1}{2}\mu_o\,\mathbf{H}^2 + \underbrace{\frac{1}{2}\Lambda(T)\,\mathbf{J}_{\mathsf{s}}^2}_{\mathbf{w_K}}\right) dv + \int_V \frac{1}{\tilde{\sigma}_o(T)}\mathbf{J}_{\mathsf{n}}^2\,dv$$

Kinetic Energy Density

With $\Lambda \equiv \frac{m^\star}{n^\star (q^\star)^2}$ and $\hat{\mathbf{J}}_s = n^\star q^\star \hat{v}$

$$w_K = \frac{1}{2}\Lambda(T)\,\mathbf{J}_s{}^2 = \underbrace{n^\star(T)}_{\text{Superelectron density}} \underbrace{\left(\frac{1}{2} m^\star \mathbf{v}_s^2\right)}_{\text{Kinetic energy of a superelectron}}$$

Energy is also stored in the kinetic energy of the superelectrons

Averaged Poynting Vector

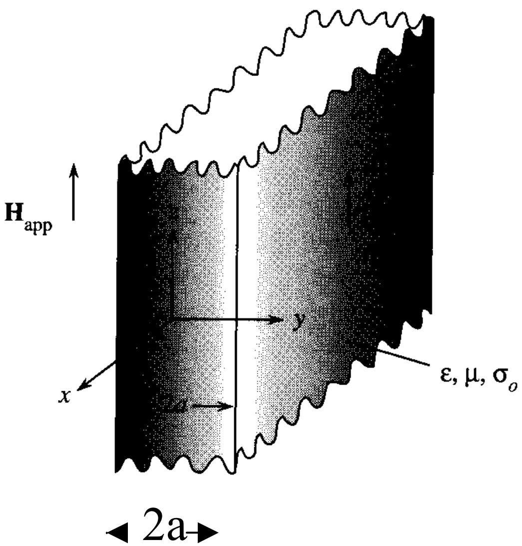
For a sinusoidal drive:

$$\langle \mathbf{S} \rangle = \frac{1}{2} \mathrm{Re} \left\{ \widehat{\mathbf{E}} \times \widehat{\mathbf{H}}^* \right\}$$ and

$$- \oint_{\Sigma} \langle \mathbf{S} \rangle \cdot d\mathbf{s} = \frac{1}{2} \int_V \frac{1}{\widetilde{\sigma}_o(T)} |\widehat{\mathbf{J}}n|^2 \, dv$$

Energy in a superconductor is dissipated through the normal channel

Power Loss in a Slab



$$\mathbf{H} = \text{Re}\left\{ \hat{H}_o \frac{\cosh ky}{\cosh ka} e^{j\omega t} \right\} \mathbf{i}_z$$

$$k^2 = j\omega\mu_o\sigma$$

$$\frac{d}{dy}\hat{E}_x(y) = j\omega\mu\hat{H}_z(y)$$

$$\mathbf{E} = \text{Re}\left\{ \hat{H}_o \frac{k}{\sigma} \frac{\sinh ky}{\cosh ka} e^{j\omega t} \right\} \mathbf{i}_x$$

Normal Metal

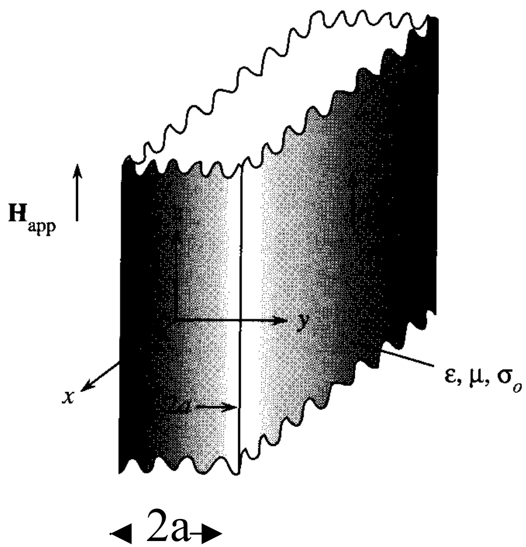
$$k^2 = \frac{j}{\delta^2}$$

$$\sigma = \sigma_o$$

Superconductor

$$k^2 = \frac{1}{\lambda^2}\left(1 + 2j\left(\frac{\lambda}{\delta}\right)^2\right)$$

$$\sigma = \sigma_o + \frac{1}{j\omega\mu_o\lambda^2}$$



For a unit area, the time averaged power is

$$P_{\text{dis}} = \text{Re}\left\{ |\hat{H}_o|^2 \frac{k}{\sigma} \frac{\sinh ka}{\cosh ka} \right\}$$

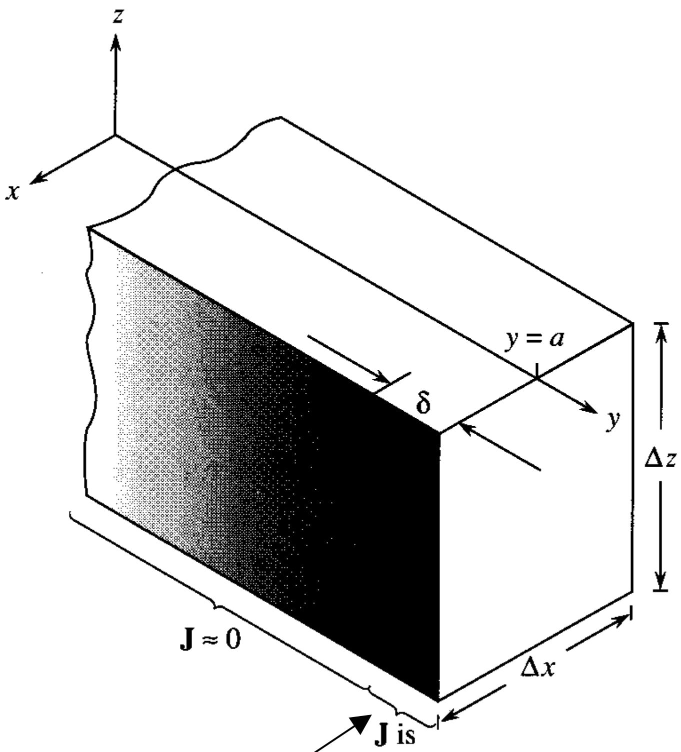
In the bulk approximation, where δ << a, or λ << a

$$P_{\text{dis}} = |\hat{H}_o|^2 \, \text{Re}\left\{ \frac{k}{\sigma} \right\}$$

For a normal metal:

$$P_{\text{dis}} = |\hat{H}_o|^2 \frac{1}{\delta \sigma_o}$$ **and surface resistance** $$R_S = \frac{1}{\delta \sigma_o}$$

Surface Resistance: normal metal



J, uniform

For an area on the surface of $\Delta x \, \Delta z$

$$R = \frac{\Delta x}{\sigma_o \, \delta \, \Delta z}$$

The current is $|\hat{\imath}| = |\hat{J}_x| \, \delta \, \Delta z$

The current density is given by

$$|\hat{J}_x| = \frac{|\hat{K}_x|}{\delta} = \frac{|\hat{H}_o|}{\delta}$$

The power dissipated per unit area is

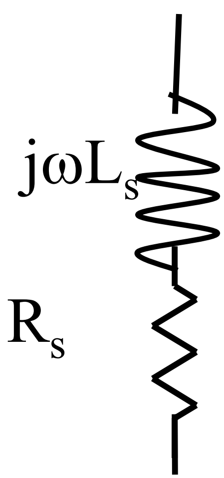
$$P_{\text{dis}} = 2 \times \frac{\frac{1}{2} |\hat{\imath}|^2 R}{\Delta x \, \Delta z} = 2 \times \frac{1}{2} |\hat{H}_o|^2 R_S = |\hat{H}_o|^2 \frac{1}{\delta \sigma_o}$$

Surface Impedance: Normal Metal

In the bulk approximation, where $\delta \ll a$, or $\lambda \ll a$, a surface impedance can be defined from

$$P_{\text{dis}} = |\hat{H}_o|^2 \, \text{Re}\left\{ \frac{k}{\sigma} \right\} \quad \text{as} \quad Z_s = \frac{k}{\sigma}$$

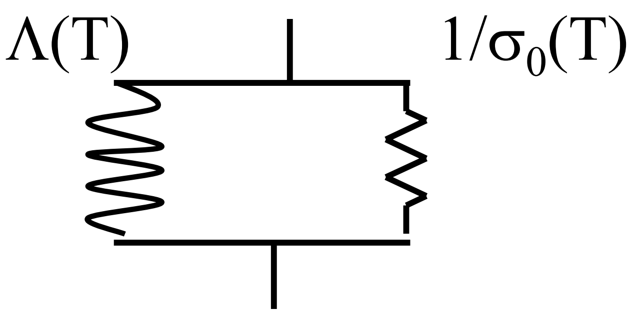
For the normal metal,
$$Z_S = \underbrace{\frac{1}{\delta \sigma_o}}_{\mathbf{R_s}} + j \underbrace{\frac{1}{\delta \sigma_o}}_{\mathbf{L_s}}$$



Surface Impedance: Superconductor

For the superconductor, with $\lambda \ll \delta$, to lowest order

$$Z_S = \underbrace{\frac{2}{\delta\tilde{\sigma}_o}\left(\frac{\lambda}{\delta}\right)^3}_{R_s \sim \omega^2} + \underbrace{j\omega\mu_o\lambda}_{L_s}$$



$$Z_s(\omega) = \frac{j\omega L_s R_s}{R_s + j\omega L_s}$$

For $R_s \gg \omega L_s$

$$Z_s(\omega) \approx j\omega L_s(1 - j\omega L_s/R_s) = \omega^2 L_s^2/R_s + j\omega L_s$$

For Pb at 2K and 100 MHz, $R_s = 10^{-10}$ Ohm/□ , and Q of cavity = 10^{10}