6.975 Lecture 2: Classical and Quantum Models of Superconductors

Thursday, February 6, 2003

1 Classical Model of a Normal Metal

The electron gas model for conductivity starts with

$$\mathbf{F} = m\frac{d\mathbf{v}}{dt} = q\mathbf{E} - \underbrace{\frac{m\mathbf{v}}{\tau}}_{\text{drag term}} \tag{1}$$

where $\mathbf{F}$ is the force on the electron, m is the electron mass, $\mathbf{v}$ is the velocity of the electron, q is the charge, and τ is known as the scattering time.

The acceleration of the electron is given by Newton's Law. The force on the electron is the Coulomb force plus a phenomenological drag term. The phenomenological drag term is interpreted in terms of electron scattering. The electron accelerates under the influence of a electric field (voltage) until it is scattered. The scattering event randomizes the velocity of the electron, and it starts accelerating again because of the electric field. The net result is that the electrons have an average velocity proportional to the strength of the electric field.

For Steady State DC Fields

Under steady state, the electrons have a constant average velocity.

$$0 = q\mathbf{E} - \frac{m\mathbf{v}}{\tau} \qquad \text{so that} \qquad \mathbf{v} = \frac{q\tau}{m}\mathbf{E} \tag{2}$$

For a material with a density of conduction electrons n, the current density $\mathbf{J}$ is given by

$$\begin{aligned} \mathbf{J} &= nq\mathbf{v} && (3) \\ &= \frac{nq^2\tau}{m}\mathbf{E} \\ \mathbf{J} &= \sigma\mathbf{E} && (4) \end{aligned}$$

Therefore, the DC conductivity σ is given as

$$\sigma = \frac{ne^2\tau}{m} \tag{5}$$

The resistivity, $\rho = \sigma^{-1}$ is given as

$$\rho = \frac{m}{ne^2\tau} \tag{6}$$

The temperature dependence of the resistivity of the normal state is approximately

$$\rho(T) = \rho_i + \rho_l(T) \tag{7}$$

where ρ_i is due to scattering off defects in the material and is independent of the temperature, $\rho_l(T)$ is due to scattering off the lattice vibrations. In this model, ρ_i is fixed and dominates at low temperatures. At high temperatures $\rho_l(T)$ dominates and is approximately linear in T. Fig. 1 shows $\rho(T)$.

2 Classical Model of a Superconductor

For a perfect conductor, Eq. 1 still holds, but the drag term is zero. Thus, the motion of the electron is governed by

$$\mathbf{F} = m\frac{d\mathbf{v}}{dt} = q\mathbf{E} \tag{8}$$

Using the definition of current given in Eq. 3, we can differentiate both sides and substitute Eq. 8 to get the time rate of change of the current density.

$$\begin{aligned} \frac{\partial \mathbf{J}}{\partial t} &= nq\frac{\partial \mathbf{v}}{\partial t} && (9) \\ &= \frac{nq^2}{m}\mathbf{E} && (10) \\ \frac{\partial \mathbf{J}}{\partial t} &= \frac{1}{\mu_0\lambda^2}\mathbf{E} && (11) \end{aligned}$$

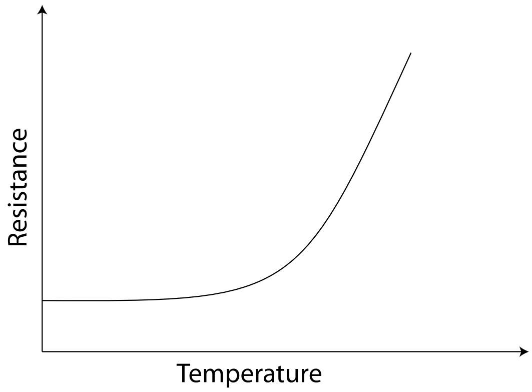

Figure 1: Resistance of a normal metal as a function of temperature.

where μ_0 is the permeability of free space and $\lambda = \sqrt{\frac{m\mu_0}{nq^2}}$ is called the penetration depth. You can see for yourself that λ has the units of length. We will see that in general fields and currents in a superconductor are restricted to within a penetration depth of the surface. Eq. 11 is referred to as the first London equation after the London brothers who used this equation to describe superconductivity.

For sinusoidal fields and currents of the form $\mathbf{J} = \mathbf{J}\, e^{i\omega t}$ and $\mathbf{E} = \mathbf{E}\, e^{i\omega t}$, one finds that

$$\mathbf{J} = \frac{nq^2}{j\omega m}\mathbf{E} \tag{12}$$

$$\mathbf{J} = \frac{1}{j\omega\Lambda}\mathbf{E} \tag{13}$$

where $\Lambda = m/nq^2$. Therefore, the AC conductivity $\sigma(\omega)$ is given as

$$\sigma(\omega) = \frac{1}{j\omega\Lambda} \tag{14}$$

Comparing Eq. 5 and Eq. 14, we see that unlike a normal metal which has finite DC conductivity, the DC conductivity of a superconductor, given as the limit of the AC conductivity as $\omega \to 0$ is infinite. The AC conductivity of the superconductor, however, looks just like that of an inductor. Thus, a superconductor can be modelled as an inductor. At nonzero temperatures, this is not exactly true, since some of the normal resistance still lingers around.

3 Two Fluid Model

Superconductivity arises from the grouping of electrons into Cooper pairs. Whereas in a normal metal, each electron moves independently; in a superconductor, an electron moves in a correlated fashion with its Cooper pair partner. (A Cooper pair consists of two electrons that have opposite spin and opposite momenta.) Thus, the effective charge carrying particles in a superconductor are Cooper pairs (superelectrons with mass $m^* = 2m_e$ and charge $q^* = -2e$), not individual electrons. Superelectrons have no resistance to flow: current carried by superelectrons does not create a voltage.

Below the critical temperature T_c, electrons begin to form Cooper pairs and the DC resistance of a superconductor drops abruptly to zero, since the current is now being carried by resistanceless superelectrons. (Under sufficiently large current densities or magnetic fields, the superconductivity can be destroyed, but we will not be concerned with high currents or fields, and so those effects are unimportant to us.) For $0 < T < T_c$, some of the electrons will be bound together in Cooper pairs and some will still be "normal" electrons. Thus, there will be a mixture of superelectrons and normal electrons. This is usually modelled in terms of having two different types of electron "fluids". The two-fluid model is a simple yet useful model for the behavior of a superconductor at nonzero temperatures. It consists of considering the superconductor as two parallel channels, one superconducting and one normal. The normal channel looks like a resistor. The superconducting channel looks like an inductor. Since the two conduction mechanisms occur in parallel, the electrical analog has a resistor and inductor in parallel as shown in Fig. 2.

The power of the two fluid model comes in the estimation of AC properties of the superconductor (for reasons we will see later, this model only works for frequencies well below the gap frequency, $\omega \ll 2\Delta/\hbar$). Correctly applied, the two fluid model is very convenient. For the DC case, the superconducting channel inductor will short out the normal channel resistance. Thus, the two fluid model reproduces the fact that a superconductor has no DC resistance. Again, we see that the DC resistance of the superconductor drops to zero as soon as the superconducting branch appears, regardless of how "weak" it is. The inductance shown in Fig. 2 is often referred to as "kinetic" inductance. Generally, an inductor stores energy in a magnetic field. For the kinetic inductance of a superconductor, the energy is stored in the kinetic energy of the Cooper pairs, hence the name. Normally, the voltage developed across an inductor represents the fact that energy must be supplied to change the

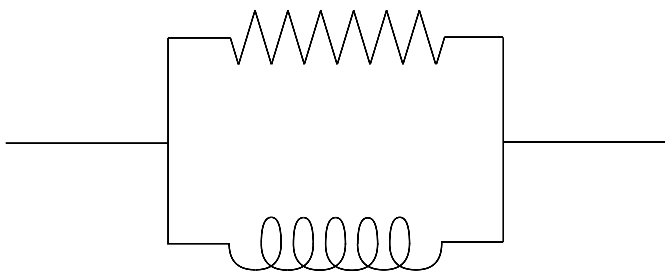

Figure 2: Two fluid electrical model of superconductor

magnetic field stored in the inductor. The voltage developed across the kinetic inductance represents energy needed to change the momentum of inertial superelectrons.

At $T = 0$, all the electrons are bound together in Cooper pairs. The density of superelectrons at finite temperatures is approximately given as

$$n_s(t) = n(1 - t^4) \tag{15}$$

where n_s is the density of superelectrons (Cooper pairs), n is the total density of conduction electrons, and $t = T/T_c$ is referred to as the "reduced" temperature. From Eq. 15, we get that the density of normal electrons is then given as

$$n_n(t) = n - n_s(t) = nt^4 \tag{16}$$

Using Eq.14, and substituting the density of superconducting electrons, we get

$$\sigma(\omega) = \frac{1}{j\omega\mu_0\lambda^2(T)} \tag{17}$$

where,

$$\lambda^2(T) = \frac{m\mu_0}{n_s(T)e^2} \tag{18}$$

The conductivity of the normal channel is correspondingly given as

$$\sigma_n(T) = \frac{n_n(T)e^2\tau}{m} = \sigma\frac{n_n(T)}{n} \tag{19}$$

At AC frequencies, the two-fluid electrical model of the superconductor gives a complex impedance. For low frequencies, most of the current goes through the inductor, i.e. is carried by superelectrons. However, some of the current goes through the normal resistance, i.e. is carried by normal electrons. There is dissipation associated with any current that goes through the normal resistive channel. For this reason, there is dissipation in a superconductor for AC currents. At high enough frequencies, the two-fluid model simply breaks down. It takes energy 2Δ to break a Cooper pair into its constituent electrons. The frequency corresponding to this energy is

$$f_{max} = 2\Delta/h \tag{20}$$

For frequencies greater than or equal to f_{max}, the AC current breaks up the Cooper pairs. We will restrict ourselves to frequencies well below this maximum frequency, and so this effect is also unimportant for us.

4 Flux Conservation

Consider Fig. 3. The flux linkage in the right hand circuit is given as

$$\lambda_{\Phi 2} = Mi_1 + L_2 i_2 \tag{21}$$

Faraday's Law for this circuit gives

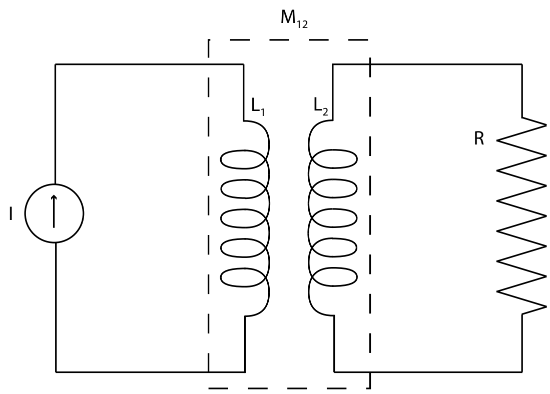

Figure 3: Current source inductively coupled to a resistor through a transformer

$$v_2 = \frac{d\lambda_{\Phi 2}}{dt} = i_2 R \tag{22}$$

$$\frac{d(L_1 i_1 + L_2 i_2)}{dt} = i_2 R \tag{23}$$

If we let $R \to 0$, Eq. 23 becomes

$$\frac{d(L_1 i_1 + L_2 i_2)}{dt} = 0 \tag{24}$$

$$L_1 i_1 + L_2 i_2 = K \tag{25}$$

where K is some constant. This simple analogy shows us that the total flux linkage from the external field (the current source) and the induced currents should remain constant for circuit with only inductance, but no resistance. However, it turns out that this model *does not* describe superconductivity. Instead, superconductors exhibit the Meissner Effect. Rather than conserving flux, superconductors *expel* flux. We will explain this effect using the macroscopic quantum model of superconductivity.

5 Review of Quantum Mechanics

5.1 Classical Equations of Motion

The classical equation of motions can be written succinctly as

$$\frac{d\mathbf{p}}{dt} = -\nabla V(\mathbf{r}) \tag{26}$$

where $\mathbf{p}$ is the canonical momentum and $V(\mathbf{r})$ is the generalized potential. The energy of this system is then given as

$$E = \frac{\mathbf{p} \cdot \mathbf{p}}{2m} + V(\mathbf{r}) \tag{27}$$

5.2 Quantum Equations of Motion

The transition from this classical system to the corresponding quantum system is achieved by changing the variables to operators.

$$\hat{\mathbf{p}} = \frac{\hbar}{i}\nabla \tag{28}$$

$$\hat{E} = i\hbar\frac{\partial}{\partial t} \tag{29}$$

Instead of having definite position and momentum values, the “particle” is described by a wavefunction $\psi(\mathbf{r}, t)$, which represents the probability of finding it at a particular location.

For a particle in a potential $V(\mathbf{r})$, substituting the operator definitions into Eq. 27 leads to the familiar Schrödinger equation.

$$i\hbar\frac{\partial}{\partial t}\psi(\mathbf{r}, t) = -\frac{\hbar^2}{2m}\nabla^2\psi(\mathbf{r}, t) + V(\mathbf{r})\psi(\mathbf{r}, t) \tag{30}$$

The probability density function ρ is given according to Born's rule as

$$\rho = \psi^*\psi = |\psi|^2 \tag{31}$$

The probability current density is given as

$$\mathbf{J} = \frac{h}{2im}\{\psi^*\nabla\psi - \psi\nabla\psi*\} \tag{32}$$

Using the definition of $\hat{\mathbf{p}}$, Eq. 32 can also be written as

$$\mathbf{J} = \frac{1}{m}\mathrm{Re}\{\psi^*\hat{\mathbf{p}}\psi\} \tag{33}$$

5.3 Classical Charged Particle in EM Field

Classically, a charged particle in an electromagnetic field is governed by Newton's equation under the Lorentz's force.

$$m\frac{d\mathbf{v}}{dt} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{34}$$

We will express Maxwell's equations as.

$$\mathbf{E} = -\nabla\phi - \frac{\partial\mathbf{A}}{\partial t} \tag{35}$$

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{36}$$

Substituting Maxwell's equations into Eq.34 gives

$$m\frac{d\mathbf{v}}{dt} = q\left(-\nabla\phi - \frac{\partial\mathbf{A}}{\partial t} + \mathbf{v} \times \nabla \times \mathbf{A}\right) \tag{37}$$

In incorporating an EM field, the momentum must be modified to include not only the momentum of the particle, but also the momentum of the field. The new momentum is thus given as

$$\mathbf{p} = m\mathbf{v} + q\mathbf{A} \tag{38}$$

Using this new definition of momentum in Eq. 37 gives after much manipulation and use of vector identities

$$\frac{d\mathbf{p}}{dt} = -\nabla \underbrace{\left(q\phi - \frac{q}{m}\mathbf{p}\cdot\mathbf{A} + \frac{q^2}{2m}\mathbf{A}\cdot\mathbf{A} \right)}_{V(\mathbf{p},\mathbf{r})} \tag{39}$$

Similar to Eq. 27, we have

$$\begin{aligned} E &= \frac{\mathbf{p}\cdot\mathbf{p}}{2m} + V(\mathbf{p},\mathbf{r}) && (40) \\ &= \frac{\mathbf{p}\cdot\mathbf{p}}{2m} + q\phi - \frac{q}{m}\mathbf{p}\cdot\mathbf{A} + \frac{q^2}{2m}\mathbf{A}\cdot\mathbf{A} && (41) \\ &= \frac{1}{2m}(\mathbf{p} - q\mathbf{A})\cdot(\mathbf{p} - q\mathbf{A}) + q\phi && (42) \\ E &= \frac{1}{2}mv^2 + q\phi && (43) \end{aligned}$$

Eq. 43 comes directly from the equations above, but it can also easily be derived from the fact that the energy of the particle is simply equal to its kinetic energy plus its potential energy. Magnetic fields do no work, so the only potential energy comes from the electric potential (voltage). For the transition to quantum mechanics, Eq. 42 is actually more useful for us.

5.4 Quantum Charged Particle in EM Field

Using the operators defined in Eqs. 28 and 29, we can rewrite Eq. 42 as

$$i\hbar\frac{\partial}{\partial t}\psi(\mathbf{r},t) = \frac{1}{2m}\left(\frac{\hbar}{i}\nabla - q\mathbf{A}\right)\cdot\left(\frac{\hbar}{i}\nabla - q\mathbf{A}\right)\psi(\mathbf{r},t) + q\phi\psi(\mathbf{r},t) \tag{44}$$

Eq. 33 still holds, but with the new definition of momentum, it is expanded out as

$$\mathbf{J} = \frac{1}{m}\mathrm{Re}\left\{\psi^*\left(\frac{\hbar}{i}\nabla - q\mathbf{A}\right)\psi\right\} \tag{45}$$

6 Macroscopic Quantum Model

6.1 Supercurrent Equation

The fundamental assumption of the macroscopic quantum model of superconductivity is that the superconductor can be described by a macroscopic quantum wave function that extends over the entire superconductor. The macroscopic wavefunction $\psi(\mathbf{r}, t)$ satisfies a Schrödinger-like equation and has the following properties:

1. The squared modulus of the wavefunction is the density of Cooper pairs $n_s^*(\mathbf{r}, t)$.
$$|\psi(\mathbf{r}, t)|^2 = n_s^*(\mathbf{r}, t) \tag{46}$$

2. The superconducting current density is
$$\mathbf{J} = \frac{q^*}{2m^*}\mathrm{Re}\left\{\psi^*\left(\frac{\hbar}{i}\nabla - q^*\mathbf{A}\right)\psi\right\} \tag{47}$$
where as before $q^* = -2e$ is the charge of the Cooper pair and $m^* = 2m_e$ is the mass of the Cooper pair.

We now make the additional assumption that the wavefunction is of the form
$$\psi(\mathbf{r}, t) = |n_s^*|^{1/2}\exp(i\theta(\mathbf{r}, t)) \tag{48}$$

Substituting this wavefunction into the superconducting current density equation, we get
$$\mathbf{J} = -\frac{1}{\mu_0\lambda^2}\left[\mathbf{A}(\mathbf{r}, \mathbf{t}) + \frac{\Phi_0}{2\pi}\nabla\theta(\mathbf{r}, t)\right] \tag{49}$$
where $\Phi_0 = h/2e = 2.07 \times 10^{-15}\mathrm{Tm}^2$ is the flux quantum. Eq. 49 is called the supercurrent equation and describes the current density of the superconducting electrons in a single piece of material.

6.2 London's Equations

The time derivative of the supercurrent equation leads (after considerable work) to the first London equation, which describes perfect conductivity.
$$\frac{\partial\mathbf{J}}{\partial t} = \frac{1}{\mu_0\lambda^2}\mathbf{E} \tag{50}$$

A careful derivation actually reveals a nonlinear term in the first London equation. The complete first London equation can be derived from the second London equation and Lorentz's law. We will not do this here, but it could be a useful exercise to do for yourself.

The second London equation is derived from the curl of the supercurrent equation.

$$\nabla \times \left\{ \mathbf{J_s}(\mathbf{r}, t) = -\frac{1}{\mu_0 \lambda^2} \left[\mathbf{A}(\mathbf{r}, t) + \frac{\Phi_0}{2\pi} \nabla \theta(\mathbf{r}, t) \right] \right\} \tag{51}$$

Noting that $\nabla \times \nabla \theta = 0$, we get

$$\nabla \times \mathbf{J_s}(\mathbf{r}, t) = -\frac{1}{\mu_0 \lambda^2} \mathbf{B}(\mathbf{r}, t) \tag{52}$$

Using Maxwell's equation $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$, we get

$$\nabla \times \nabla \times \mathbf{B}(\mathbf{r}, t) = -\frac{1}{\lambda^2} \mathbf{B}(\mathbf{r}, t) \tag{53}$$

This leads to the second London equation

$$\nabla^2 \mathbf{B}(\mathbf{r}, t) = \frac{1}{\lambda^2} \mathbf{B}(\mathbf{r}, t) \tag{54}$$

As mentioned earlier, λ is referred to as the penetration depth. The solutions to the second London equation are magnetic fields that decay exponentially within a characteristic length λ of the surface of the superconductor. The second London equation correctly describes the Meissner effect mentioned earlier. The electric field and the current density also decay exponentially within the penetration depth. In the bulk superconductor, therefore, there are no fields or currents.

6.3 Flux Quantization

Taking the closed line integral of the supercurrent equation (49) gives

$$\oint_C d\mathbf{s} \cdot \left\{ \mathbf{J} = -\frac{1}{\mu_0 \lambda^2} \left[\mathbf{A}(\mathbf{r}, \mathbf{t}) + \frac{\Phi_0}{2\pi} \nabla \theta(\mathbf{r}, t) \right] \right\} \tag{55}$$

$$-\frac{\Phi_0}{2\pi} \oint_C \nabla \theta(\mathbf{r}, t) = \mu_0 \lambda^2 \oint_C \mathbf{J} \cdot d\mathbf{s} + \int_A \nabla \times \mathbf{A}(\mathbf{r}, t) \cdot d\mathbf{a} \tag{56}$$

$$-\frac{\Phi_0}{2\pi}\oint_C \nabla\theta(\mathbf{r},t) = \mu_0\lambda^2 \oint_C \mathbf{J}\cdot d\mathbf{s} + \Phi \tag{57}$$

where Φ is the flux through the contour C. Ordinarily, for a single-valued function the left hand integral would be identically zero. θ however only has a definite value modulus 2π. Therefore, the left hand integral gives $2\pi n$, which leads to the fluxoid quantization equation

$$n\Phi_0 = \underbrace{\Phi + \oint_C \mathbf{J}\cdot d\mathbf{s}}_{\text{fluxoid}} \tag{58}$$

where n is any integer and Φ_0 is the flux quantum defined earlier.

Fluxoid quantization is the central result that comes from a quantum treatment of superconducting material. If we take the path C to be in the bulk superconductor, there are virtually no currents flowing along C since they decay exponentially away from the surface. In this case, the second term on the right hand side of Eq. 58 is essentially zero and we get flux quantization

$$\Phi = n\Phi_0 \tag{59}$$

In many cases of interest, the approximate flux quantization equation applies. We will shift between these two cases but it should be implied from context whether we are discussing fluxoid quantization or flux quantization.