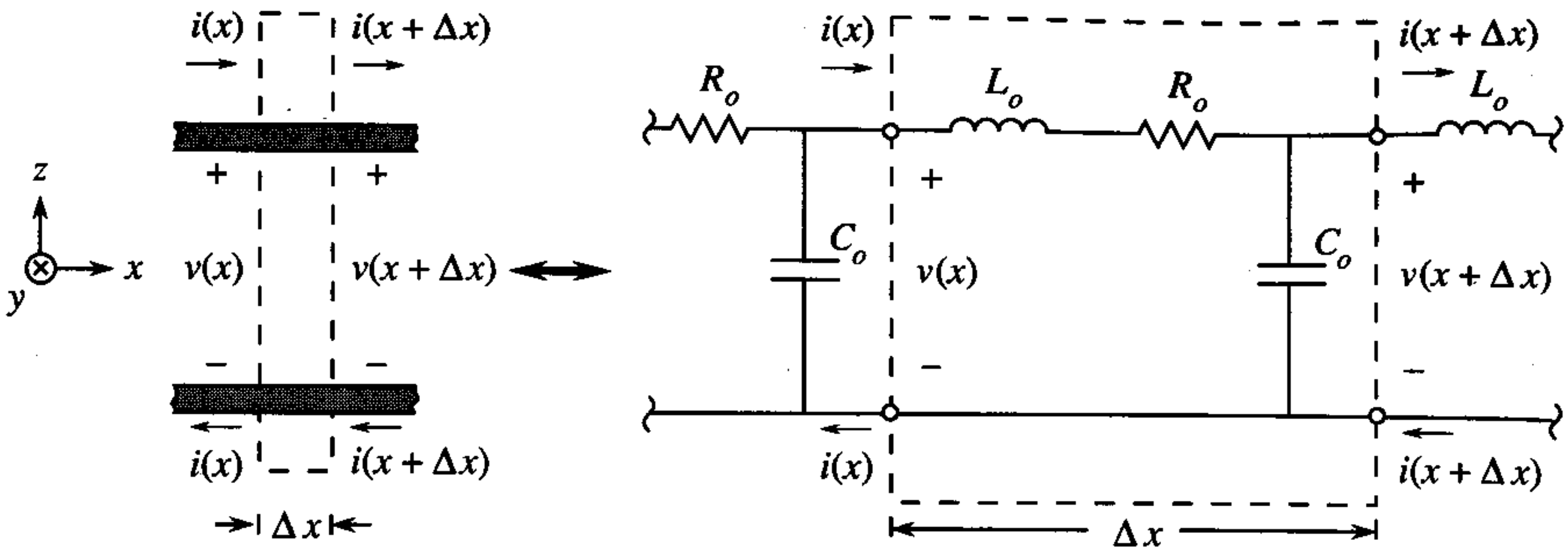
Lecture 7: Transmission Lines

Outline

1. **Ladder Network Approximation**
2. **Inductance**
3. **Superconducting Transmission Line**
4. **Comparison with normal transmission line**

September 25, 2003

Transmission Line: circuit model



Actual Transmission Line

Model Transmission Line

$$\hat{v}(x) - \hat{v}(x+\Delta x) = (j\omega L_o + R_o)\,\Delta x\,\hat{\imath}(x)$$

$$\hat{\imath}(x) - \hat{\imath}(x+\Delta x) = j\omega C_o\,\Delta x\,\hat{v}(x+\Delta x)$$

Transmission Line

$$\frac{d\hat{v}}{dx} = -(j\omega L_o + R_o)\,\hat{\imath} \qquad \frac{d\hat{\imath}}{dx} = -j\omega C_o\,\hat{v}$$

A wave equation is obtained

$$\frac{d^2\hat{v}}{dx^2} = -\left(\omega^2 L_o C_o - j\omega R_o C_o\right)\hat{v}$$

Which has solutions of the form $\hat{v}(x) = \hat{V}\, e^{-jk_o x}$ with

$$k_o = \omega\sqrt{L_o C_o}\,\sqrt{1 - j(R_o/\omega L_o)}$$

Transmission line parameters

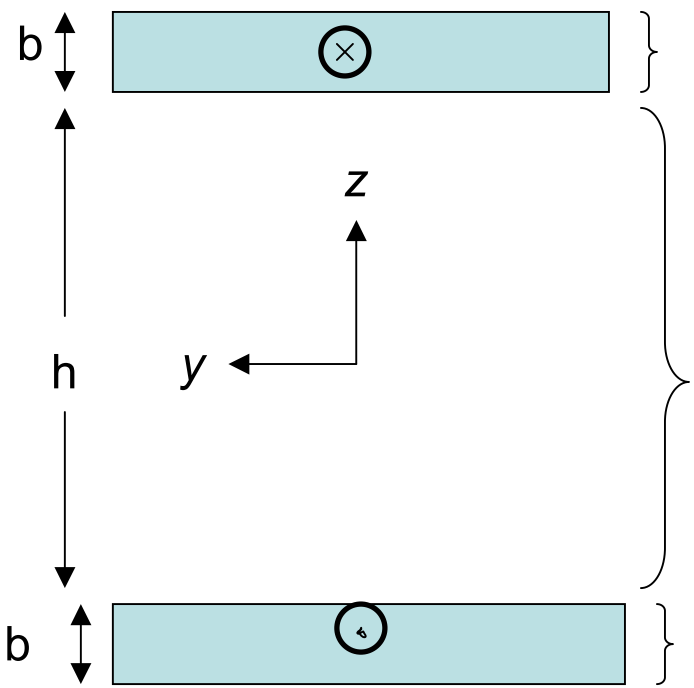
In the limit where the inductive impedance dominates,

$$\lim_{\omega\tau_{LR}\gg 1} k_o = \omega\sqrt{L_o C_o} - j\,\frac{1}{2}\,\frac{R_o}{\sqrt{L_o/C_o}}$$

So that $v(x,t) = \mathrm{Re}\left\{\hat{V}\, e^{-\alpha x}\, e^{j\omega(t-(x/u_p))}\right\}$

TEM Waveguide Characteristics		
Symbol	Name	Value
u_p	Phase Velocity	$\frac{1}{\sqrt{L_o C_o}}$
2α	Power Attenuation per Unit Length	$\frac{R_o}{\sqrt{L_o/C_o}}$
Z_o	Characteristic Impedance	$\sqrt{\frac{L_o}{C_o}}\left(1 - j\,\frac{1}{2}\,\frac{R_o}{\omega L_o}\right)$

Fields in the Transmission Line

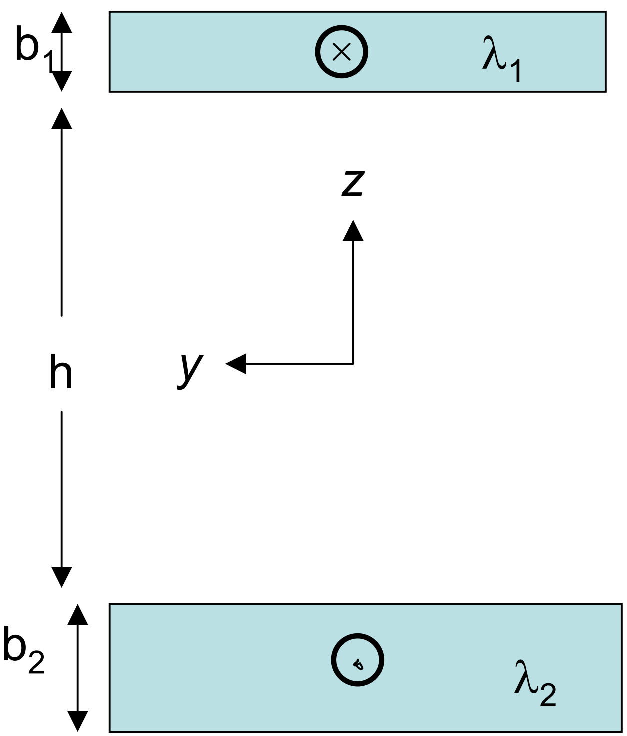


$$\hat{H}_y = \frac{\hat{\imath}}{d}\frac{\sinh\,(b - z + h/2)\,/\lambda}{\sinh b/\lambda} \qquad \text{for } \; 0 \le z - (h/2) \le b$$

$$\hat{H}_y = \hat{\imath}/d \qquad \text{for } \; |z| \le h/2$$

$$\hat{H}_y = \frac{\hat{\imath}}{d}\frac{\sinh\,(b + z + h/2)\,/\lambda}{\sinh b/\lambda} \qquad \text{for } \; -b \le z + (h/2) \le 0$$

Fields in the Transmission Line

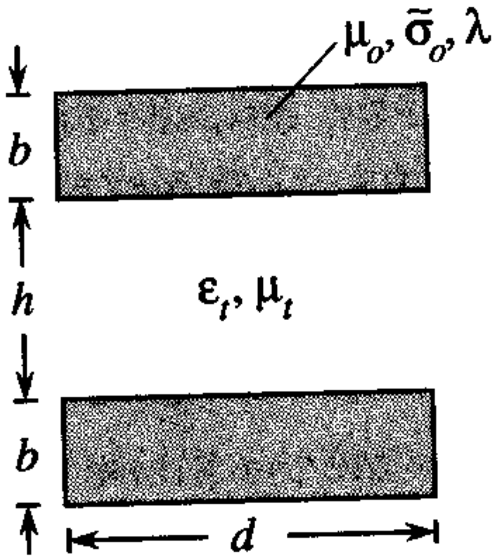


$$\hat{H}_y = \frac{\hat{\imath}}{d}\frac{\sinh\left(b_1 - z + h/2\right)/\lambda_1}{\sinh b_1/\lambda_1} \qquad \text{for } \; 0 \le z - (h/2) \le b_1$$

$$\hat{H}_y = \hat{\imath}/d \qquad \text{for } \; |z| \le h/2$$

$$\hat{H}_y = \frac{\hat{\imath}}{d}\frac{\sinh\left(b_2 + z + h/2\right)/\lambda_2}{\sinh b_2/\lambda_2} \qquad \text{for } \; -b \le z + (h/2) \le 0$$

Bulk Superconducting Transmission Line



Two identical, thick ($\lambda \ll b$) superconducting plates.

$$C_o = \epsilon_t \frac{d}{h}$$

$$\lim_{\substack{\lambda \ll \delta \\ \lambda \ll b}} R_o = \frac{4}{d\delta\tilde{\sigma}_o}\left(\frac{\lambda}{\delta}\right)^3 = \frac{2}{d}\,\mathrm{Re}\{Z_S\}$$

$$\lim_{\substack{\lambda \ll \delta \\ \lambda \ll b}} L_o = \mu_t \frac{h}{d} + \underbrace{2\mu_o \frac{\lambda}{d}}$$

$$\lim_{\substack{\lambda \ll \delta \\ \lambda \ll b}} L_o = \frac{2}{d}\,\mathrm{Im}\{Z_S\}$$

?

Inductance

Inductance per unit length is found from

$$\frac{1}{4}\int dy \int dz \left(\mu\,|\widehat{\mathbf{H}}|^2 + \Lambda\,|\widehat{\mathbf{J}}_{\mathrm{S}}|^2\right) = \frac{1}{4}L_o\,|\widehat{\imath}|^2\,,$$

Inside the transmission line space

$$\widehat{H}_y = \widehat{\imath}/d \qquad \text{for} \quad |z| \le h/2$$

$$L_{o,in} = \mu(\frac{1}{d})^2\,dh = \mu\frac{h}{d}$$

Inside the transmission line material

$$\widehat{H}_y = \frac{\widehat{\imath}}{d}\,\frac{\sinh k\,(b - z + (h/2))}{\sinh kb} \qquad \text{for} \quad 0 \le z - (h/2) \le b$$

$$\lim_{\substack{\lambda \ll \delta \\ \lambda \ll b}} L_{o,material} = 2L_s = 2\mu_o\frac{\lambda}{d}$$

Inductance for a thin slab

The current density is uniform for then slab so that

$$\mathbf{J} = \frac{\hat{i}}{b\,d}\,\mathbf{i}_x$$

The energy stored in the slab is

$$W = \frac{1}{2}\mu_o\lambda^2(\mathbf{J})^2\,b = \mu_o\lambda^2(\frac{\hat{i}}{b\,d})^2\,bd\Delta x = \frac{1}{2}L_o\Delta x\hat{i}^2$$

Therefore, $L_o = \dfrac{\mu_o\lambda^2}{db}$

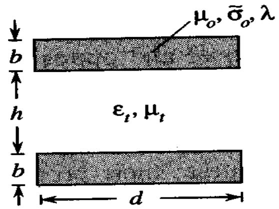
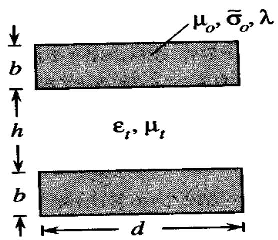
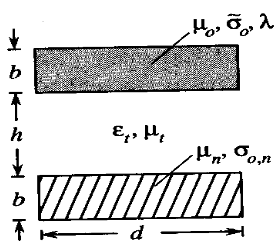
For each slab and the total inductance per unit length is twice this. This is the *kinetic inductance.*

Dispersionless Transmission Lines

Because L_o and C_o do not depend on frequency for a superconductor, the phase velocity is independent of frequency. So that a pulse will propagate down a superconducting transmission line without dispersing. Also, the amount of attenuation is extremely small, since this is due to R_o.

For a normal metal, $L_o$depends on frequency so that there is dispersion, in addition to a much greater loss.

Summary

Transmission Line Geometry	L_o	C_o	R_o
 Two identical, thin ($\lambda \gg b$) superconducting plates.	$\frac{\mu_t h}{d} + \frac{2\mu_o \lambda^2}{db}$	$\frac{\epsilon_t d}{h}$	$\frac{8}{db\tilde{\sigma}_o}\left(\frac{\lambda}{\delta}\right)^4$
 Two identical, thick ($\lambda \ll b$) superconducting plates.	$\frac{\mu_t h}{d} + \frac{2\mu_o \lambda}{d}$	$\frac{\epsilon_t d}{h}$	$\frac{4}{d\delta\tilde{\sigma}_o}\left(\frac{\lambda}{\delta}\right)^3$
 One thick ($\lambda \ll b$) superconducting plate and one thick ($\lambda \ll b$) ohmic plate.	$\frac{\mu_t h}{d} + \frac{\mu_o \lambda}{d} + \frac{\mu_n \delta_n}{2d}$	$\frac{\epsilon_t d}{h}$	$\frac{1}{d\delta_n \sigma_{o,n}}$