Lecture 8: Perfect Diamagnetism

Outline

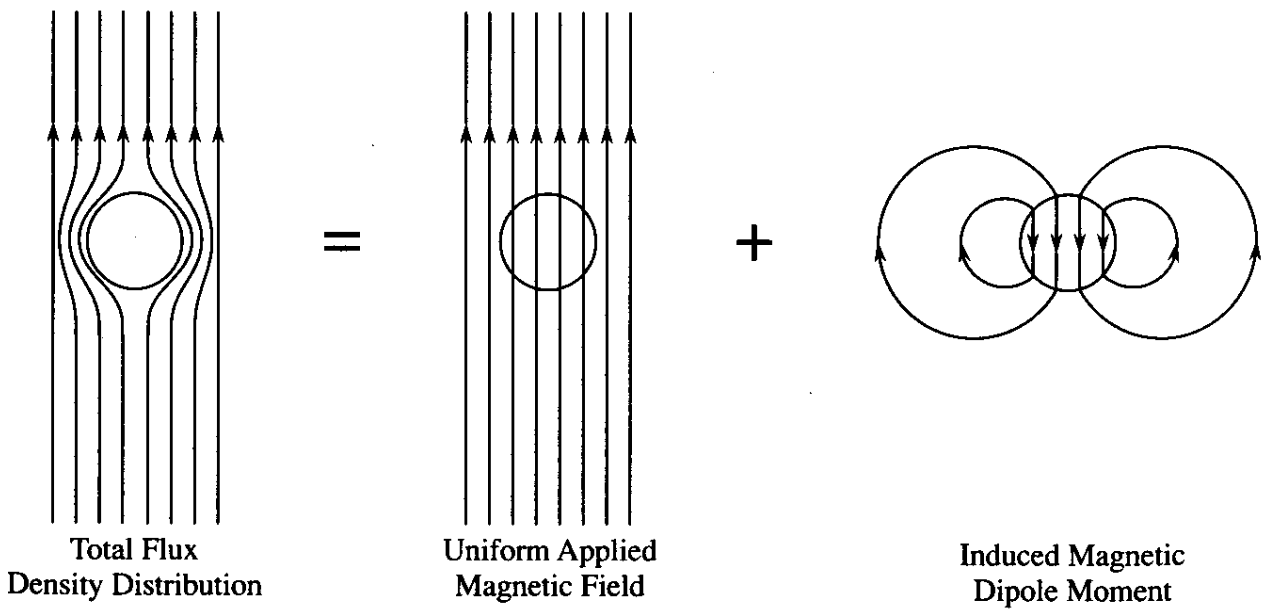
1. **Description of a Perfect Diamagnet**
   - **Method I and Method II**
   - **Examples**

2. **Energy and Coenergy in Methods I and II**

3. **Levitating magnets and Maglev trains**

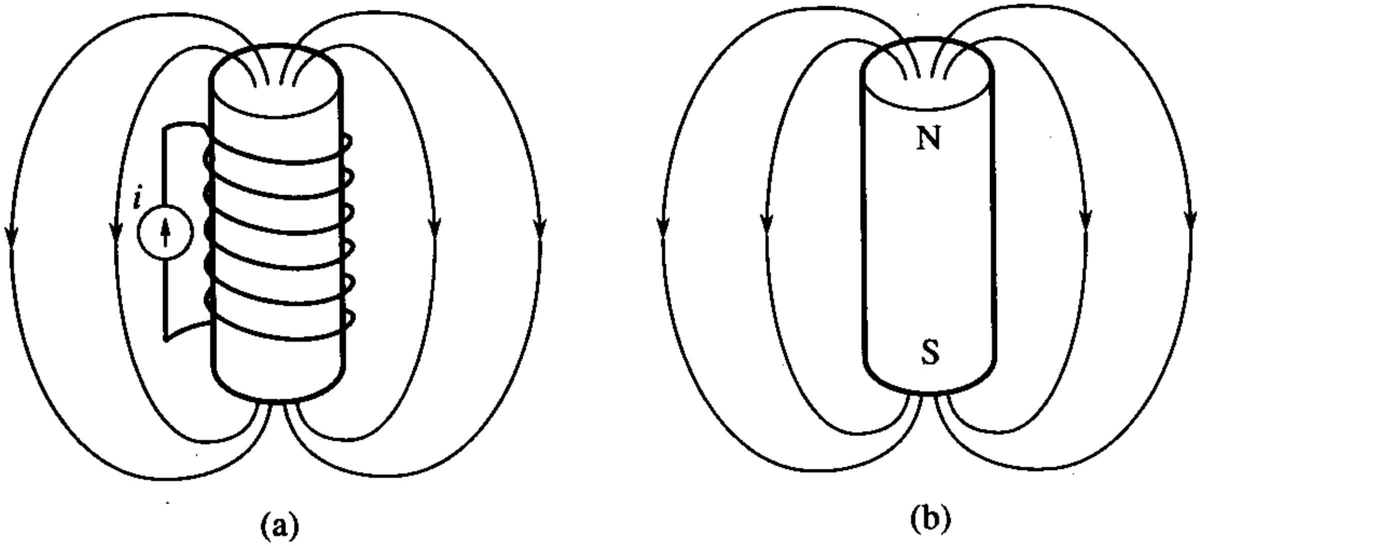
September 30, 2003

Description of Perfect Diamagetism



Surface currents or internal induced magnetization?

Methods I and II



$$\mathbf{B} = \mu_o \mathbf{H}^I$$

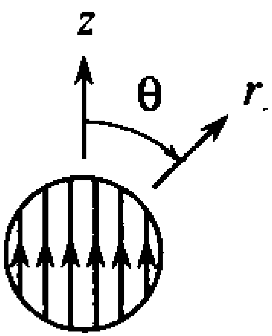
Method I

$$\mathbf{B} = \mu_o \left(\mathbf{H}^{II} + \mathbf{M}\right)$$

Method II

B field is the same in both methods.

Example: Magnetized Sphere



$$\mathbf{M} = M_o\,\mathbf{i}_z = M_o\,(\cos\theta\,\mathbf{i}_r - \sin\theta\,\mathbf{i}_\theta) \qquad \text{for } r \le R.$$

M Lines

$$\nabla \times \mathbf{H} = \mathbf{J} = 0 \quad \Rightarrow \quad \mathbf{H} = -\nabla\psi$$

$$\left.\begin{array}{c} \nabla \cdot \mathbf{B} = 0 \\ \mathbf{B} = \mu_o\,(\mathbf{H} + \mathbf{M}) \end{array}\right\} \quad \Rightarrow \quad \nabla^2\psi = \nabla \cdot \mathbf{M}$$

For this example: $\nabla^2\psi = 0$ Laplace's equation

Boundary Conditions

Inside the sphere:

$$\psi(r \leq R) = C_1\, r \cos\theta \quad \Rightarrow \quad \mathbf{H}(r \leq R) = -C_1\,(\cos\theta\,\mathbf{i}_r - \sin\theta\,\mathbf{i}_\theta)$$

Outside the sphere:

$$\psi(r \geq R) = C_2 \frac{\cos\theta}{r^2} \quad \Rightarrow \quad \mathbf{H}(r \geq R) = \frac{C_2}{r^3}(2\cos\theta\,\mathbf{i}_r + \sin\theta\,\mathbf{i}_\theta)$$

Boundary Conditions:

$$\mathbf{n} \times (\mathbf{H}_2 - \mathbf{H}_1) = \mathbf{K} \quad \Rightarrow \quad C_1 R^3 = C_2$$

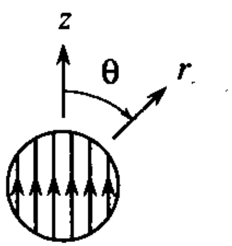
$$\mathbf{n} \cdot (\mathbf{B}_2 - \mathbf{B}_1) = 0 \quad \Rightarrow \quad \mathbf{i}_r \cdot \left(\mathbf{H}|_{r=R^+} - \mathbf{H}|_{r=R^-}\right) = \mathbf{i}_r \cdot \mathbf{M}|_{r=R^-}$$

Therefore, $\mathbf{M} = M_o\,(\cos\theta\,\mathbf{i}_r - \sin\theta\,\mathbf{i}_\theta) \quad \Rightarrow \quad C_1 = \dfrac{M_o}{3}$

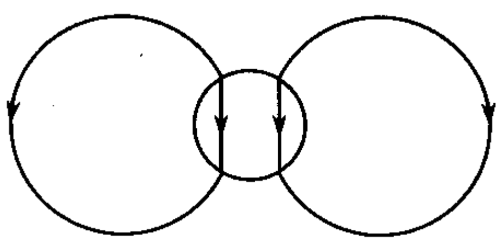
Magnetized Sphere

$$\mathbf{H}(r \le R) = -\frac{M_o}{3}\mathbf{i}_z = -\frac{M_o}{3}(\cos\theta\,\mathbf{i}_r - \sin\theta\,\mathbf{i}_\theta)$$

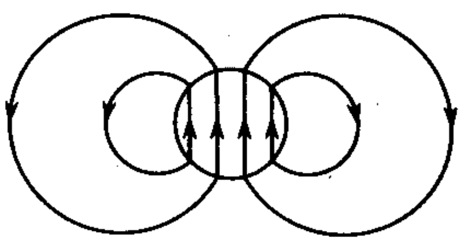
$$\mathbf{H}(r \ge R) = \frac{M_o}{3}\left(\frac{R}{r}\right)^3 (2\cos\theta\,\mathbf{i}_r + \sin\theta\,\mathbf{i}_\theta)$$



M Lines

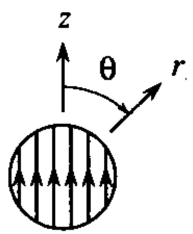


H Lines



B Lines

Magnetized Sphere in field



$$\mathbf{H}_{\mathsf{app}} = H_o\,\mathbf{i}_z$$

$$\mathbf{H}(r \le R) = \left(H_o - \frac{M_o}{3}\right)\mathbf{i}_z$$

$$\mathbf{H}(r \ge R) = \left(H_o + \frac{2}{3} M_o \left(\frac{R}{r}\right)^3\right)\cos\theta\,\mathbf{i}_r - \left(H_o - \frac{1}{3} M_o \left(\frac{R}{r}\right)^3\right)\sin\theta\,\mathbf{i}_\theta$$

For this to describe a superconductor (bulk limit), then B=0 inside.

Therefore, $0 = \mathbf{M} + \mathbf{H}(r \le R) = M_o\mathbf{i}_z + \left(H_o - \frac{M_o}{3}\right)\mathbf{i}_z$

So that $M = -\frac{3}{2} H_o$

Comparison of Methods

Method I

$$\mathbf{B}^I = \mu_o \mathbf{H}^I$$

Method II

$$\mathbf{B} = \mu_o \left(\mathbf{H}^{II} + \mathbf{M} \right)$$

Inside a bulk superconducting sphere:

Method I:

$$\mathbf{B}^I = 0$$

$$\mathbf{H}^I = 0$$

$$\mathbf{K}^I = -\mathrm{Re}\left\{ \frac{3}{2} \hat{H}_o \sin\theta \right\} \mathbf{i}_\phi$$

Method II:

$$\mathbf{B}^{II} = 0$$

$$\mathbf{H}^{II} = \frac{3}{2} H_o \mathbf{i}_z$$

$$\mathbf{M} = -\frac{3}{2} H_o \mathbf{i}_z$$

$$\mathbf{K}^{II} = 0$$

B field is the same, but not **H**.

Methods I and II: Summary

Maxwell's Equations

$$\nabla \cdot \mathbf{B} = 0$$

Method I

$$\nabla \times \mathbf{H}^I = \mathbf{J}^I$$

$$\mathbf{B} = \mu_o \mathbf{H}^I$$

$$\nabla \times \mathbf{B} = \mu_o \mathbf{J}^I$$

$$\mathbf{J}^{\mathrm{I}} = \underbrace{\left(\mathbf{J}_{\mathsf{s,app}} + \mathbf{J}_{\mathsf{s,ind}}\right)}_{\mathbf{J}_{\mathsf{s}}^{\mathrm{I}}} + \mathbf{J}_{\mathsf{n}}$$

Method II

$$\nabla \times \mathbf{H}^{II} = \mathbf{J}^{II}$$

$$\mathbf{B} = \mu_o \left(\mathbf{H}^{II} + \mathbf{M}\right)$$

$$\nabla \times \mathbf{B} = \mu_o \mathbf{J}^{II} + \mu_o \underbrace{\nabla \times \mathbf{M}}_{\mathbf{J}_{\mathsf{s,ind}}}$$

$$\mathbf{J}^{\mathrm{II}} = \mathbf{J}_{\mathsf{s,app}} + \mathbf{J}_{\mathsf{n}}$$

London Equations

Method I

$$\mathbf{E} = \frac{\partial}{\partial t}\left(\Lambda \mathbf{J}_{\mathsf{s}}^{\mathrm{I}}\right)$$

$$\nabla \times \left(\Lambda \mathbf{J}_{\mathsf{s}}^{\mathrm{I}}\right) = -\mathbf{B}$$

Method II

$$\mathbf{E} = \frac{\partial}{\partial t}\left(\Lambda \mathbf{J}_{\mathsf{s}}^{\mathrm{II}}\right) + \frac{\partial}{\partial t}\left(\Lambda\left(\nabla \times \mathbf{M}\right)\right)$$

$$\nabla \times \left(\Lambda \mathbf{J}_{\mathsf{s}}^{\mathrm{II}}\right) + \nabla \times \left(\Lambda\left(\nabla \times \mathbf{M}\right)\right) = -\mathbf{B}$$

Why Method II

Question:
The constitutive relations and London's Equations have gotten much more difficult. So why do Method II?

Answer:
The Energy and Thermodynamics are easier, especially when there is no applied current.

So we will find the energy stored in both methods.

Poynting's theorem is a result of Maxwell's equation, so both methods give

$$-\oint_{\Sigma} (\mathbf{E}\times\mathbf{H})\cdot d\mathbf{s} = \int_V \left(\mathbf{E}\cdot\frac{\partial \mathbf{D}}{\partial t} + \mathbf{H}\cdot\frac{\partial \mathbf{B}}{\partial t} + \mathbf{E}\cdot\mathbf{J} \right) dv$$

Method I: The Energy

Combining the constitutive relations with Poynting's theorem,

$$-\oint_{\Sigma}(\mathbf{E}\times\mathbf{H}^{\mathrm{I}})\cdot d\mathbf{s} = \underbrace{\int_V\left(\mathbf{E}\cdot\frac{\partial\mathbf{D}}{\partial t}+\mathbf{H}^{\mathrm{I}}\cdot\frac{\partial\mathbf{B}}{\partial t}+\mathbf{J}_{\mathsf{s}}^{\mathrm{I}}\cdot\frac{\partial}{\partial t}\left(\Lambda\mathbf{J}_{\mathsf{s}}^{\mathrm{I}}\right)\right)dv}_{\text{Power dW/dt in the E\&M field}}+\int_V\mathbf{E}\cdot\mathbf{J}_{\mathsf{n}}\,dv$$

Power dW/dt in the E&M field

The energy stored in the electromagnetic field is

$$dW = \int_V\left(\mathbf{E}\cdot d\mathbf{D}+\mathbf{H}^{\mathrm{I}}\cdot d\mathbf{B}+\mathbf{J}_{\mathsf{s}}^{\mathrm{I}}\cdot d\left(\Lambda\mathbf{J}_{\mathsf{s}}^{\mathrm{I}}\right)\right)dv$$

So that the energy W is a function of **D**, **B**, and $\Lambda\mathbf{J}_s^{\mathrm{I}} = \mathbf{v}_s$. However, one rarely has control over these variables, but rather over their conjugates **E**, $\mathbf{H}^{\mathrm{I}}$, and $\mathbf{J}_s^{\mathrm{I}}$.

Method I: The Coenergy

The coenergy is a function of **E**, $\mathbf{H}^{\mathrm{I}}$, and $\mathbf{J}_s{}^{\mathrm{I}}$ is defined by

$$W + \widetilde{W} = \int_V \left(\mathbf{E}\cdot\mathbf{D} + \mathbf{H}^{\mathrm{I}}\cdot\mathbf{B} + \mathbf{J}_{\mathsf{s}}^{\mathrm{I}}\cdot\left(\Lambda\mathbf{J}_{\mathsf{s}}^{\mathrm{I}}\right)\right)dv$$

and with

$$dW = \int_V \left(\mathbf{E}\cdot d\mathbf{D} + \mathbf{H}^{\mathrm{I}}\cdot d\mathbf{B} + \mathbf{J}_{\mathsf{s}}^{\mathrm{I}}\cdot d\left(\Lambda\mathbf{J}_{\mathsf{s}}^{\mathrm{I}}\right)\right)dv$$

gives

← EQS → ←——— MQS ———→

$$d\widetilde{W} = \int_V \left(\mathbf{D}\cdot d\mathbf{E} + \mathbf{B}\cdot d\mathbf{H}^{\mathrm{I}} + \Lambda\mathbf{J}_{\mathsf{s}}^{\mathrm{I}}\cdot d\mathbf{J}_{\mathsf{s}}^{\mathrm{I}}\right)dv$$

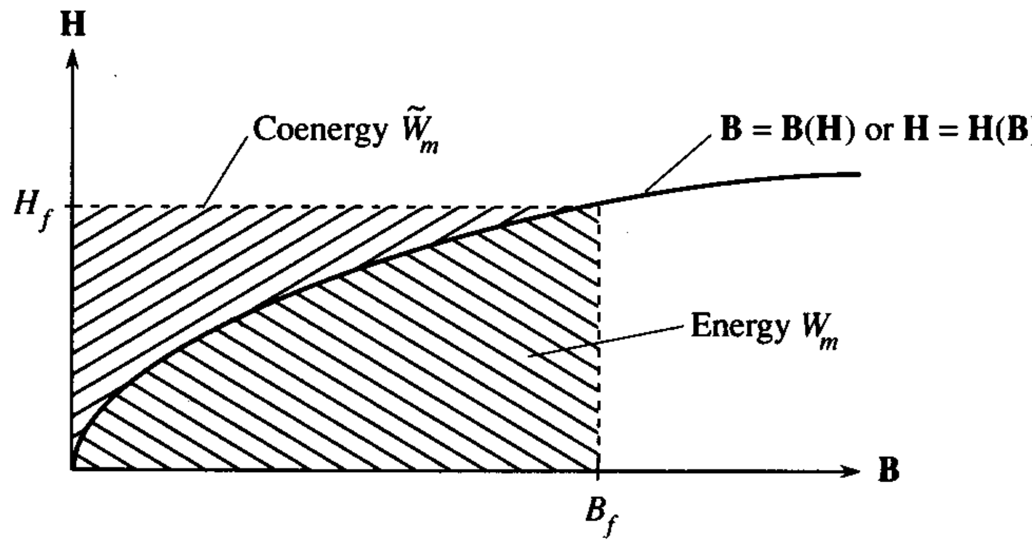
(The coenergy is the Free Energy at zero temperature.)

Interpretation of the Coenergy

Consider the case where there are only magnetic fields:

$$\widetilde{W} = \int_{\mathrm{H}^{\mathrm{I}}} d\widetilde{W} = \int_V \mathrm{B} \cdot d\mathrm{H}^{\mathrm{I}}\, dv \qquad W = \int_{\mathrm{B}} dW = \int_{\mathrm{B}} \int_V \mathrm{H}^{\mathrm{I}} \cdot d\mathrm{B}\, dv$$

$$W_m + \widetilde{W}_m = H_f B_f$$



The energy and coenergy contain the same information.

Method II

$$dW = \int_V \left(\mathbf{E} \cdot d\mathbf{D} + \mathbf{H}^{\mathrm{II}} \cdot d\mathbf{B} + \mathbf{J}_{\mathsf{s}}^{\mathrm{II}} \cdot d\left(\Lambda \mathbf{J}_{\mathsf{s}}^{\mathrm{II}} + \Lambda \nabla \times \mathbf{M} \right) \right) dv$$

$$d\widetilde{W} = \int_V \left(\mathbf{D} \cdot d\mathbf{E} + \mathbf{B} \cdot d\mathbf{H}^{\mathrm{II}} + \left(\Lambda \mathbf{J}_{\mathsf{s}}^{\mathrm{II}} + \Lambda \nabla \times \mathbf{M} \right) \cdot d\mathbf{J}_{\mathsf{s}}^{\mathrm{II}} \right) dv$$

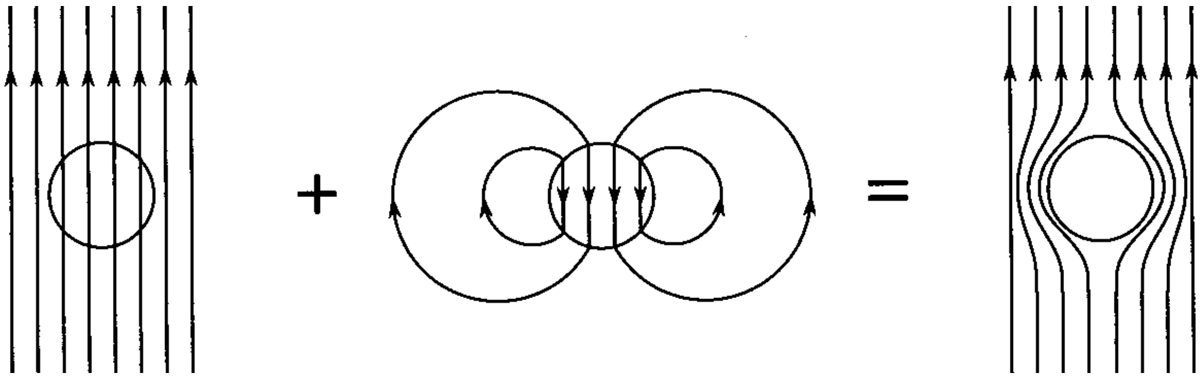
In the important case when of the MQS limit and $\mathbf{J}_{\mathsf{s}}^{\mathrm{II}} = \mathbf{J}_{\mathsf{s,app}} = 0$

$$dW\Big|_{\mathbf{J}_{\mathsf{s,app}}=0} = \int_V \left(\mathbf{H}^{\mathrm{II}} \cdot d\mathbf{B} \right) dv$$

$$d\widetilde{W}\Big|_{\mathbf{J}_{\mathsf{s,app}}=0} = \int_V \left(\mathbf{B} \cdot d\mathbf{H}^{\mathrm{II}} \right) dv$$

Note that these two relations apply also in free space.

Example: Energy of a Superconducting Sphere



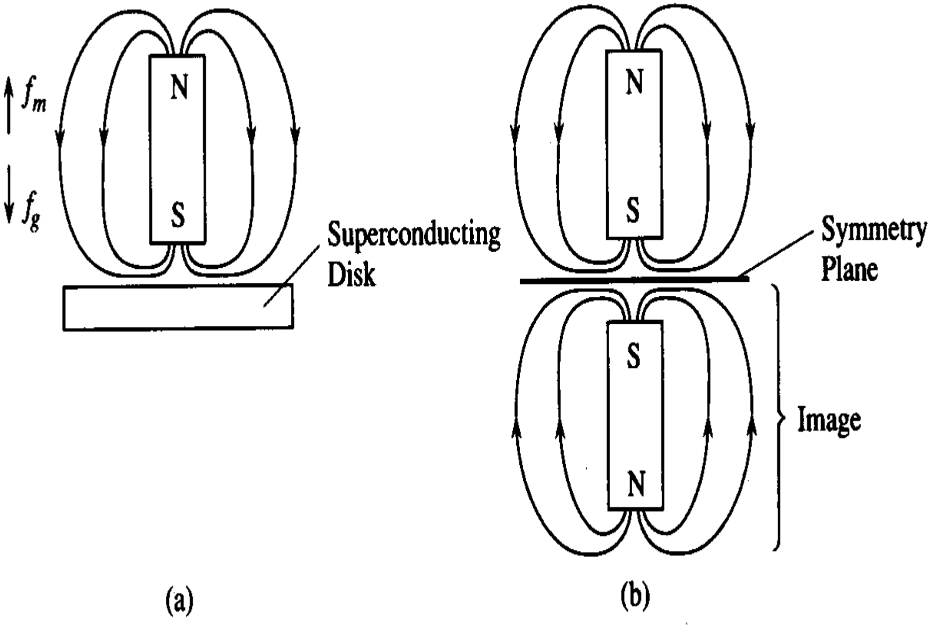
$$\mathbf{B}(r \le R) = 0 \qquad \mathbf{H}^{II}(r < R) = \frac{3}{2} H_o \mathbf{i}_z$$

$$\mathbf{H}(r \ge R) = \mathrm{Re}\left\{\hat{H}_o\left(1 - \left(\frac{R}{r}\right)^3\right)\cos\theta\right\}\mathbf{i}_r - \mathrm{Re}\left\{\hat{H}_o\left(1 + \frac{1}{2}\left(\frac{R}{r}\right)^3\right)\sin\theta\right\}\mathbf{i}_\theta$$

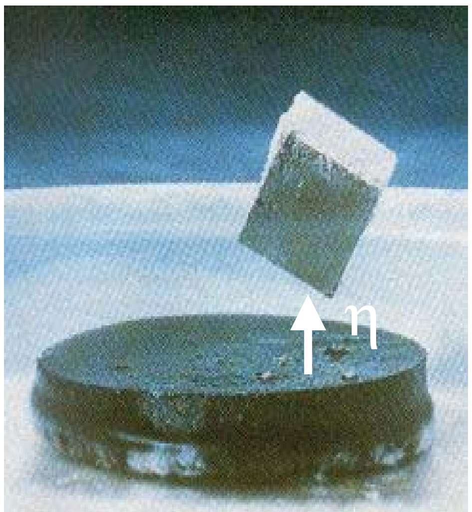
$$W_{inside} = \int_B \int_V \left(\mathbf{H}^{\mathrm{II}} \cdot d\mathbf{B}\right) dv = 0$$

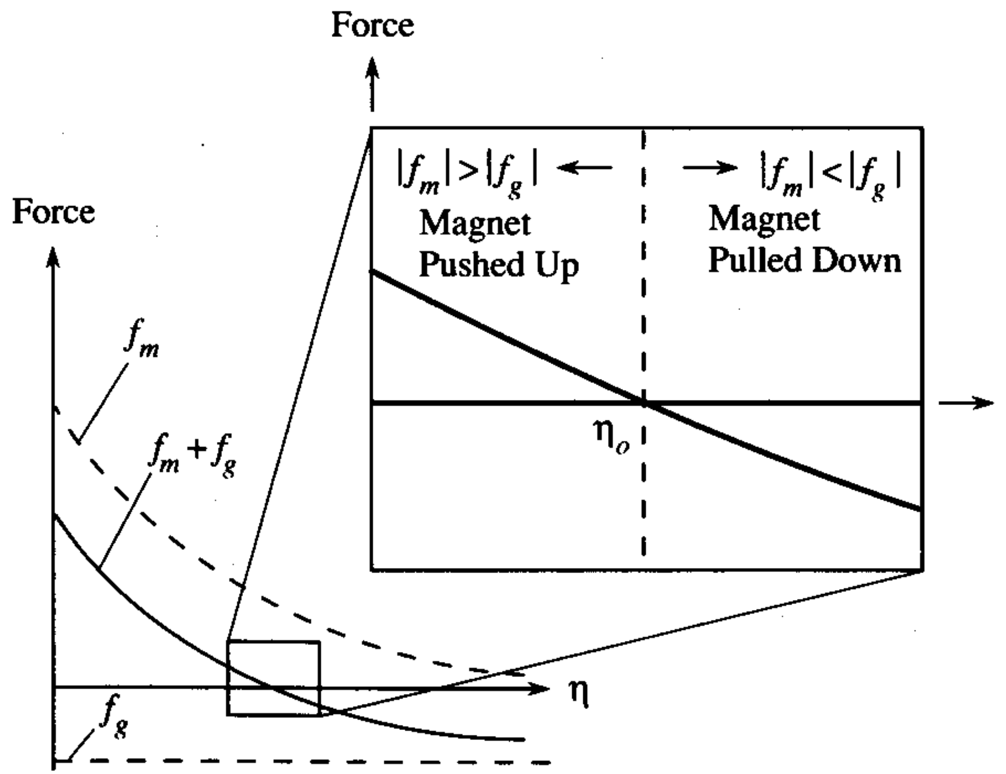
$$W_{outside} = \int_{\mathbf{B}} \int_V \left(\mathbf{H}^{\mathrm{II}} \cdot d\mathbf{B}\right) dv = \int_V \frac{1}{\mu_o} \mathbf{B}^2 \, dv \neq 0$$

Magnetic Levitation

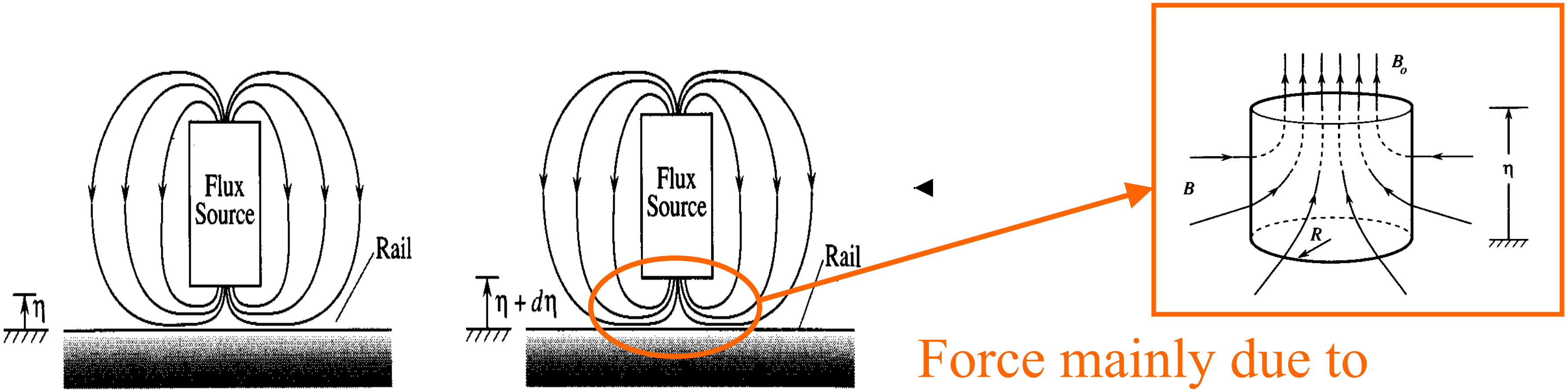


Magnetic vs. Gravitational Forces





Magnetic Levitation Equilibrium Point



Force mainly due to bending of flux lines here.

$$\Phi = \pi R^2 B_o = 2\pi R \eta B$$

$$W_m = \int_V \frac{1}{\mu_o} B^2 \, dv \approx \frac{\pi R^2 \eta}{\mu_o} B^2 = \frac{\Phi^2}{8\pi\mu_o\eta}$$

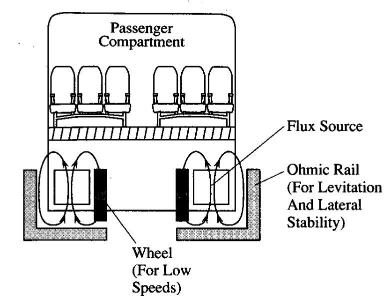
$$f_m = -\frac{\partial}{\partial\eta} W_m(\Phi, \eta) \approx \frac{\Phi^2}{8\pi\mu_o\eta^2}$$

Force is upwards

Equilibrium point $f_m(\eta_0) = mg$ ⇒ $\eta_o = \sqrt{\dfrac{\Phi^2}{8\pi\mu_o mg}}$

Levitating magnets and trains

$$f_m \approx \frac{\Phi^2}{8\pi\mu_o\eta^2}$$



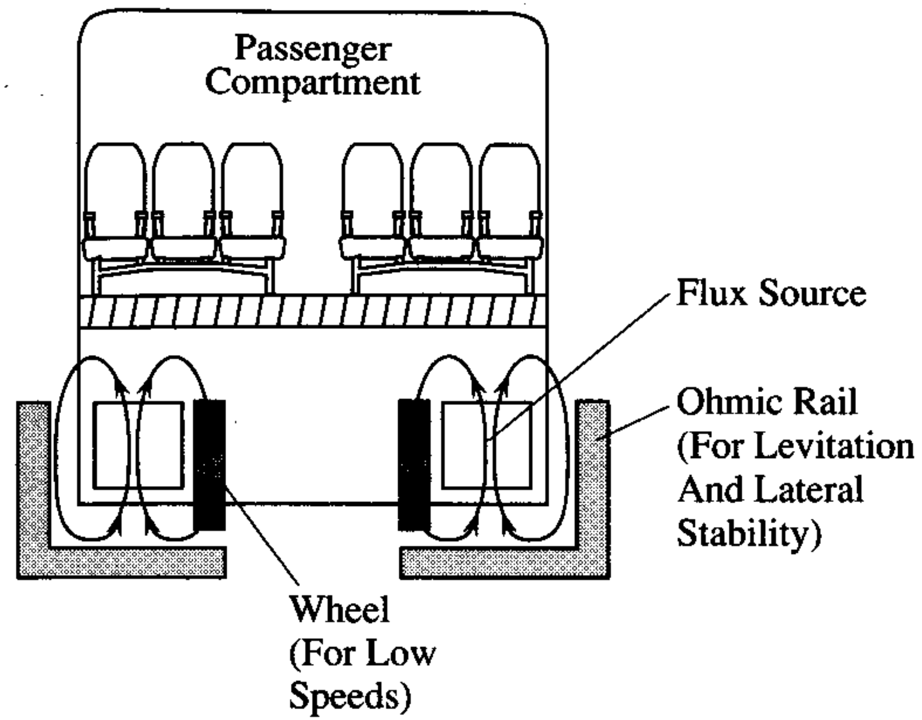
B = 1 Tesla
Area = 0.5 cm^2
η_0 = 1 cm
Force = 1 Newton

Enough to lift magnet, but not a train

B = 2 Tesla
Area = 100 cm^2
η_0 = 10 cm
Force = 1,200 Newtons

Enough to lift a train

Maglev Train



Magnets on the train are superconducting magnets;
the rails are ohmic!

Principle of Maglev

The train travels at a velocity U, and the moving flux lines and the rails "see" a moving magnetic field at a frequency of

$$\omega \sim U/R.$$

If this frequency is much larger than the inverse of the magnetic diffusion time,

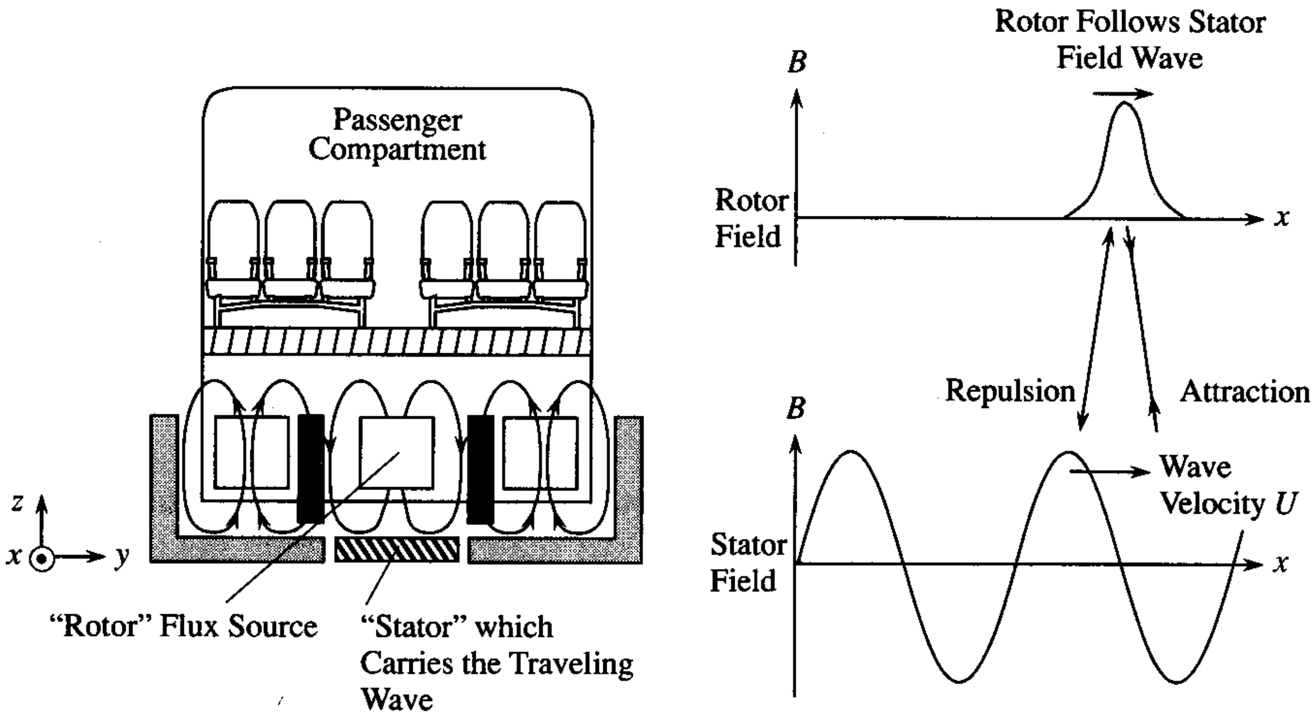
$$\tau_m = \mu \sigma_o R d$$

then the flux lines are "repelled" from the ohmic rails.

$$\omega \tau_m \gg 1 \qquad\qquad U \gg \frac{1}{\mu \sigma_o d}$$

From the previous numbers, U > 40 km/hr for levitation.

Real trains have wheels and high-voltage rails



Synchronous motor action down the rails provides thrust to accelerate train to the needed velocity to levitate, and provides a source of energy to further accelerate the train and to overcome the losses due to drag from the wind.

Our Approach to Superconductivity