Coding Bounds on Low Peak-to-Average Power Ratio Codes

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Outline

I. Orthogonal Frequency Division Multiplexing (OFDM)

II. Peak-to-Average Power Ratio (PAPR) problem

III. Bounds on Minimum Distance for Block Codes

IV. Main Results

V. Conclusion

OFDM SYSTEM BLOCK DIAGRAM

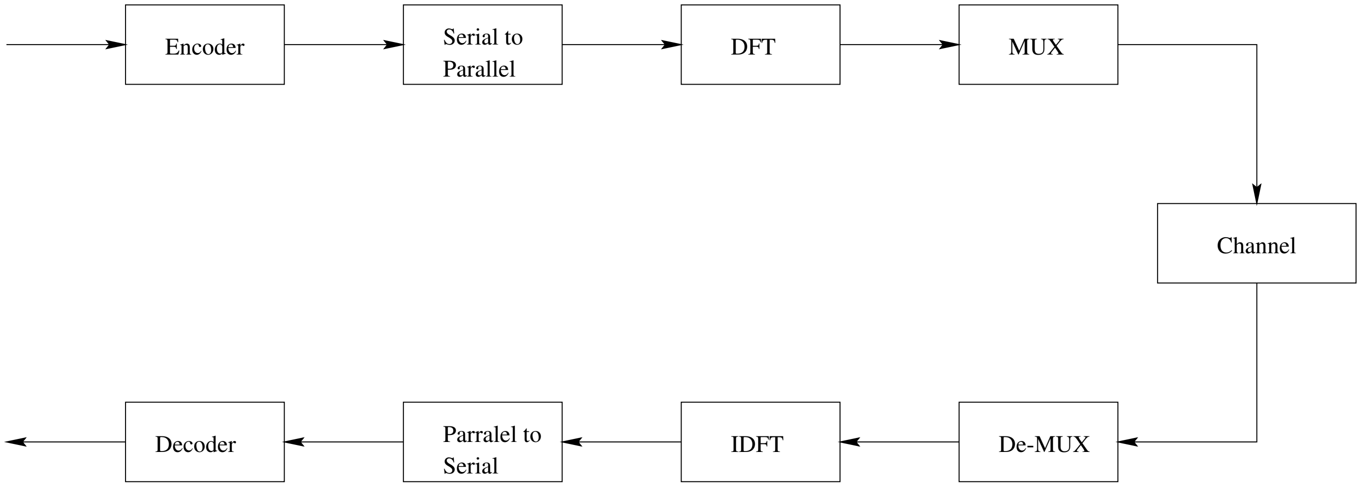

Figure 1 : OFDM SYSTEM

Signals

Output from DFT:

$$C_l = \sum_{i=0}^{n-1} c_i \exp(\frac{-2\pi jl}{n}) \quad \text{for} \quad 0 \leq l \leq n-1$$

Envelope:

$$S_c(t) = \sum_{i=0}^{n-1} c_i \exp(-2\pi j(f_0 + if_s)t) \qquad 0 \leq t \leq 1/f_s$$

f_0 = carrier frequency

f_s = bandwidth of each tone

$f_0 >> f_s$

$\Re(S_c(t))$ = transmitted signal

Power

$\Re(S_c(t))^2$ = instantaneous power of the transmitted signal

$|S_c(t)|^2$ = the envelope power of the signal

$||c||^2$ = average envelope power

For a code C and codeword $c \in C$

$$\text{PAPR}(c) = \max_{0 \leq t \leq 1} \frac{\left| \Re\left(\sum_{i=0}^{n-1} c_i \exp(-2\pi j(f_0 + i f_s)t) \right) \right|^2}{||c||^2}$$

$$\text{PAPR}(c) \leq \max_{0 \leq t \leq 1} \frac{|S_c(t)|^2}{||c||^2}$$

$$\text{PAPR}(C) = \max_{c \in C} \text{PAPR}(c)$$

Note: $\frac{P_{max}}{P_{ave}} \neq PAPR(C)$. Suppose Q is equal energy, then $\frac{P_{max}}{P_{ave}} = 1$ and $PAPR(C) \leq n$.

Approaches to solve PAPR Problem

- Only transmit those codewords with low PAPR (some fraction of the codewords will be ommitted).
- Clipping and Peak Windowing
- Peak Cancellation
- Coding and Scrambling
- Offset by phase shift

OFFSET SYSTEM BLOCK DIAGRAM

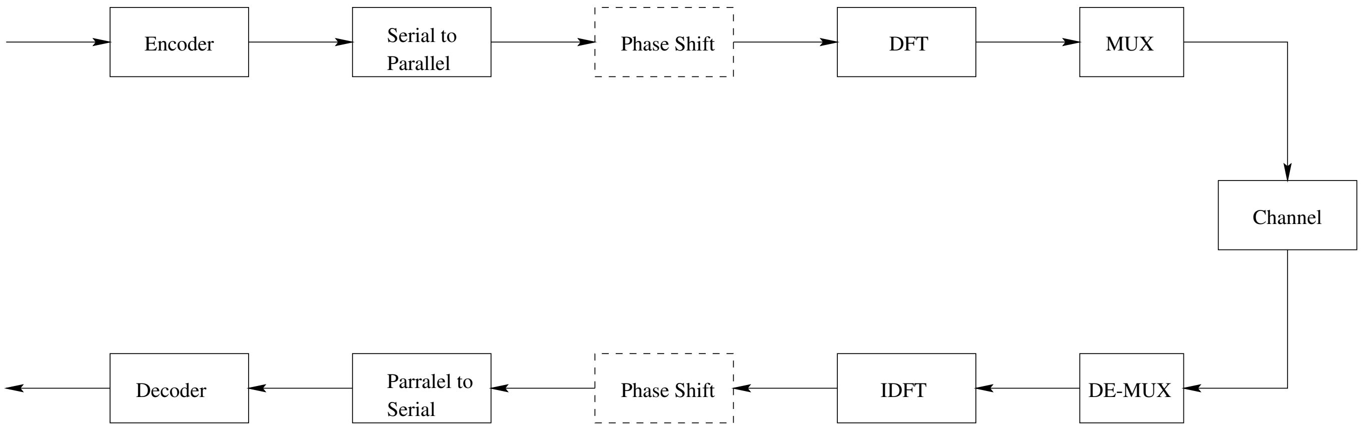

Figure 2 : OFDM SYSTEM WITH OFFSET

Gilbert-Varshamov Bound

A lower bound on the minimum distance d as a function of the code rate R, length n, alphabet q.

(i) List all codewords (q^n of them)

(ii) Choose arbitrary codeword from the list.

(iii) Delete from the list the chosen codeword and all codewords that are distance $d-1$ or less from that codeword. Stop when there are no codewords left.

Let V be the number of points in a sphere of radius d around any given point.

$$V = \binom{n}{0} + (q-1)\binom{n}{1} + \cdots + (q-1)^d \binom{n}{d}$$

- For a code C with q^k codewords, the average number of codewords is a sphere of radius d around an arbitrary point in space is $q^k \frac{V}{q^n}$. (We know $q^{k-n}V \geq 1$ otherwise can enlarge our codebook)
- There exists codes of rate R, minimum distance d, alphabet q that satisfy $R \geq 1 - \frac{1}{n}\log_q V$
- For binary codewords ($q = 2$), $M \geq 2^n \left[\sum_{i=0}^{d-1} \binom{n}{i}\right]^{-1}$

Definitions for any code C

- Classical coding problem:
  What is the achievable region for (n, k, d)
- OFDM coding problem:
  What is the achievable rate region of $(n, k, d, PAPR)$
- Aymptotically good code:
  sequence of codes $(n_1, k_1, d_1), (n_2, k_2, d_2), \ldots, (n_m, k_m, d_m)$
  such that $\lim_{m\to\infty}(\frac{d_m}{n_m})(\frac{k_m}{n_m}) > 0$
  (d_m/n_m) = normalized minimum distance
  (k_m/n_m) = rate

MAIN RESULTS

For a constant energy code constellation:

- A lower bound on PAPR as a function of (n, R, d)
- A lower bound on R given $(n, d, PAPR)$
  (given the maximum tolerable PAPR)
- Asymptotically good sequences of codes exist with PAPR growth of order $8 \log n$;
- Explicit construction of codes (trace codes) with upper bound on the PAPR of order $(\log n)^2$

Geometric Interpretation

Codewords in a constant energy code C are points on the n-dimensional complex sphere with radius $\sqrt{n}$. Define a curve by

$$\Omega = \{\omega(t,\zeta), 0 \leq t < 1\}$$

$$\omega(t,\zeta) = \{(\exp(2\pi j\zeta t), \exp(2\pi j(\zeta+1)t), \ldots, \exp(2\pi j(\zeta+n-1))t)\}$$

The closer a codeword is to $\Omega \cup -\Omega$, the larger the PAPR

$H(S,r)$ = the surface of points on the sphere within a distance r from S.
$A(S,r)$ = the area of $H(S,r)$
Then for any point x, $H(x,r)$ is a spherical cap

$$4\lceil 2\pi(n-1)\rceil A(d_*) + (l+1)A\left(\frac{d}{2}\right) \geq \frac{2\pi^n n^{n-\frac{1}{2}}}{(n-1)!}$$

Packing Relationship Bound

$$A\left(\Omega \cup -\Omega, d_* - \frac{d}{2}\right) + 2^{nR} A\left(\frac{d}{2}\right) \geq \frac{2\pi^n n^{n-\frac{1}{2}}}{(n-1)!}$$

d_* = minimum Euclidean distance between the codewords of C and the points of $\Omega \cup -\Omega$.