Massachusetts Institute of Technology
Department of Electrical Engineering and Computer Science
6.826 Principles of Computer Systems

PROBLEM SET 1 SOLUTIONS

Problem 1: Matrix Multiplication

a) Each element `mc(i,j)` of the matrix is equal to a sum of products. We calculate the sum by generating the sequence of the elements in the sum and then folding the elements of the sequence using the `+ :` construct.

```
FUNC isMatMul(ma: Matrix, mb: Matrix, mc: Matrix) -> Bool =
<< RET (ALL i: Range | ALL j: Range |
        mc(i,j) = + : { k :IN 0 .. n-1 | | ma(i,k)*mb(k,j) }) >>
```

b) The following implementation corresponds to an implementation of matrix multiplication in a conventional imperative language.

```
APROC MatMul(ma: Matrix, mb: Matrix) -> Matrix =
<< VAR mc: Matrix |
   VAR i: Int := 0 |
   DO i < n =>
     VAR j: Int := 0 |
     DO j < n =>
       VAR sum: Int := 0 |
       VAR k: Int := 0 |
       DO k < n =>
         sum := sum + ma(i,k)*mb(k,j);
         k := k+1
       OD;
       mc(i,j) := sum;
       j := j + 1
     OD;
     i := i+1
   OD;
   RET mc
>>
```

Problem 2: Distribution of Prime Numbers

a) The following `Spec` function closely follows the mathematical definitions of prime numbers. (Operator `//` denotes the remainder in division of integers.)

```
FUNC isPrime(p: Int) -> Bool =
 (p > 1) /\
 { n:Int | n > 0 /\ p // n = 0 } = {1, p}

FUNC isPrimeBetween(p: Int, n: Int) -> Bool =
  isPrime(p)  /\  n < p  /\  p < 2*n
```

b) This is a simple-minded implementation of the specification in the previous part. The atomic procedure `primeBetween` does a linear search for prime numbers from $n+1$ to $2n-1$ and returns the least number that is prime. The primality test is implemented in the `isPrimeImpl` atomic procedure by a linear search that attempts to find the smallest factor k of p where $2 < k \leq \sqrt{p}$.

```
APROC isPrimeImpl(p: Int) -> Bool =
<< VAR k: Int := 2 |
   DO (k*k <= p) =>
     IF (p // k = 0) => RET false [*] SKIP FI;
     k := k+1
   OD;
   RET true
>>
APROC primeBetween(n: Int) -> Int =
<< VAR x: Int := n+1 |
   DO x < 2*n =>
      IF isPrimeImpl(x) => RET x
      [*] x := x+1
      FI
   OD
>>
```

c) For example, let $n = 7$ and $p = 13$. Procedure `primeBetween` returns always 11, never 13.

Problem 3. Shortest Path

a) The shortest path predicate considers the set of all paths from n_1 to n_2 and then ensures that `path` has the minimum length.

```
FUNC isPathFromTo(g: Graph[Node].G,
                  n1: Node, n2: Node,
                  path: SEQ Node) -> Bool =
  g.paths(path) /\
  path.head=n1 /\ path.last=n2

FUNC isShortestPath(g: Graph[Node].G,
                    n1: Node, n2: Node,
                    path: SEQ Node) -> Bool =
 isPathFromTo(g,n1,n2,path) /\
 path.size = { path2: SEQ Node | isPathFromTo(g,n1,n2,path2)
                               | path2.size }.min
```

b) The implementation performs a breadth-first search in the graph finding the shortest distance to every reachable node from the node `n1`. The breadth-first search is implemented using a queue represented as a list of nodes `queue`. After reaching the target node `n2`, the path is reconstructed using the atomic procedure `recoverPath`. The reconstruction traverses the path backwards using the fact that `dist(path(i+1))=dist(path(i))+1` on the shortest path.

```
APROC recoverPath(g : Graph[Node].G,
                  dist : Node -> IN 0 .. n+1,
                  n2 : Node) -> SEQ Node =
<< IF dist(n2)=0 => RET {n2}
   [*] VAR nd: Node := { nd2: Node |
                 g(nd2,n2) /\ dist(n2)=dist(nd2)+1 }.min |
```

```
      RET recoverPath(g,dist,nd) + {nd}
   FI
>>

APROC shortestPath(g: Graph[Node].G,
                   n1: Node, n2: Node) -> SEQ Node =
<< VAR queue: SEQ Node := { n1 } |
   VAR dist:  Node -> IN 0 .. n+1 := (\ nd:Node | n+1 ) |
   dist(n1) := 0;
   DO queue.size > 0 =>
      VAR first: Node := queue.head |
      IF first=n2 => RET recoverPath(g,dist,n2)
      [*] queue := queue.tail;
          VAR succ: SEQ Node :=
            { nd :IN 1 .. n | g(first,nd) /\ dist(nd) = n+1 } |
          queue := queue + succ;
          DO succ.size > 0 =>
             dist(succ.head) := dist(first) + 1;
             succ := succ.tail;
          OD
      FI
   OD;
   % There is no path from n1 to n2.  Procedure fails.
   false => SKIP
>>
```

c) One of the examples is the following. Let $n = 4$, `n1` $= 1$, `n2` $= 4$, and let the graph `g` be

$$\mathtt{g} = \{(1,2),(1,3),(2,4),(3,4)\}$$

The path `path` $= [1,3,4]$ satisfies `isShortestPath(n1,n2,path)` but the result of `shortestPath(g,n1,n2)` is always the path $[1,2,4]$.