# Knowing and Teaching Elementary Mathematics

## *Teachers' Understanding of Fundamental Mathematics in China and the United States*

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# Introduction

Chinese students typically outperform U.S. students on international comparisons of mathematics competency. Paradoxically, Chinese teachers seem far less mathematically educated than U. S. teachers. Most Chinese teachers have had 11 to 12 years of schooling—they complete ninth grade and attend normal school for two or three years. In contrast, most U.S. teachers have received between 16 and 18 years of formal schooling—a bachelor's degree in college and often one or two years of further study.

In this book I suggest an explanation for the paradox, at least at the elementary school level. My data suggest that Chinese teachers begin their teaching careers with a better understanding of elementary mathematics than that of most U.S. elementary teachers. Their understanding of the mathematics they teach and—equally important—of the ways that elementary mathematics can be presented to students continues to grow throughout their professional lives. Indeed, about 10% of those Chinese teachers, despite their lack of formal education, display a depth of understanding which is extraordinarily rare in the United States.

I document the differences between Chinese and U.S. teachers' knowledge of mathematics for teaching and I suggest how Chinese teachers' understanding of mathematics and of its teaching contributes to their students' success. I also document some of the factors that support the growth of Chinese teachers' mathematical knowledge and I suggest why at present it seems difficult, if not impossible, for elementary teachers in the United States to develop a deep understanding of the mathematics they teach. I shall begin with some examples that motivated the study.

In 1989, I was a graduate student at Michigan State University. I worked as a graduate assistant in the Teacher Education and Learning to Teach Study (TELT) at the National Center for Research on Teacher Education (NCRTE) coding transcripts of teachers' responses to questions like the following:

> Imagine that you are teaching division with fractions. To make this meaningful for kids, something that many teachers try to do is relate mathematics to other things. Sometimes they try to come up with real-world situations or story-problems to show the application of some particular piece of content. What would you say would be a good story or model for $1\frac{3}{4} \div \frac{1}{2}$ ?

I was particularly struck by the answers to this question. Very few teachers gave a correct response. Most, more than 100 preservice, new, and experienced teachers, made up a story that represented $1\frac{3}{4} \times \frac{1}{2}$, or $1\frac{3}{4} \div 2$. Many other teachers were not able to make up a story.

The interviews reminded me of how I learned division by fractions as an elementary student in Shanghai. My teacher helped us understand the relationship between division by fractions and division by positive integers—division remains the inverse of multiplication, but meanings of division by fractions extend meanings of whole-number division: the measurement model (finding how many halves there are in $1\frac{3}{4}$) and the partitive model (finding a number such that half of it is $1\frac{3}{4}$).[1] Later, I became an elementary school teacher. The understanding of division by fractions shown by my elementary school teacher was typical of my colleagues. How was it then that so many teachers in the United States failed to show this understanding?

Several weeks after I coded the interviews, I visited an elementary school with a reputation for high-quality teaching that served a prosperous White suburb. With a teacher-educator and an experienced teacher, I observed a mathematics class when a student teacher was teaching fourth graders about measurement. During the class, which went smoothly, I was struck by another incident. After teaching measurements and their conversions, the teacher asked a student to measure one side of the classroom with a yardstick. The student reported that it was 7 yards and 5 inches. He then worked on his calculator and added, "7 yards and 5 inches equals 89 inches." The teacher, without any hesitation, jotted down "(89 inches)" beside the "7 yards and 5 inches" that she had just written on the chalkboard. The apparent mismatch of the two lengths, "7 yards and 5 inches" and "89 inches," seemed conspicuous on the chalkboard. It was obvious, but not surprising, that the student had misused conversion between feet and inches in calculating the number of inches in a yard. What surprised me, however, was that the apparent mismatch remained on the chalkboard until the end of the class without any discussion. What surprised me even more was that the mistake was never revealed or corrected, nor even mentioned after the class in a discussion of the student teacher's teaching. Neither the cooperating teacher nor the teacher-educator who was supervising the student teacher even noticed the mistake. As an elementary teacher and as a researcher who worked with teachers for many years, I had developed certain expectations about elementary teachers' knowledge of mathematics. However, the expectations I had developed in China did not seem to hold in the United States.

[1]For more information about the two models, see chapter 3, p. 72.



The more I saw of elementary mathematics teaching and research in the United States, the more intrigued I became. Even expert teachers, experienced teachers who were mathematically confident, and teachers who actively participated in current mathematics teaching reform did not seem to have a thorough knowledge of the mathematics taught in elementary school. Apparently, the two incidents that had amazed me were only two more examples of an already widespread and well-documented phenomenon.[2]

Later, I read international studies of mathematics achievement.[3] These studies found that students of some Asian countries, such as Japan and China, consistently outperformed their counterparts in the United States.[4] Researchers have described various factors that contribute toward this "learning gap": differences in cultural contexts, such as parental expectations or number-word systems[5]; school organization, or amount of time spent learning mathematics; content and content allocation in mathematics curricula.[6] As I read this research, I kept thinking about the issue of teachers' mathematical knowledge. Could it be that the "learning gap" was not limited to students? If so, there would be another explanation for U.S. students' mathematical performance. Unlike factors outside of classroom teaching, teachers' knowledge might directly affect mathematics

[2]For more information about research on teacher subject matter knowledge, see Ball (1988a), Cohen (1991), Leinhardt and Smith (1985), NCRTE (1991), Putnam (1992), and Simon (1993).

[3]The International Association for the Evaluation of Educational Achievement (IEA) conducted the First International Mathematics Study in 1964. The study measured achievement in various mathematical topics in each of 12 different countries at Grades 8 and 12. In the early 1980s, IEA carried out another study. The Second International Mathematics Study compared 17 countries in the Grade 8 component and 12 in the Grade 12 component. The Third International Mathematics and Science Study (TIMSS), in which more than 40 countries participated, has recently started to release its reports. (For more information about the three studies, see Chang & Ruzicka, 1986; Coleman, 1975; Crosswhite, 1986; Crosswhite et al., 1985; Husen, 1967a, 1967b; LaPointe, Mead, & Philips, 1989; Lynn, 1988; McKnight et al., 1987; National Center for Education Statistics, 1997; Robitaille & Garden, 1989; Schmidt, McKnight, & Raizen, 1997.)

[4]TIMMS results follow this pattern. For example, five Asian countries participated in the Grade 4 mathematics component. Singapore, Korea, Japan, and Hong Kong had the top average scores. These were significantly higher than the U.S. score. (Thailand was the fifth Asian country participating.)

[5]For example, the Chinese word for the number 20 means "two tens," the Chinese word for the number 30 means "three tens," and so on. The consensus is that the Chinese number-word system illustrates the relationship between numbers and their names more straightforwardly than the English number-word system.

[6]For more information, see Geary, Siegler, and Fan (1993); Husen (1967a, 1967b); Lee, Ichikawa, and Stevenson (1987); McKnight et al. (1987); Miura and Okamoto (1989); Stevenson, Azuma, and Hakuta (1986); Stevenson and Stigler (1991, 1992); Stigler, Lee, and Stevenson (1986); Stigler and Perry (1988a, 1988b); Stigler and Stevenson (1981).

teaching and learning. Moreover, it might be easier to change than cultural factors, such as the number-word system[7] or ways of raising children.

It seemed strange that Chinese elementary teachers might have a better understanding of mathematics than their U.S. counterparts. Chinese teachers do not even complete high school; instead, after ninth grade they receive two or three more years of schooling in normal schools. In contrast, most U.S. teachers have at least a bachelor's degree. However, I suspected that elementary teachers in the two countries possess differently structured bodies of mathematical knowledge, that aside from subject matter knowledge "equal to that of his or her lay colleague" (Shulman, 1986), a teacher may have another kind of subject matter knowledge. For example, my elementary teacher's knowledge of the two models of division may not be common among high school or college teachers. This kind of knowledge of school mathematics may contribute significantly to what Shulman (1986) called pedagogical content knowledge—"the ways of representing and formulating the subject that make it comprehensible to others" (p. 9).

I decided to investigate my suspicion. Comparative research allows us to see different things—and sometimes to see things differently. My research did not focus on judging the knowledge of the teachers in two countries, but on finding examples of teachers' sufficient subject matter knowledge of mathematics. Such examples might stimulate further efforts to search for sufficient knowledge among U.S. teachers. Moreover, knowledge from teachers rather than from conceptual frameworks might be "closer" to teachers and easier for them to understand and accept.

Two years later, I completed the research described in this book. I found that although U.S. teachers may have been exposed to more advanced mathematics during their high school or college education,[8] Chinese teachers display a more comprehensive knowledge of the mathematics taught in elementary school.

In my study, I used the TELT interview questions. The main reason for using these instruments is their relevance to mathematics teaching. As Ed Begle recounts in *Critical Variables in Mathematics Education*, earlier studies often measured elementary and secondary teachers' knowledge by the number and type of mathematics courses taken or degrees obtained—and found little correlation between these measures of teacher knowledge and various measures of student learning. Since the late 1980s, researchers have

[7]However, instruction can successfully address irregularities in number-word systems. See Fuson, Smith, and Lo Cicero (1997) for an example of instruction that addresses the irregularities of the English and Spanish number-word systems.

[8]For information on the preparation of U.S. teachers, see Lindquist (1997).

Chapter 3

# Generating Representations: Division By Fractions

**Scenario**

People seem to have different approaches to solving problems involving division with fractions. How do you solve a problem like this one?

$$1\tfrac{3}{4} \div \tfrac{1}{2} =$$

Imagine that you are teaching division with fractions. To make this meaningful for kids, something that many teachers try to do is relate mathematics to other things. Sometimes they try to come up with real-world situations or story-problems to show the application of some particular piece of content. What would you say would be a good story or model for $1\tfrac{3}{4} \div \tfrac{1}{2}$ ?

This time the teachers are required to accomplish two tasks: to compute $1\tfrac{3}{4} \div \tfrac{1}{2}$, and to represent meaning for the resulting mathematical sentence. The mathematical topics discussed in the previous two chapters are relatively elementary, but division by fractions is an advanced topic in arithmetic. Division is the most complicated of the four operations. Fractions are often considered the most complex numbers in elementary school mathematics. Division by fractions, the most complicated operation with the most complex numbers, can be considered as a topic at the summit of arithmetic.

## THE U.S. TEACHERS' PERFORMANCE ON CALCULATION

The weaknesses of the U.S. teachers' subject matter knowledge were more noticeable in this advanced topic than in the two topics discussed earlier. Their discussions of whole number subtraction and multiplication had all displayed correct procedural knowledge, but even this was lacking in many of their discussions of division by fractions. Of the 23 U.S. teachers, 21 tried to calculate $1\frac{3}{4} \div \frac{1}{2}$. Only nine (43%) completed their computations and reached the correct answer. For example, Mr. Felix, a beginning teacher, gave this explanation:

> I would convert the $1\frac{3}{4}$ to fourths, which would give me $\frac{7}{4}$. Then to divide by $\frac{1}{2}$, I would invert $\frac{1}{2}$ and multiply. So, I would multiply $\frac{7}{4}$ by 2 and I would get $\frac{14}{4}$, and then I would divide 14 by 4 to get it back to my mixed number, $3\frac{2}{4}$ or then I would reduce that into $3\frac{1}{2}$.

For teachers like Mr. Felix, the computational procedure was clear and explicit: Convert the mixed number into an improper fraction, invert the divisor and multiply it by the dividend, reduce the product, $\frac{14}{4}$, and change it to a proper fraction, $3\frac{1}{2}$.

Two out of the 21 teachers (9%) correctly conducted the algorithm, but did not reduce their answer or turn it into a proper fraction. Their answer, $\frac{14}{4}$, was an incomplete one.

Four out of 21 teachers (19%) were either unclear about the procedure, or obviously unsure of what they were doing:

> The first thing you'd *have to do* is change them into sync. Well, you're supposed to multiply that and add that. So that's 4, plus it's $\frac{7}{4}$, and then you *have to make it the same.* Divided by $\frac{2}{4}$. Right? And then you just cross multiply like that. You get $\frac{28}{8}$? (Ms. Felice, italics added)

To change the dividend and divisor into like fractions and then perform the division is an alternative to the standard division by fractions algorithm. For example, by converting a problem of dividing $1\frac{3}{4}$ pizzas by $\frac{1}{2}$ pizza into dividing $\frac{7}{4}$ pizzas by $\frac{2}{4}$ pizza, one divides 7 quarters of pizza by 2 quarters of pizza. This "common denominator" approach converts division by a fraction into division by a whole number (7 pieces divided by 2 pieces). Ms. Felice's difficulty, however, was that she did not present a sound knowledge of the standard algorithm yet thought that you "have to" change the numbers into like fractions. She might have seen the common denominator approach before, but seemed to understand neither its rationale nor the relationship between the alternative approach and the standard algorithm.



She might also have confused the standard algorithm for division by fractions with that for adding fractions, which requires a common denominator. In any case she was not confident during computation. Moreover, she did not reduce the quotient and convert it into a proper fraction.

Tr. Blanche, an experienced teacher, was extremely unsure about what she remembered of the algorithm:

> It seems that you need to, you cannot work with a fraction and a mixed number, so the first thing I would do, I turn this into some number of fourths. So you would have $\frac{7}{4}$ divided by $\frac{1}{2}$. Is this, is being the same as multiplying it by 2 as my understanding. So that the steps that I would take, now I am starting to wonder if I am doing this right. Would be that I have $\frac{7}{4}$ that I have converted this divided by $\frac{1}{2}$ is the same as doing $\frac{7}{4}$ times 2, I think. So that gives you 14, let me see if this . . . wait a second—Now let me think through this process . . . I cannot tell if it makes sense because I cannot remember . . . And for some reason I thought that that was exactly the formula that I remembered. But I'm not sure if it is logical.

Tr. Blanche started to wonder if she was doing this right at the beginning of the computation and ended up with "I'm not sure if it is logical."

While the memories of teachers like Ms. Felice and Tr. Blanche were confused or unsure, those of five others (24%) were even more fragmentary. They recalled vaguely that "you should flip it over and multiply" (Ms. Fawn), but were not sure what "it" meant:

> For some reason it is in the back of my mind that you invert one of the fractions. Like, you know, either $\frac{7}{4}$ becomes $\frac{4}{7}$, or $\frac{1}{2}$ becomes $\frac{2}{1}$. I am not sure. (Ms. Frances)

These five teachers' incomplete memories of the algorithm impeded their calculations. Tr. Bernadette, the experienced teacher who was very articulate about the rationale for subtraction with regrouping, tried a completely incorrect strategy:

> I would try to find, oh goodness, the lowest common denominator. I think I would change them both. Lowest common denominator, I think that is what it is called. I do not know how I am going to get the answer. Whoops. Sorry.

Like Ms. Felice, Tr. Bernadette first mentioned finding a common denominator. Her understanding was more fragmentary than Ms. Felice's, however. She did not know what the next step would be.

TABLE 3.1
The U.S. Teachers' Computation of $1\frac{3}{4} \div \frac{1}{2}$ ($N = 21$)

| *Response* | % | *N* |
|---|---|---|
| Correct algorithm, complete answer | 43 | 9 |
| Correct algorithm, incomplete answer | 9 | 2 |
| Incomplete algorithm, unsure, incomplete answer | 19 | 4 |
| Fragmentary memory of the algorithm, no answer | 24 | 5 |
| Wrong strategy, no answer | 5 | 1 |

The remaining teacher simply admitted that she did not know how to do the calculation after taking a look at it. Table 3.1 summarizes the 21[1] U.S. teachers' performance in computing $1\frac{3}{4} \div \frac{1}{2}$.

## THE CHINESE TEACHERS' PERFORMANCE ON CALCULATION

All of the 72 Chinese teachers computed correct and complete answers to the problem. Instead of "invert and multiply," most of the Chinese teachers used the phrase "dividing by a number is equivalent to multiplying by its reciprocal":

> Dividing by a number is equivalent to multiplying by its reciprocal. So, to divide $1\frac{3}{4}$ by $\frac{1}{2}$ we multiply $1\frac{3}{4}$ by the reciprocal of $\frac{1}{2}$, $\frac{2}{1}$, and we get $3\frac{1}{2}$. (Ms. M.)

> The reciprocal of a fraction with numerator 1 is the number in its denominator. The reciprocal of $\frac{1}{2}$ is 2. We know that dividing by a fraction can be converted to multiplying by its reciprocal. Therefore, dividing $1\frac{3}{4}$ by $\frac{1}{2}$ is equivalent to multiplying $1\frac{3}{4}$ by 2. The result will be $3\frac{1}{2}$. (Tr. O.)

Some teachers mentioned the connection between division by fractions and division by whole numbers. Tr. Q. explained why the rule that "dividing by a number is equivalent to multiplying by its reciprocal" is not taught to students until the concept of fraction is introduced:[2]

> Dividing by a number is equivalent to multiplying by its reciprocal, as long as the number is not zero. Even though this concept is introduced when learning how to divide by fractions, it applies to dividing by whole numbers as well. Dividing by 5 is equivalent to multiplying by $\frac{1}{5}$. But the reciprocal of any whole number is a fraction—a fraction with 1 as its numerator and the original number as its denominator—so we have to wait until fractions to introduce this concept.

[1]As indicated earlier, 21 of the 23 teachers attempted the calculation.

[2]According to the current national mathematics curriculum of China, the concept of fractions is not taught until Grade 4. Division by fractions is taught in Grade 6, the last year of elementary education.



"Dividing by a number is equivalent to multiplying by its reciprocal" is used in Chinese textbooks to justify the division by fractions algorithm. This is consistent with the Chinese elementary mathematics curriculum's emphasis on relationships between operations and their inverses. Most teachers did not refer to the property to remind themselves of the computational procedure. They referred to it to justify their calculations.

### Making Sense of the Algorithm

The original interview question only asked teachers to calculate the division problem. During interviews, however, some Chinese teachers tended to elaborate how the algorithm would make sense. Then after interviewing two thirds of the Chinese teachers, I started to ask teachers if the algorithm made sense to them. Most fourth- and fifth-grade teachers were able to say more than "dividing by a number is equivalent to multiplying by its reciprocal." They elaborated their understanding from various perspectives. Some teachers argued that the rationale for the computational procedure can be proved by converting the operation with fractions into one with whole numbers:

> We can use the knowledge that students have learned to prove the rule that dividing by a fraction is equivalent to multiplying by its reciprocal. They have learned the commutative law. They have learned how to take off and add parentheses. They have also learned that a fraction is equivalent to the result of a division, for example, $\frac{1}{2} = 1 \div 2$. Now, using these, to take your example, we can rewrite the equation in this way:

$$\begin{aligned} 1\tfrac{3}{4} \div \tfrac{1}{2} &= 1\tfrac{3}{4} \div (1 \div 2) \\ &= 1\tfrac{3}{4} \div 1 \times 2 \\ &= 1\tfrac{3}{4} \times 2 \div 1 \\ &= 1\tfrac{3}{4} \times (2 \div 1) \\ &= 1\tfrac{3}{4} \times 2 \end{aligned}$$

> It is not difficult at all. I can even give students some equations with simple numbers and ask them to prove the rule on their own. (Tr. Chen)

Other teachers justified the algorithm by drawing on another piece of knowledge that students had learned—the rule of "maintaining the value of a quotient":[3]

> OK, fifth-grade students know the rule of "maintaining the value of a quotient." That is, when we multiply both the dividend and the divisor with the same number, the quotient will remain unchanged. For example, dividing 10 by 2 the quotient is 5. Given that we multiply both 10 and 2 by a number, let's say 6, we will get 60 divided by 12, and the quotient will remain the same, 5. Now if both the dividend and the divisor are multiplied by the reciprocal of the divisor, the divisor will become 1. Since dividing by 1 does not change a number, it can be omitted. So the equation will become that of multiplying the dividend by the reciprocal of the divisor. Let me show you the procedure:
>
> $$\begin{aligned} 1\tfrac{3}{4} \div \tfrac{1}{2} &= (1\tfrac{3}{4} \times \tfrac{2}{1}) \div (\tfrac{1}{2} \times \tfrac{2}{1}) \\ &= (1\tfrac{3}{4} \times \tfrac{2}{1}) \div 1 \\ &= 1\tfrac{3}{4} \times \tfrac{2}{1} \\ &= 3\tfrac{1}{2} \end{aligned}$$
>
> With this procedure we can explain to students that this seemingly arbitrary algorithm is reasonable. (Tr. Wang)

There are various ways that one can show the equivalence of $1\frac{3}{4} \div \frac{1}{2}$ and $1\frac{3}{4} \times \frac{2}{1}$. Tr. Chen and Tr. Wang demonstrated how they used the knowledge that students had already learned to justify the division by fractions algorithm. Other teachers reported that their explanation of why $1\frac{3}{4} \div \frac{1}{2}$ equals $1\frac{3}{4} \times 2$ would draw on the meaning of the expression $1\frac{3}{4} \div \frac{1}{2}$:

> Why is it equal to multiplying by the reciprocal of the divisor? $1\frac{3}{4} \div \frac{1}{2}$ means that $\frac{1}{2}$ of a number is $1\frac{3}{4}$. The answer, as one can imagine, will be $3\frac{1}{2}$, which is exactly the same as the answer for $1\frac{3}{4} \times 2$. 2 is the reciprocal of $\frac{1}{2}$. This is how I would explain it to my students. (Tr. Wu)

### Alternative Computational Approaches

The interview question reminded the teachers that "people seem to have different approaches to solving problems involving division with fractions." Yet the U.S. teachers only mentioned one approach—"invert and multiply"—the standard algorithm. The Chinese teachers, however, proposed at least three other approaches: dividing by fractions using decimals, applying the distributive law, and dividing a fraction without multiplying by the reciprocal of the divisor.

[3]In China, the rule of "maintaining the value of a quotient" is introduced as a part of whole number division. The rule is: While the dividend and the divisor are multiplied, or divided, by the same number, the quotient remains unchanged. For example, 15 ÷ 5 = 3, so (15 × 2) ÷ (5 × 2) = 3 and (15 ÷ 2) ÷ (5 ÷ 2) = 3.



#### *Alternative I: Dividing by Fractions Using Decimals*[4]

A popular alternative way of dividing by fractions used by the Chinese teachers was to compute with decimals. More than one third reported that the equation could also be solved by converting the fractions into decimal numbers:

$$1\tfrac{3}{4} \div \tfrac{1}{2} = 1.75 \div 0.5 = 3.5$$

Many teachers said that the equation was actually easier to solve with decimals:

> I think this problem is easier to solve with decimals. Because it is so obvious that $1\frac{3}{4}$ is 1.75 and $\frac{1}{2}$ is 0.5, and any number can be divisible by the digit 5. You divide 1.75 by 0.5 and get 3.5. It is so straightforward. But if you calculate it with fractions, you have to convert $1\frac{3}{4}$ into a improper fraction, invert $\frac{1}{2}$ into $\frac{2}{1}$, multiply, reduce numerators and denominators, and, at last, you need to convert the product from an improper fraction into a mixed number. The process is much longer and more complicated than that with decimals. (Ms. L.)

Not only may decimals make a fraction problem easier, fractions may also make a decimal problem easier. The problem is to know the features of both approaches and be able to judge according to the context:

> Even though dividing by a decimal is sometimes easier than dividing by a fraction, this is not always the case. Sometimes converting fractions into

[4]In the Chinese national curriculum, topics related to fractions are taught in this order:

1. Introduction of "primary knowledge about fractions" (the concept of fraction) without operations.
2. Introduction of decimals as "special fractions with denominators of 10 and powers of 10."
3. Four basic operations with decimals (which are similar to those of whole numbers).
4. Whole number topics related to fractions, such as divisors, multiples, prime number, prime factors, highest common divisors, lowest common multiples, etc.
5. Topics such as proper fractions, improper fractions, mixed numbers, reduction of a fraction, and finding common denominators.
6. Addition, subtraction, multiplication, and division with fractions.

decimals is complex and difficult, sometimes the decimal might not terminate. Still sometimes, it is easier to solve a division with decimals problem by converting it into fractions. Like, $0.3 \div 0.8$, it is easier to solve by fractions: you will easily get $\frac{3}{8}$. It is important for us and also for our students, though, to know alternative ways of approaching a problem, and to be able to judge which way is more reasonable for a particular problem. (Tr. B.)

Teachers' comprehensive knowledge of a topic may contribute to students' opportunities to learn it. The teachers reported that students were also encouraged to solve fraction problems with decimals:

We also encourage students to solve fraction problems with decimals, or vice versa for all four operations. There are several advantages in doing this. Since they have already learned operations with decimals, this is a chance for them to review knowledge learned before. In addition, converting between fractions and decimals will deepen their understanding of these two representations of numbers and foster their number sense. Moreover, it is a practice of solving a problem through alternative ways. (Tr. S.)

### *Alternative II: Applying the Distributive Law*

Seven teachers said that the distributive law could be used to calculate $1\frac{3}{4} \div \frac{1}{2}$. Instead of considering $1\frac{3}{4}$ as a mixed number and converting it into an improper fraction, they wrote it as $1 + \frac{3}{4}$, divided each part by $\frac{1}{2}$, then added the two quotients together. Two slightly different procedures were reported:

A)

$$\begin{aligned} 1\tfrac{3}{4} \div \tfrac{1}{2} &= (1 + \tfrac{3}{4}) \div \tfrac{1}{2} \\ &= (1 + \tfrac{3}{4}) \times \tfrac{2}{1} \\ &= (1 \times 2) + (\tfrac{3}{4} \times 2) \\ &= 2 + 1\tfrac{1}{2} \\ &= 3\tfrac{1}{2} \end{aligned}$$

B)

$$\begin{aligned} 1\tfrac{3}{4} \div \tfrac{1}{2} &= (1 + \tfrac{3}{4}) \div \tfrac{1}{2} \\ &= (1 \div \tfrac{1}{2}) + (\tfrac{3}{4} \div \tfrac{1}{2}) \\ &= 2 + 1\tfrac{1}{2} \\ &= 3\tfrac{1}{2} \end{aligned}$$

After presenting version A, Tr. Xie commented that this seemingly complicated procedure actually made the computation simpler than the standard procedure:



In this case applying the distributive law makes the operation simpler. The computational procedure I put on paper looks complicated but I wanted to show you the logic of the process. But, when you conduct the operation, it is very simple. You just think that 1 times 2 is 2 and $\frac{3}{4}$ times 2 is $1\frac{1}{2}$, then you add them together and get $3\frac{1}{2}$. One can do it even without a pencil. When working on whole numbers my students learned how to solve certain kinds of problems in a simpler way by applying the distributive law. This approach applies to operations with fractions as well.

The teachers' use of the distributive law provided evidence of their comprehension of the law and their confidence in using it. It also demonstrated their comprehensive understanding of a mixed number, a concept which was as we shall see, an obstacle for some U.S. teachers during computations.

### *Alternative III: "You Don't Have to Multiply"*

Three teachers pointed out that even though multiplying by the reciprocal of the divisor is the conventional way to perform division by fractions, one does not always need to do this. Sometimes division by fractions problems can be solved without using multiplication. The equation that I required them to solve was one such example:

$$\begin{aligned} 1\tfrac{3}{4} \div \tfrac{1}{2} &= \tfrac{7}{4} \div \tfrac{1}{2} \\ &= \tfrac{7 \div 1}{4 \div 2} \\ &= \tfrac{7}{2} \\ &= 3\tfrac{1}{2} \end{aligned}$$

Again, the teachers who proposed this approach argued that for the equation presented in the interview, their method was easier than the standard method. Two steps, inverting the divisor and reducing the final answer, were eliminated. However, the teachers explained that this approach is only applicable to the problems in which both the numerator and denominator of the dividend are divisible by those of the divisor. For example, in $1\frac{3}{4} \div \frac{1}{2}$, 7 is divisible by 1, and 4 is divisible by 2. However, if the problem is $1\frac{2}{3} \div \frac{1}{2}$, since the denominator of the dividend, 3, is not divisible by the denominator, 2, this approach will not apply. Tr. T. said:

In fact, division is more complicated than multiplication. Just think about the cases when one number can't be divided by another number without a remainder. Even if you use decimals, you may encounter repeating decimals. But in multiplication you never have the problem of remainders. Probably that is why the way of multiplying by the reciprocal of the divisor was accepted as the standard way. But in this case, because 4 divided by 2 is easy and so is 7 divided by 1, conducting division directly is even simpler.

Tr. Xie was the first teacher I met who described this nonstandard method of solving a division by fractions problem without performing multiplication. I told him that I had never thought about it that way and asked him to explain how it worked. He said that it could be proved easily:

$$\begin{aligned} 1\tfrac{3}{4} \div \tfrac{1}{2} &= \tfrac{7}{4} \div \tfrac{1}{2} \\ &= (7 \div 4) \div (1 \div 2) \\ &= 7 \div 4 \div 1 \times 2 \\ &= 7 \div 1 \div 4 \times 2 \\ &= (7 \div 1) \div (4 \div 2) \\ &= \tfrac{7 \div 1}{4 \div 2} \end{aligned}$$

Again, he deduced the result by drawing on basic principles such as that of the order of operations and the equivalence between a fraction and a division expression.

All the teachers who suggested alternative methods argued that their methods were "easier" or "simpler" for this calculation. In fact, they not only knew alternative ways of calculating the problem, but also were aware of the meaning of these ways for the calculation—to make the procedure of calculation easier or simpler. To solve a complex problem in a simple way is one of the aesthetic standards of the mathematical community. The teachers argued that not only should students know various ways of calculating a problem but they should also be able to evaluate these ways and to determine which would be the most reasonable to use.

## THE U.S. TEACHERS' REPRESENTATIONS OF DIVISION BY FRACTIONS

### The Mathematical Concepts that the Teachers Represented

Although 43% of the U.S. teachers successfully calculated $1\frac{3}{4} \div \frac{1}{2}$, almost all failed to come up with a representation of division by fractions. Among the 23 teachers, 6 could not create a story and 16 made up stories with misconceptions. Only one teacher provided a conceptually correct but pedagogically problematic representation. The teachers displayed various misconceptions about the meaning of division by fractions.

#### *Confounding Division by $\frac{1}{2}$ with Division by 2*

Ten U.S. teachers confounded division by $\frac{1}{2}$ with division by 2. The teachers with this misconception generated stories about dividing the quantity $1\frac{3}{4}$ evenly between two people, or into two parts. The most common subject of these stories was circular food, such as pie or pizza:



> You could be using pie, a whole pie, one, and then you have three fourths of another pie and *you have two people,* how will you make sure that this gets *divided evenly,* so that each person gets an equal share. (Ms. Fiona, italics added)

The phrases the teachers used, "divide evenly between two," or "divide into half," corresponded to division by 2, not division by $\frac{1}{2}$. When we say that we are going to divide ten apples evenly between two people, we divide the number of apples by 2, not by $\frac{1}{2}$. However, most teachers did not notice that this difference.

#### *Confounding Division by $\frac{1}{2}$ with Multiplication by $\frac{1}{2}$*

Six teachers provided stories that confused dividing by $\frac{1}{2}$ with multiplying by $\frac{1}{2}$. This misconception, although not as common as the previous one, was also substantial. Taking another example with pies:

> Probably the easiest would be pies, with this small number. It is to use the typical pie for fractions. You would have a whole pie and a three quarters of it like someone stole a piece there somewhere. But you would happen to divide it into fourths and then have to *take one half of the total.* (Tr. Barry, italics added)

While the teachers we discussed earlier talked about "dividing between two," Tr. Barry suggested "take half of the total." To find a certain portion of a unit we would use *multiplication by fractions.* Suppose we want to take $\frac{2}{3}$ of a two-pound sack of flour, we multiply 2 by $\frac{2}{3}$ and get $1\frac{1}{3}$ pounds of flour. What teachers like Tr. Barry represented was multiplying by a fraction: $1\frac{3}{4} \times \frac{1}{2}$, not $1\frac{3}{4} \div \frac{1}{2}$. The stories that confused dividing by $\frac{1}{2}$ with multiplying by $\frac{1}{2}$ also revealed weaknesses in the teachers' conceptions of multiplication by fractions.

#### *Confusing the Three Concepts*

Tr. Bernadette and Tr. Beatrice, who were not in either of the above two groups, confused the three concepts, dividing by $\frac{1}{2}$, dividing by 2, and multiplying $\frac{1}{2}$:

> Dividing the one and three-fourths into the half. OK. Let us see . . . You would have all of this whole, you would have the three fourths here. And then you want only half of the whole. (Tr. Bernadette)

> You get one and three-quarters liquid in a pitcher, you want to divide it in half, to visually, each one of you is going to have, get half of it to drink. (Tr. Beatrice)

When Tr. Bernadette and Tr. Beatrice phrased the problem as "dividing the one and three-quarters into the half" or "divide it in half," they were confusing division by $\frac{1}{2}$ with division by 2. Then, when they proposed that "you want only half of the whole" or "get half of it," they confused division by $\frac{1}{2}$ with multiplication by $\frac{1}{2}$. For them, there seemed to be no difference among division by $\frac{1}{2}$, division by 2, and multiplication by $\frac{1}{2}$.

### *No Confusion, But No Story Either*

Two other teachers failed to provide a story but noticed that dividing by $\frac{1}{2}$ is different from dividing by 2. Tr. Belinda, an experienced sixth grade teacher, was aware of the deficiency in her knowledge and the pitfall of the problem:

> I am not quite sure I understand it well enough, except in terms of computation. I know how to do it, but I do not really know what it means to me.

Mr. Felix also noticed a difference between the two concepts. After trying and failing to invent a story, he explained:

> Dividing something by one half and so I confused myself with the two, thinking it meant dividing by two, but it doesn't . . . It means something totally different . . . Well, for me what makes it difficult is not being able to envision it, what it represents in the real world. I can't really think of what dividing by a half means.

Although Tr. Belinda and Mr. Felix were not able to provide a representation of the conception of division by fractions, they did not confuse it with something else. They were the only U.S. teachers who did not confuse division by fractions with another operation.

### *Correct Conception and Pedagogically Problematic Representation*

Tr. Belle, an experienced teacher, was the only one who provided a conceptually correct representation of the meaning of division by fractions. She said:[5]

[5]Tr. Belle used $2\frac{1}{4}$ instead of $1\frac{3}{4}$. However, her understanding of the concept of dividing by a fraction is correct.



> Let's take something like, two and a quarter Twinkies. And, I want to give each child a half a Twinkie. How many kids can get, will get a piece of Twinkie. Of course, I've got a half a child there at the end, but. OK, that's the problem with using children there, because then you have four and a half kids. You know, four kids, and one child's only going to get half the amount of the others. I guess they could figure that out.

Tr. Belle represented the concept correctly. To divide the number A by the number B is to find how many Bs are contained in A. However, as Tr. Belle herself indicated, this representation results in a fractional number of children. The answer will be $3\frac{1}{2}$ students. It is pedagogically problematic because in real life a number of persons will never be a fraction.

## **Dealing with the Discrepancy: Correct Computation Versus Incorrect Representation**

Even though the stories created by the teachers illustrated misconceptions about division by fractions, there were opportunities during the interviews that might have led some of them to find the pitfall. Of the 16 teachers who created a conceptually incorrect story, 9 had computed correct or incomplete answers. Because most teachers discussed the results of their stories, these discrepancies between the answers from the conceptually wrong stories ($\frac{7}{8}$) and from computations ($3\frac{1}{2}$, $\frac{7}{2}$, or $\frac{28}{8}$) might call for their reflection. Although four teachers did not notice any discrepancy, the remaining five did. Unfortunately, none of the five was led to a correct conception by discovering the discrepancy.

The five teachers reacted in three ways to the discrepancy. Three teachers doubted the possibility of creating a representation for the equation and decided to give up. Ms. Fleur was frustrated that "the problem doesn't turn out the way you think it would." Tr. Blanche was "totally baffled" when she noticed that the two answers were different. Tr. Barry concluded that "[the story] is not going to work. I do not know what I did."

Ms. Felice, however, seemed to be more assertive. She created a story for $1\frac{3}{4}\times\frac{1}{2}$ to represent $1\frac{3}{4}\div\frac{1}{2}$: "That's one and three-fourths cups of flour and you'd want half of that, so you could make a half a batch of cookies." In estimating the result of the story problem, she noticed that it would be "a little over three quarters" rather than three and a half. Because she had been unsure during her procedural calculations, she soon decided that $\frac{28}{8}$, the answer she had attained earlier, was wrong. She thought that "a real-world thing" that she came up with held more authority than a solution she obtained using the algorithm:

> It makes, it [the calculation that she did] was wrong. Because you have a half of a one would be one half, and a half of three fourths would be

[lengthy pause] if you estimated it be a fourth and then a little bit more. Let's see, that the answer is a little over three fourths . . . When I did it in a real-world thing, I would realize that I had done it wrong, and then I'd just go over it again. When you do that without a real-world thing you might be doing them real wrong, and you might do the problem wrong that way.

Unfortunately, Ms. Felice's "real-world thing" represented a misconception. Because of her unsureness with computation and her blind inclination to "real-world things," finding the discrepancy did not lead her to reflect on the misconception, but to discard the correct, though incomplete, result that she had computed.

The remaining teacher, Ms. Francine, eventually found a way to explain away the discrepancy. The story problem she made up represented $1\frac{3}{4} \div \frac{1}{2}$:

So some kind of food, graham cracker maybe, because it has the four sections. You have one whole, four fourths, and then break off a quarter, we only have one and three fourths, and then we want, how are we going to divide this up so that let us say we have two people and we want to give half to one, half to the other, see how they would do it.

By dividing one and three fourths crackers between two people, she expected that she would get the same answer as she did with the equation $1\frac{3}{4} \div \frac{1}{2}$, "three and one half." However, it came out that each person would get three and one half quarters of crackers:

*Would we get three and one half, did I do that right?* [She is looking at what she wrote and mumbling to herself.] Let us see one, two, three, yes, that is right, one, two, three. *They would each get three quarters and then one half of the other quarter.* (italics added)

Even though Ms. Francine noticed that it was "two different [answers]," she finally explained how the latter, three and one half *quarters,* made sense with the previous answer, three and one half. She seemed to find this a satisfactory explanation of why the dividend $1\frac{3}{4}$ was smaller than the quotient $3\frac{1}{2}$:

You wonder how could one and three fourths which is smaller than three and a half see, so it is, here one and three quarters is referring to what you have completely, three and one half is, is according to the fraction of the one and the three fourths, so if you just took the equation, it would not make sense, I mean it would not make sense.

The way that Ms. Francine explained the discrepancy conflated the number $3\frac{1}{2}$ (the answer of $1\frac{3}{4} \div \frac{1}{2}$) with $3\frac{1}{2}$ quarters (the answer of $1\frac{3}{4} \div 2$).



Since the number $\frac{1}{2}$ is one quarter of the number 2, the quotient of a number divided by 2 will be one quarter of the quotient of the number divided by $\frac{1}{2}$. For instance, $2 \div 2 = 1$, $2 \div \frac{1}{2} = 4$, or $\frac{1}{2} \div 2 = \frac{1}{4}$, $\frac{1}{2} \div \frac{1}{2} = 1$. That is why $\frac{7}{8}$, the quotient of the equation $1\frac{3}{4} \div 2$, happened to be $3\frac{1}{2}$ quarters. Ms. Francine, of course, did not confuse them on purpose. She did not even notice the coincidence. Her inadequate knowledge of fractions and her ignorance that the result of dividing by a fraction less than 1 will be larger than the dividend led her to an incorrect explanation of the discrepancy.

The reason that finding discrepancies did not lead Ms. Francine or Ms. Felice to reflect on their representations was that their computational knowledge was limited and flimsy. Even though their calculations were correct, they were not solidly supported by conceptual understanding. As the teachers said during interviews, they did not understand why the computational algorithm worked. Therefore, results obtained from computation were unable to withstand a challenge, nor could they serve as a point from which to approach the meaning of the operation.

### An Inadequate Understanding of Procedure Impedes Creating a Representation

The case of Ms. Fay was another example of how knowledge of a computational skill may influence one's conceptual approach to the meaning of the operation. Ms. Fay seemed likely to reach an understanding of the meaning of division by fractions. While computing, she described the procedure clearly, and got a correct answer:

I would copy the first fraction as it reads, then I would change the sign from division to multiplication. And then I would invert the second fraction. Then because the first fraction was a mixed fraction, I would change it from a mixed to a whole fraction. So I would take 1 times 4 which is 4 and then add it to 3 which would be $\frac{7}{4}$ times 2 . . . With fractions you multiply straight across so it would be 7 times 2 is $\frac{14}{4}$. And then I would reduce that.

Moreover, Ms. Fay phrased the problem correctly, using "dividing by one half" ($\div \frac{1}{2}$), rather than "into half" ($\div 2$). However, when she started to divide $1\frac{3}{4}$ pizza by $\frac{1}{2}$ pizza, she got "lost" and did not know where she "would go from there":

Well, it would be one whole pizza and then three fourths of a pizza. Which would be kind of like this. And it would be divided by one half of a pizza. And then . . . I am lost after that, actually. If I combine those [the whole pizza and the three fourths of a pizza], I do not know what I would do next with a student. I would say that we would have to combine these because I know that you have to, that you need to. It is very hard, it is almost impossible

> to divide a mixed fraction by whole fraction to me and I cannot explain why, but that is the way I was told. That you have to change the mixed numeral to a fraction . . . So you would have to show the student how to combine these two. And that is kind of hard. I do not know where I would go from there.

Ms. Fay had made an appropriate start. The story which she tried to make up, dividing $1\frac{3}{4}$ pizza by $\frac{1}{2}$ pizza, was likely to be a correct model for dividing $1\frac{3}{4}$ by $\frac{1}{2}$. However, she got "lost" in the middle and gave up finishing the story. What impeded Ms. Fay from completing the story was her inadequate understanding of the computational procedure that she wanted to use: change the mixed fraction into a improper fraction, and divide.

When computing, Ms. Fay dealt with the mixed number according to what she "was told." She executed the first part of the procedure, converting $1\frac{3}{4}$ into $\frac{7}{4}$. However, she could not explain why it should be changed. Moreover, she did not understand what was going on during the procedure of changing a mixed number into an improper fraction. This deficiency of understanding caused her to become "lost." If Ms. Fay had understood what is meant by changing a mixed number into an improper fraction—to change the whole number into an improper fraction with the same denominator as the fraction and combine it with the latter—she would have been able to conduct this procedure for the $1\frac{3}{4}$ pizzas. What she needed to do was only to cut the whole pizza into quarters so that the whole, 1, becomes $\frac{4}{4}$, and $1\frac{3}{4}$ pizza becomes $\frac{7}{4}$ pizza. It would take her at least one more step toward completing the representation. In addition to Ms. Fay, at least three other teachers reported that they had difficulty working with mixed numbers. Their inadequate knowledge of the computational procedure impeded their approach to the meaning of the operation.

### Can Pedagogical Knowledge Make Up for Ignorance of the Concept?

The teachers' deficiency in understanding the meaning of division by fractions determined their inability to generate an appropriate representation. Even their pedagogical knowledge could not make up for their ignorance of the concept. Circular foods are considered appropriate for representing fraction concepts. However, as we have seen, the representations teachers generated with pizzas or pies displayed misconceptions. Ms. Francine's use of graham crackers with four sections was also pedagogically thoughtful in representing quarters. However, it did not remedy her misunderstanding of the meaning of division by fractions. To generate a representation, one should first know what to represent. During the interviews the teachers reported various pedagogical ideas for generating



representations. Unfortunately, because of their inadequate subject matter knowledge, none of these ideas succeeded in leading them to a correct representation.

Ms. Florence was a teacher who claimed that she liked fractions. She would use "articles right in the classroom to represent a conception." The representation she proposed was:

> José has one and three fourths box of crayons and he wants to divide them between two people or divide the crayons in half, and then, first we could do it with the crayons and maybe write it on the chalkboard or have them do it in numbers.

Other contexts using measures, such as recipes, mileage, money, and capacity, were also used by teachers to represent fraction concepts. Ms. Francesca said she would use money: "I would tell them, 'You have got so much money, you have two people, and you have to divide it up evenly.' "

Tr. Blanche, an experienced teacher who was very confident in her mathematical knowledge, thought that she could use anything for the representation: "I would have one and three-quarters something, whatever it is, and if I needed to divide it by two I want to divide it into two groups . . ."

While the previously mentioned teachers represented the concept of dividing by 2, other teachers represented the concept of multiplying by $\frac{1}{2}$. Tr. Barbara was an experienced teacher who was proud of her mathematical knowledge and said she enjoyed "the challenge of math." She said she used to have a hard time with fractions when she was a student, but ever since one of her teacher taught her fractions by bringing in a recipe, she "got it" and "loved working" on it. She would teach her students in the way she was taught—using a recipe:

> Well if I were to have this type of equation, I would say well using one and three quarters cup of butter. And you want to *take a half of it*, how would you do it. Or it could be used in any, you know, I have flour or, or sugar or something like that.

Ms. Fawn, a beginning teacher, created several representations with different subjects, such as money, recipes, pies, apples, etc. However, all of her stories represented a misconception—that of multiplying by $\frac{1}{2}$ rather than dividing by $\frac{1}{2}$. There was no evidence that these teachers lacked pedagogical knowledge. The subjects of their stories—circular food, recipes, classroom articles, etc.—were suitable for representing fraction concepts. However, because of their misconceptions about the meaning of division by fractions, these teachers failed to create correct representations.

## THE CHINESE TEACHERS' APPROACH TO THE MEANING OF DIVISION BY FRACTIONS

The deficiency in the subject matter knowledge of the U.S. teachers on the advanced arithmetical topic of division by fractions did not appear among the Chinese teachers. While only one among the 23 U.S. teachers generated a conceptually correct representation for the meaning of the equation, 90% of the Chinese teachers did. Sixty-five of the 72 Chinese teachers created a total of more than 80 story problems representing the meaning of division by a fraction. Twelve teachers proposed more than one story to approach different aspects of the meaning of the operation. Only six (8%) teachers said that they were not able to create a story problem, and one teacher provided an incorrect story (which represented $\frac{1}{2} \div 1\frac{3}{4}$ rather than $1\frac{3}{4} \div \frac{1}{2}$). Figure 3.1 displays a comparison of teachers' knowledge about this topic.

The Chinese teachers represented the concept using three different models of division: measurement (or quotitive), partitive, and product and factors.[6] For example, $1\frac{3}{4} \div \frac{1}{2}$ might represent:

- $1\frac{3}{4}$ feet $\div \frac{1}{2}$ feet $= \frac{7}{2}$ (measurement model)
- $1\frac{3}{4}$ feet $\div \frac{1}{2} = \frac{7}{2}$ feet (partitive model)
- $1\frac{3}{4}$ square feet $\div \frac{1}{2}$ feet $= \frac{7}{2}$ feet (product and factors)

which might correspond to:

- How many $\frac{1}{2}$-foot lengths are there in something that is 1 and $\frac{3}{4}$ feet long?
- If half a length is 1 and $\frac{3}{4}$ feet, how long is the whole?
- If one side of a $1\frac{3}{4}$ square foot rectangle is $\frac{1}{2}$ feet, how long is the other side?

### The Models of Division by Fractions

#### *The Measurement Model of Division: "Finding How Many $\frac{1}{2}$s There Are in $1\frac{3}{4}$" or "Finding How Many Times $1\frac{3}{4}$ is of $\frac{1}{2}$"*

Sixteen stories generated by the teachers illustrated two ideas related to the measurement model of division: "finding how many $\frac{1}{2}$s there are in $1\frac{3}{4}$" and "finding how many times $1\frac{3}{4}$ is of $\frac{1}{2}$." Eight stories about five topics corresponded to "finding how many $\frac{1}{2}$s there are in $1\frac{3}{4}$." Here are two examples:

[6]Greer (1992) gives an extensive discussion of models of multiplication and division. His category "rectangular area" is included in "product and factors."



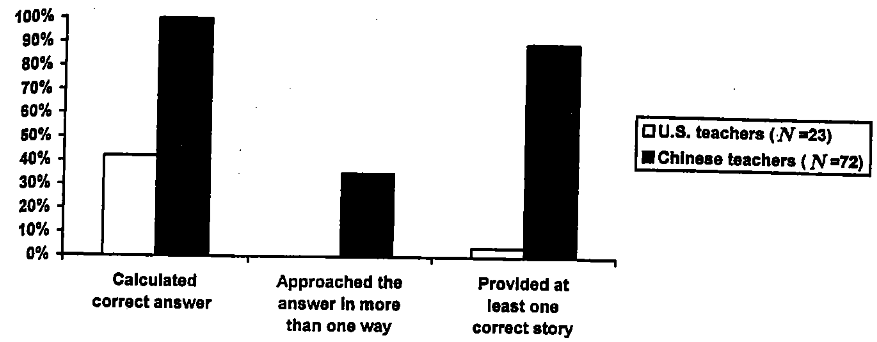


FIG. 3.1. Teachers' knowledge of division by fractions.

> Illustrating it with the measurement model of division, $1\frac{3}{4} \div \frac{1}{2}$ can be articulated as how many $\frac{1}{2}$s there are in $1\frac{3}{4}$. To represent it we can say, for example, given that a team of workers construct $\frac{1}{2}$ km of road each day, how many days will it take them to construct a road of $1\frac{3}{4}$ km long? The problem here is to find how many pieces of $\frac{1}{2}$ km, which they can accomplish each day, are contained in $1\frac{3}{4}$ km. You divide $1\frac{3}{4}$ by $\frac{1}{2}$ and the result is $3\frac{1}{2}$ days. It will take them $3\frac{1}{2}$ days to construct the road. (Tr. R.)

> Cut an apple into four pieces evenly. Get three pieces and put them together with a whole apple. Given that $\frac{1}{2}$ apple will be a serving, how many servings can we get from the $1\frac{3}{4}$ apples? (Ms. I.)

"Finding how many $\frac{1}{2}$s there are in $1\frac{3}{4}$" parallels the approach of Tr. Belle, the U.S. teacher who had a conceptual understanding of the topic. There were eight other stories that represented "finding how many times $1\frac{3}{4}$ is of $\frac{1}{2}$." For example:

> It was planned to spend $1\frac{3}{4}$ months to construct a bridge. But actually it only took $\frac{1}{2}$ month. How many times is the time that was planned of the time that actually was taken? (Tr. K.)

"Finding how many $\frac{1}{2}$s there are in $1\frac{3}{4}$" and "finding how many times $1\frac{3}{4}$ is of $\frac{1}{2}$" are two approaches to the measurement model of division by fractions. Tr. Li indicated that though the measurement model is consistent for whole numbers and fractions when fractions are introduced the model needs to be revised:

> In whole number division we have a model of finding how many times one number is of another number. For example, how many times the number 10 is of the number 2? We divide 10 by 2 and get 5. 10 is 5 times 2. This is what we call the measurement model. With fractions, we can still say, for example, what times $\frac{1}{2}$ is $1\frac{3}{4}$? Making a story problem, we can say for instance, there are two fields. Field A is $1\frac{3}{4}$ hectares, and field B is $\frac{1}{2}$ hectare. What

times the area of field B is the area of field A? To calculate the problem we divide $1\frac{3}{4}$ hectares by $\frac{1}{2}$ hectare and get $3\frac{1}{2}$. Then we know that the area of the field A is $3\frac{1}{2}$ times that of the field B. The equation you asked me to represent fits this model. However, when fractions are used this division model of measurement need to be revised. In particular, when the dividend is smaller than the divisor and then the quotient becomes a proper fraction. Then the model should be revised. The statement of "finding what fraction one number is of another number," or, "finding what fractional times one number is of another number" should be added on the original statement. For example, for the expression $2 \div 10$, we may ask, what fraction of 10 is 2? Or, what fractional times is 2 of 10? We divide 2 by 10 and get $\frac{1}{5}$. 2 is $\frac{1}{5}$ of 10. Similarly, we can also ask: What is the fractional part that $\frac{1}{4}$ is of $1\frac{1}{2}$? Then you should divide $\frac{1}{4}$ by $1\frac{1}{2}$ and get $\frac{1}{6}$.

### *The Partitive Model of Division: Finding a Number Such That $\frac{1}{2}$ of It is $1\frac{3}{4}$*

Among more than 80 story problems representing the meaning of $1\frac{3}{4} \div \frac{1}{2}$, 62 stories represented the partitive model of division by fractions—"finding a number such that $\frac{1}{2}$ of it is $1\frac{3}{4}$":

> Division is the inverse of multiplication. Multiplying by a fraction means that we know a number that represents a whole and want to find a number that represents a certain fraction of that. For example, given that we want to know what number represents $\frac{1}{2}$ of $1\frac{3}{4}$, we multiply $1\frac{3}{4}$ by $\frac{1}{2}$ and get $\frac{7}{8}$. In other words, the whole is $1\frac{3}{4}$, and $\frac{1}{2}$ of it is $\frac{7}{8}$. In division by a fraction, on the other hand, the number that represents the whole becomes the unknown to be found. We know a fractional part of it and want to find the number that represents the whole. For example, $\frac{1}{2}$ of a jump-rope is $1\frac{3}{4}$ meters, what is the length of the whole rope? We know that a part of a rope is $1\frac{3}{4}$ meters, and we also know that this part is $\frac{1}{2}$ of the rope. We divide the number of the part, $1\frac{3}{4}$ meters, by the corresponding fraction of the whole, $\frac{1}{2}$, we get the number representing the whole, $3\frac{1}{2}$ meters. Dividing $1\frac{3}{4}$ by $\frac{1}{2}$, we will find that the whole rope is $3\frac{1}{2}$ meters long . . . But I prefer not to use dividing by $\frac{1}{2}$ to illustrate the meaning of division by fractions. Because one can easily see the answer without really doing division by fractions. If we say $\frac{4}{5}$ of a jump-rope is $1\frac{3}{4}$ meters, how long is the whole rope? The division operation will be more significant because then you can't see the answer immediately. The best way to calculate it is to divide $1\frac{3}{4}$ by $\frac{4}{5}$ and get $2\frac{3}{16}$ meters. (Ms. G.)

> Dividing by a fraction is finding a number when a fractional part of it is known. For example, given that we know that $\frac{1}{2}$ of a number is $1\frac{3}{4}$, dividing $1\frac{3}{4}$ by $\frac{1}{2}$, we can find out that this number is $3\frac{1}{2}$. Making a story problem to illustrate this model, let's say that one kind of wood weighs $1\frac{3}{4}$ tons per cubic meter, it is just $\frac{1}{2}$ of the weight of per cubic meter of one kind of marble. How much does one cubic meter of the marble weigh? So we know that $\frac{1}{2}$ cubic meter of the marble weighs $1\frac{3}{4}$ tons. To find the weight of one cubic meter of it, we divide $1\frac{3}{4}$, the number that represents the fractional part, by $\frac{1}{2}$, the fraction which $1\frac{3}{4}$ represents, and get $3\frac{1}{2}$, the number of the whole. Per cubic meter the marble weighs $3\frac{1}{2}$ tons. (Tr. D.)



> My story will be: A train goes back and forth between two stations. From Station A to Station B is uphill and from Station B back to Station A is downhill. The train takes $1\frac{3}{4}$ hours going from Station B to Station A. It is only $\frac{1}{2}$ time of that from Station A to Station B. How long does the train take going from Station A to Station B? (Tr. S.)

> The mom bought a box of candy. She gave $\frac{1}{2}$ of it which weighed $1\frac{3}{4}$ kg to the grandma. How much did the box of the candy originally weigh? (Ms. M.)

The teachers above explained the fractional version of the partitive model of division. Tr. Mao discussed in particular how the partitive model of division by integers is revised when fractions are introduced:

> With integers students have learned the partitive model of division. It is a model of finding the size of each of the equal groups that have been formed from a given quantity. For example, in our class we have 48 students, they have been formed into 4 groups of equal size, how many students are there in each group? Here we know the quantity of several groups, 48 students. We also know the number of groups, 4. What to be found is the size of one group. So, *a partitive model is finding the value of a unit when the value of several units is known.* In division by fractions, however, the condition has been changed. Now what is known is not the value of several units, rather, the value of a part of the unit. For example, given that we paid $1\frac{3}{4}$ Yuan to buy $\frac{1}{2}$ of a cake, how much would a whole cake cost? Since we know that $\frac{1}{2}$ of the whole price is $1\frac{3}{4}$ Yuan, to know the whole price we divide $1\frac{3}{4}$ by $\frac{1}{2}$ and get $3\frac{1}{2}$ Yuan. In other words, *the fractional version of the partitive model is to find a number when a part of it is known.* (italics added)

Tr. Mao's observation was true. Finding a number when several units is known and finding a number when a fractional part of it is known are represented by a common model—finding the number that represents a unit when a certain amount of the unit is known. What differs is the feature of the amount: with a whole number divisor, the condition is that "*several times* the unit is known," but with a fractional divisor the condition is that "*a fraction* of the unit is known." Therefore, conceptually, these two approaches are identical.

This change in meaning is particular to the partitive model. In the measurement model and the factors and product model, division by fractions keeps the same meaning as whole number division. This may explain why so many of the Chinese teachers' representations were partitive.

### *Factors and Product: Finding a Factor That Multiplied by $\frac{1}{2}$ Will Make $1\frac{3}{4}$*

Three teachers described a more general model of division—to find a factor when the product and another factor are known. The teachers articulated it as "to find a factor that when multiplied by $\frac{1}{2}$ makes $1\frac{3}{4}$":

> As the inverse operation of multiplication, division is to find the number representing a factor when the product and the other factor are known. From this perspective, we can get a word problem like "Given that the product of $\frac{1}{2}$ and another factor is $1\frac{3}{4}$, what is the other factor?" (Tr. M.)

> We know that the area of a rectangle is the product of length and width. Let's say that the area of a rectangle board is $1\frac{3}{4}$ square meters, its width is $\frac{1}{2}$ meters, what is its length? (Mr. A.)

These teachers regarded the relationship between multiplication and division in a more abstract way. They ignored the particular meaning of the multiplicand and multiplier in multiplication and related models of division. Rather, they perceived the multiplicand and multiplier as two factors with the same status. Their perspective, indeed, was legitimized by the commutative property of multiplication.

The concept of fractions as well as the operations with fractions taught in China and U.S. seem different. U.S. teachers tend to deal with "real" and "concrete" wholes (usually circular or rectangular shapes) and their fractions. Although Chinese teachers also use these shapes when they introduce the concept of a fraction, when they teach operations with fractions they tend to use "abstract" and "invisible" wholes (e.g., the length of a particular stretch of road, the length of time it takes to complete a task, the number of pages in a book).

## Meaning of Multiplication by a Fraction: The Important Piece in the Knowledge Package

Through discussion of the meaning of division by fractions, the teachers mentioned several concepts that they considered as pieces of the knowledge package related to the topic: the meaning of whole number multiplication, the concept of division as the inverse of multiplication, models of whole number division, the meaning of multiplication with fractions, the concept of a fraction, the concept of a unit, etc. Figure 3.2 gives an outline of the relationships among these items.

The learning of mathematical concepts is not a unidirectional journey. Even though the concept of division by fractions is logically built on the previous learning of various concepts, it, in turn, plays a role in reinforcing and deepening that previous learning. For example, work on the meaning



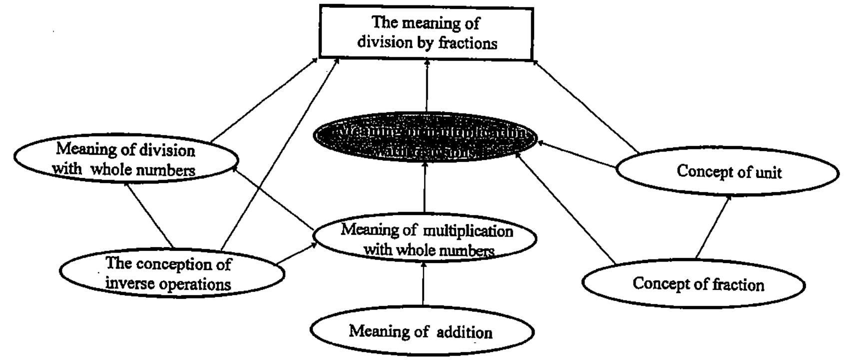


FIG. 3.2. A knowledge package for understanding the meaning of division by fractions.

of division by fractions will intensify previous concepts of rational number multiplication. Similarly, by developing rational number versions of the two division models, one's original understanding of the two whole number models will become more comprehensive:

> This is what is called "gaining new insights through reviewing old ones." The current learning is supported by, but also deepens, the previous learning. The meaning of division by fractions seems complicated because it is built on several concepts. On the other hand, however, it provides a good opportunity for students to deepen their previous learning of these concepts. I am pretty sure that after approaching the meaning and the models of division by fractions, students' previous learning of these supporting concepts will be more comprehensive than before. Learning is a back and forth procedure. (Tr. Sun)

From this perspective, learning is a continual process during which new knowledge is supported by previous knowledge and the previous knowledge is reinforced and deepened by new knowledge.

During the interviews, "the meaning of multiplication with fractions" was considered a key piece of the knowledge package. Most teachers considered multiplication with fractions the "necessary basis" for understanding the meaning of division by fractions:

> The meaning of multiplication with fractions is particularly important because it is where the concepts of division by fractions are derived . . . Given that our students understand very well that multiplying by a fraction means finding a fractional part of a unit, they will follow this logic to understand how the models of its inverse operation work. On the other hand, given that they do not have a clear idea of what multiplication with fractions means, concepts of division by a fraction will be arbitrary for them and very difficult to understand. Therefore, in order to let our students grasp the

> meaning of division by fractions, we should first of all devote significant time and effort when teaching multiplication with fractions to make sure students understand thoroughly the meaning of this operation . . . Usually, my teaching of the meaning of division by fractions starts with a review of the meaning of multiplication with fractions. (Tr. Xie)

> The concepts of division by fractions, such as "finding a number when a fractional part is known" or "finding what fraction one number is of another number," etc. sound complicated. But once one has a comprehensive understanding of the meaning of multiplication with fractions, one will find that these concepts are logical and easy to understand. Therefore, to help students to understand the meaning of division by fractions, many of our efforts are not devoted directly to the topic, but rather, to their thorough understanding of the meaning of multiplication with fractions, and the relationship between division and multiplication. (Tr. Wu)

The meaning of multiplication with fractions is also important in the knowledge package because it "connects several relevant conceptions":

> The concept of multiplication with fractions is like a "knot." It "ties" several other important concepts together. As the operation of multiplication, it is connected with concepts of whole number addition and division. Moreover, in the sense that it deals with fractional numbers, it is related to the conception of a fraction, and those of addition and division with fractions. A grasp of the meaning of multiplication with fractions depends on comprehension of several concepts. At the same time, it substantially reinforces one's previous learning and contributes to one's future learning. (Ms. I.)

Indeed, from the teachers' perspective, the importance of pieces of knowledge in mathematics is not the same. Some of them "weigh" more than others because they are more significant to students' mathematical learning. In addition to "the power of supporting" that we have discussed earlier, another aspect that contributes to the importance of a piece of knowledge is its "location" in a knowledge network. For example, multiplication with fractions is important also because it is at an "intersection" of several mathematical concepts.

## The Representations of the Models of Division by Fractions

The Chinese teachers' profound understanding of the meaning of division by fractions and its connections to other models in mathematics provided them with a solid base on which to build their pedagogical content knowledge of the topic. They used their vivid imaginations and referred to rich topics to represent a single concept of division by fractions. On the other hand, some teachers used one subject to generate several story problems to represent various aspects of the concept. Teachers also drew on knowledge of elementary geometry—the area of a rectangle—to represent division.



### *Rich Topics Representing the Partitive Model*

Even though the operation of division has two models, it appears that the two models do not receive the same attention. For most of the teachers in our research, the partitive model was substantially more impressive than the measurement model. Teachers referred to about thirty subjects in generating more than sixty story problems to represent the fractional version of the partitive model of division. In addition to those discussed earlier, here are a few other examples:

> A factory that produces machine tools now uses $1\frac{3}{4}$ tons of steel to make one machine tool, $\frac{1}{2}$ of what they used to use. How much steel did they used to use for producing one machine tool? (Ms. H.)

> Uncle Wang ploughed $1\frac{3}{4}$ mus[7] in $\frac{1}{2}$ a day; with this speed, how many mus can he plough in a whole day? (Mr. B.)

> Yesterday I rode a bicycle from town A to town B. I spent $1\frac{3}{4}$ hour for $\frac{1}{2}$ of my journey, how much time did I take for the whole journey? (Tr. R.)

> A farm has $1\frac{3}{4}$ mus of experimental fields growing wheat. It is $\frac{1}{2}$ of the area of the experimental field growing cotton. How big is the field of cotton? (Tr. N.)

> In a river with swift current a downriver boat takes only $\frac{1}{2}$ the time of an upriver boat to go the same long journey. Now we have a downriver boat which took $1\frac{3}{4}$ hour going from place A to place B, how long it will take an upriver boat to go from place B to place A? (Tr. Mao)

> Given that we want to know how much vegetable oil there is in a big bottle, but we only have a small scale. We draw $\frac{1}{2}$ of the oil from the bottle, weigh it, and find that it is $1\frac{3}{4}$ kg. Can you tell me how much all the oil in the bottle originally weighed? (Ms. R.)

> One day Xiao-Min went to downtown to see a movie. On his way he ran into his aunt. Xiao-Min asked her, "Do you know how far is it from our village to downtown?" His aunt said, "I am not going to tell you the number but I will give you a clue. You have walked $1\frac{3}{4}$ lis[8] and it is exactly $\frac{1}{2}$ of the whole distance. Figure out your question on your own." (Ms. K.)

While the U.S. teachers tended to use a concrete whole (such as round food) and its parts to represent a whole and a fraction, most Chinese teachers

[7] "Mu" is a Chinese measurement for area. Fifteen mus is one hectare.
[8] "Li" is a traditional measurement for distance. One li is $\frac{1}{2}$ kilometer.

represented these concepts in a more abstract way. Only 3 of the 72 teachers used round food as the subject of their representation. In many story problems created by the Chinese teachers, $3\frac{1}{2}$, the quotient of the division, was treated as a unit, and $1\frac{3}{4}$, the dividend, was regarded as $\frac{1}{2}$ of the unit.

While food and money were the two main subjects of U.S. teachers' representations, those used by the Chinese teachers were more diverse. In addition to topics in students' lives, those related to students' lives were also included, such as what happens in a farm, in a factory, in the family, etc. Teachers' solid knowledge of the meaning of division by fractions made them comfortable using a broad range of topics in representations.

#### *Several Stories With a Single Subject*

Among the teachers who created more than one story to illustrate various aspects of the concept of division by fractions, Ms. D. stood out. She generated three stories about the same subject:

> The equation of $1\frac{3}{4} \div \frac{1}{2} =$ can be represented from different perspectives. For instance, we can say, here is $1\frac{3}{4}$ kg sugar and we want to wrap it into packs of $\frac{1}{2}$ kg each. How many packs can we wrap? Also, we can say that here we have two packs of sugar, one of white sugar and the other of brown sugar. The white sugar is $1\frac{3}{4}$ kg and the brown sugar is $\frac{1}{2}$ kg. How many times is the weight of white sugar of that of brown sugar? Still, we can say that here is some sugar on the table that weighs $1\frac{3}{4}$ kg; it is $\frac{1}{2}$ of all the sugar we now have at home, so how much sugar do we have at home? All three stories are about sugar, and all of them represent $1\frac{3}{4} \div \frac{1}{2}$. But the numerical models they illustrate are not the same. I would put the three stories on the board and invite my students to compare the different meanings they represent. After the discussion I would ask them to try to make up their own story problems to represent the different models of division by fractions. (Ms. D.)

In order to involve students in a comparison of the different concepts associated with $1\frac{3}{4} \div \frac{1}{2}$, Ms. D. created several representations with a single subject. The similarity in the subject and the similarity in the numbers included in the operation would make the difference in the numerical models that the stories represented more obvious to students.

## DISCUSSION

### Calculation: How Did It Reveal Teachers' Understanding of Mathematics?

The difference between the mathematical knowledge of the U.S. teachers and that of the Chinese teachers became more striking with the topic of division by fractions. The first contrast was presented in calculation. The



interview question of this chapter asked the teachers to calculate $1\frac{3}{4} \div \frac{1}{2}$. The process of calculation revealed features of teachers' procedural knowledge and of their understanding of mathematics, as well as of their attitude toward the discipline.

In the two previous chapters all teachers presented a sound procedural knowledge. This time, only 43% of the U.S. teachers succeeded in calculation and none of them showed an understanding of the rationale of the algorithm. Most of these teachers struggled. Many tended to confound the division by fractions algorithm with those for addition and subtraction or for multiplication. These teachers' procedural knowledge was not only weak in division with fractions, but also in other operations with fractions. Reporting that they were uncomfortable doing calculation with mixed numbers or improper fractions, these teachers' knowledge about the basic features of fractions was also very limited.

All of the Chinese teachers succeeded in their calculations and many of them showed enthusiasm in doing the problem. These teachers were not satisfied by just calculating and getting an answer. They enjoyed presenting various ways of doing it—using decimals, using whole numbers, applying the three basic laws, etc. They went back and forth across subsets of numbers and across different operations, added and took off parentheses, and changed the order of operations. They did this with remarkable confidence and amazingly flexible skills. In addition, many teachers made comments on various calculation methods and evaluated them. Their way of "doing mathematics" showed significant conceptual understanding.

Another interesting feature of the Chinese teachers' mathematics is that they tended to provide "proofs" for their calculation procedures. Most teachers justified their calculations by mentioning the rule that "dividing by a number is equivalent to multiplying by its reciprocal." Others converted the fraction $\frac{1}{2}$ into $1 \div 2$ and proved step by step that dividing by $\frac{1}{2}$ is equivalent to multiplying by 2. Still other teachers used the meaning of dividing by $\frac{1}{2}$ to explain the calculating procedure. Their performance is mathematician-like in the sense that to convince someone of a truth one needs to prove it, not just assert it.

#### "A Concept Knot": Why It is Important

In addition to their performance in "doing mathematics," the Chinese teachers showed a knowledge of fractions that was markedly more solid than that of the U.S. teachers in other ways. The Chinese teachers were aware of abundant connections between fractions and other mathematical topics. They were aware of how a fraction can be written as a division expression in which the numerator is the dividend and the denominator is the divisor. They were also aware of the relationship between decimals and fractions, and were very skillful in converting between the two number forms. Moreover, they were aware of how the models of division by fractions are connected to the meaning of multiplication with fractions and to whole number models of division.

As in the two previous chapters, the Chinese teachers did not regard the topic of this chapter as the key piece of the knowledge package in which it is included. The key piece in the package was the meaning of multiplication with fractions. The teachers regarded it as a "knot" that ties a cluster of concepts that support the understanding of the meaning of division by fractions. In the previous chapters we noted that the Chinese teachers tend to pay significant attention to the occasion when a concept is first introduced and tend to regard it as a key piece in a knowledge package. In addressing the key piece in the knowledge package of this chapter, they still adhered to this principle. However, since the mathematical topic discussed in this chapter is more advanced and complex, its stepping stone is not a single concept but a connection of several concepts.

One of the reasons why the U.S. teachers' understanding of the meaning of division of fractions was not built might be that their knowledge lacked connections and links. The understanding of most of the U.S. teachers was supported by only one idea—the partitive model of whole number division. Because other necessary concepts for understanding and their connections with the topic were missing, these teachers were not able to generate a conceptual representation of the meaning of division by fractions.

### Relationship Between Teachers' Subject Matter Knowledge and Their Representations

Generating representations for a mathematical concept is a common teaching task. Most of the U.S. teachers tended to represent the meaning of division by fractions with a real-world example. The topics that the Chinese teachers used, however, were broader and less connected with students' lives. Doubtless connecting school mathematics learning with students' out-of-school lives may help them make more sense of mathematics. However, the "real world" cannot produce mathematical content by itself. Without a solid knowledge of what to represent, no matter how rich one's knowledge of students' lives, no matter how much one is motivated to connect mathematics with students' lives, one still cannot produce a conceptually correct representation.

## SUMMARY

This chapter investigated teachers' subject matter knowledge of two aspects of the same topic—division by fractions. Teachers were asked to calculate $1\frac{3}{4} \div \frac{1}{2}$ and to illustrate the meaning of the operation, an aspect of subject matter knowledge not approached in previous chapters. The U.S. teachers'



knowledge of division by fractions was obviously weaker than their knowledge of the two previous topics. Although 43% of the U.S. teachers succeeded in correctly calculating a complete answer, none showed an understanding of the rationale underlying their calculations. Only Tr. Belle, an experienced teacher, succeeded in generating a representation that correctly illustrated the meaning of division by fractions.

The Chinese teachers' performance on the task for this chapter was not noticeably different from that on the previous tasks. All of their calculations were correct and a few teachers went a step further to discuss the rationale underlying the algorithm. Most of the teachers generated at least one correct and appropriate representation. Their ability to generate representations that used a rich variety of subjects and different models of division by fractions seemed to be based on their solid knowledge of the topic. On the other hand, the U.S. teachers, who were unable to represent the operation, did not correctly explain its meaning. This suggests that in order to have a pedagogically powerful representation for a topic, a teacher should first have a comprehensive understanding of it.

Chapter 5

# Teachers' Subject Matter Knowledge: Profound Understanding Of Fundamental Mathematics

The previous four chapters depicted U.S. and Chinese teachers' knowledge of four topics in elementary mathematics. There was a striking contrast in the knowledge of the two groups of teachers studied. The 23 "above average" U.S. teachers tended to be procedurally focused. Most showed sound algorithmic competence in two beginning topics, whole number subtraction and multiplication, but had difficulty with two more advanced topics, division by fractions, and perimeter and area of a rectangle. Although they came from schools whose quality ranged from excellent to mediocre, most of the 72 Chinese teachers demonstrated algorithmic competence as well as conceptual understanding of all four topics. This chapter is devoted to discussion of the teachers' knowledge across the particular topics.

Considered as a whole, the knowledge of the Chinese teachers seemed clearly coherent while that of the U.S. teachers was clearly fragmented. Although the four topics in this study are located at various levels and subareas of elementary mathematics, while interviewing the Chinese teachers I could perceive interconnections among their discussions of each topic. From the U.S. teachers' responses, however, one can hardly see any connection among the four topics. Intriguingly, the fragmentation of the U.S. teachers' mathematical knowledge coincides with the fragmentation of mathematics curriculum and teaching in the U.S. found by other researchers as major explanations for unsatisfactory mathematics learning in the United States (Schmidt, McKnight, & Raizen, 1997; Stevenson & Stigler, 1992). From my perspective, however, this fragmentation and coherence are effects, not causes. Curricula, teaching, and teachers' knowledge reflect the terrains of elementary mathematics in the United States and in China. What caused the coherence of the Chinese teachers' knowledge, in fact, is the mathematical substance of their knowledge.

## A CROSS-TOPIC PICTURE OF THE CHINESE TEACHERS' KNOWLEDGE: WHAT IS ITS MATHEMATICAL SUBSTANCE?

Let us take a bird's eye view of the Chinese teachers' responses to the interview questions. It will reveal that their discussions shared some interesting features that permeated their mathematical knowledge and were rarely, if ever, found in the U.S. teachers' responses.

### To Find the Mathematical Rationale of an Algorithm

During their interviews, the Chinese teachers often cited an old saying to introduce further discussion of an algorithm: "Know how, and also know why." In adopting this saying, which encourages people to discover a reason behind an action, the teachers gave it a new and specific meaning—*to know how to carry out an algorithm and to know why it makes sense mathematically.* Arithmetic contains various algorithms—in fact it is often thought that knowing arithmetic means being skillful in using these algorithms. From the Chinese teachers' perspective, however, to know a set of rules for solving a problem in a finite number of steps is far from enough—one should also know why the sequence of steps in the computation makes sense. For the algorithm of subtraction with regrouping, while most U.S. teachers were satisfied with the pseudoexplanation of "borrowing," the Chinese teachers explained that the rationale of the computation is "decomposing a higher value unit."[1] For the topic of multidigit multiplication, while most of the U.S. teachers were content with the rule of "lining up with the number by which you multiplied," the Chinese teachers explored the concepts of place value and place value system to explain why the partial products aren't lined up in multiplication as addends are in addition. For the calculation of division by fractions for which the U.S. teachers used "invert and multiply," the Chinese teachers referred to "dividing by

[1]In teaching, Chinese teachers tend to use mathematical terms in their verbal explanations. Terms such as *addend, sum, minuend, subtrahend, difference, multiplicand, multiplier, product, partial product, dividend, divisor, quotient, inverse operation,* and *composing* and *decomposing,* are frequently used. For example, Chinese teachers do not express the additive version of the commutative law as "The order in which you add two numbers doesn't matter." Instead, they say "When we add two addends, if we exchange their places in the sentence, the sum will remain the same."



a number is equivalent to multiplying by its reciprocal" as the rationale for this seemingly arbitrary algorithm.

The predilection to ask "Why does it make sense?" is the first stepping stone to conceptual understanding of mathematics. Exploring the mathematical reasons underlying algorithms, moreover, led the Chinese teachers to more important ideas of the discipline. For example, the rationale for subtraction with regrouping, "decomposing a higher value unit," is connected with the idea of "composing a higher value unit," which is the rationale for addition with carrying. A further investigation of composing and decomposing a higher value unit, then, may lead to the idea of the "rate of composing and decomposing a higher value unit," which is a basic idea of number representation. Similarly, the concept of place value is connected with deeper ideas, such as place value system and basic unit of a number. Exploring the "why" underlying the "how" leads step by step to the basic ideas at the core of mathematics.

### To Justify an Explanation with a Symbolic Derivation

Verbal explanation of a mathematical reason underlying an algorithm, however, seemed to be necessary but not sufficient for the Chinese teachers. As displayed in the previous chapters, after giving an explanation the Chinese teachers tended to justify it with a symbolic derivation. For example, in the case of multidigit multiplication, some of the U.S. teachers explained that the problem 123 × 645 can be separated into three "small problems": 123 × 600, 123 × 40, and 124 × 5. The partial products, then, are 73800, 4920, and 615, instead of 738, 492, and 615. Compared with most U.S. teachers' emphasis on "lining up," this explanation is conceptual. However, the Chinese teachers gave explanations that were even more rigorous. First, they tended to point out that the distributive law[2] is the rationale underlying the algorithm. Then, as described in chapter 2, they showed how it could be derived from the distributive law in order to

[2]In the Chinese mathematics curriculum, the additive versions of commutative and associative laws are first introduced in third grade. The commutative, associative, and distributive laws of multiplication are introduced in fourth grade. They are introduced as alternatives to the standard method. For example, the textbook says of the commutative law of addition, "When two numbers are added, if the locations of the addends are exchanged, the sum remains the same. This is called the commutative law of addition. If the letters $a$ and $b$ represent two arbitrary addends, we can write the commutative law of addition as: $a + b = b + a$. The method we learned of checking a sum by exchanging the order of addends is drawn from this law" (Beijing, Tianjin, Shanghai, and Zhejiang Associate Group for Elementary Mathematics Teaching Material Composing, 1989, pp. 82–83). The textbook illustrates how the two laws can be used as "a way for fast computation." For example, students learn that a faster way of solving 258 + 791 + 642 is to transform it into (258 + 642) + 791, a faster way of solving 1646 − 248 − 152 is to transform it into 1646 − (248 + 152).

illustrate how the distributive law works in this situation and why it makes sense:

$$\begin{aligned} 123 \times 645 &= 123 \times (600 + 40 + 5) \\ &= 123 \times 600 + 123 \times 40 + 123 \times 5 \\ &= 73800 + 4920 + 615 \\ &= 78720 + 615 \\ &= 79335 \end{aligned}$$

For the topic of division by fractions, the Chinese teachers' symbolic representations were even more sophisticated. They drew on concepts that "students had learned" to prove the equivalence of $1\frac{3}{4} \div \frac{1}{2}$ and $1\frac{3}{4} \times \frac{2}{1}$ in various ways. The following is one proof based on the relationship between a fraction and a division ($\frac{1}{2} = 1 \div 2$):

$$\begin{aligned} 1\tfrac{3}{4} \div \tfrac{1}{2} &= 1\tfrac{3}{4} \div (1 \div 2) \\ &= 1\tfrac{3}{4} \div 1 \times 2 \\ &= 1\tfrac{3}{4} \times 2 \div 1 \\ &= 1\tfrac{3}{4} \times (2 \div 1) \\ &= 1\tfrac{3}{4} \times \tfrac{2}{1} \end{aligned}$$

A proof drawing on the rule of "maintaining the value of a quotient" is:

$$\begin{aligned} 1\tfrac{3}{4} \div \tfrac{1}{2} &= (1\tfrac{3}{4} \times \tfrac{2}{1}) \div (\tfrac{1}{2} \times \tfrac{2}{1}) \\ &= (1\tfrac{3}{4} \times \tfrac{2}{1}) \div 1 \\ &= 1\tfrac{3}{4} \times \tfrac{2}{1} \\ &= 3\tfrac{1}{2} \end{aligned}$$

Moreover, as illustrated in chapter 3, the Chinese teachers used mathematical sentences to illustrate various nonstandard ways to solve the problem $1\frac{3}{4} \div \frac{1}{2}$, as well as to derive these solutions. Symbolic representations are widely used in Chinese teachers' classrooms. As Tr. Li reported, her first-grade students used mathematical sentences to describe their own way of regrouping: $34 - 6 = 34 - 4 - 2 = 30 - 2 = 28$. Other Chinese teachers in this study also referred to similar incidents.

Researchers have found that elementary students in the United States often view the equal sign as a "do-something signal" (see e.g., Kieran, 1990, p. 100). This reminds me of a discussion I had with a U.S. elementary teacher. I asked her why she accepted student work like "$3 + 3 \times 4 = 12 = 15$." She said, "Well, they did the calculational order correctly and got the correct answer, what is wrong?" From the Chinese teachers' perspective, however, the semantics of mathematical operations should be represented rigorously. It is intolerable to have two different values on each side of an equal sign. As my elementary teacher once said to her class, "The equal sign is the soul of mathematical operations." In fact, changing one or both sides of an equal sign for certain purposes while preserving the "equals" relationship is the "secret" of mathematical operations.



The Chinese teachers were skilled in adding and removing parentheses and in changing the order of operations in a mathematical sentence. Drawing on a few simple properties such as the three basic laws, the rule of maintaining the value of a quotient, and the meaning of fractions they developed clever symbolic justifications of the arithmetic algorithms they encountered in the interviews.

As Schoenfeld (1985) indicated, "proof" as a form of explanation is mandatory, an accepted standard of the discipline of mathematics. The Chinese teachers tended to justify mathematical statements both verbally and symbolically. Verbal justification tended to come before symbolic justification, but the latter tended to be more rigorous. After the Chinese teachers reported their investigations of the student's claim, as discussed in chapter 4, they all justified their ideas. All of those who presented an invalid idea only gave verbal justifications. If they had used symbolic representations, I suspect some might have avoided or at least found the pitfalls in their arguments.

### Multiple Approaches to a Computational Procedure: Flexibility Rooted in Conceptual Understanding

Although proofs and explanations should be rigorous, mathematics is not rigid. Mathematicians use and value different approaches to solving problems (Pólya, 1973), even arithmetic problems. Dowker (1992) asked 44 professional mathematicians to estimate mentally the results of products and quotients of 10 multiplication and division problems involving whole numbers and decimals. The most striking result of her investigation "was the number and variety of specific estimation strategies used by the mathematicians." "The mathematicians tended to use strategies involving the understanding of arithmetical properties and relationships" and "rarely the strategy of 'Proceeding algorithmically.' "

"To solve a problem in multiple ways" is also an attitude of Chinese teachers. For all four topics, they discussed alternative as well as standard approaches. For the topic of subtraction, they described at least three ways of regrouping, including the regrouping of subtrahends. For the topic of multidigit multiplication, they mentioned at least two explanations of the algorithm. One teacher showed six ways of lining up the partial products. For the division with fractions topic the Chinese teachers demonstrated at least four ways to prove the standard algorithm and three alternative methods of computation.

For all the arithmetic topics, the Chinese teachers indicated that although a standard algorithm may be used in all cases, it may not be the best method for every case. Applying an algorithm and its various versions flexibly allows one to get the best solution for a given case. For example, the Chinese teachers pointed out that there are several ways to compute $1\frac{3}{4} \div \frac{1}{2}$. Using decimals, the distributive law, or other mathematical ideas, all the alternatives were faster and easier than the standard algorithm. Being able to calculate in multiple ways means that one has transcended the formality of an algorithm and reached the essence of the numerical operations—the underlying mathematical ideas and principles. The reason that one problem can be solved in multiple ways is that mathematics does not consist of isolated rules, but connected ideas. Being able to and tending to solve a problem in more than one way, therefore, reveals the ability and the predilection to make connections between and among mathematical areas and topics.

Approaching a topic in various ways, making arguments for various solutions, comparing the solutions and finding a best one, in fact, is a constant force in the development of mathematics. An advanced operation or advanced branch in mathematics usually offers a more sophisticated way to solve problems. Multiplication, for example, is a more sophisticated operation than addition for solving some problems. Some algebraic methods of solving problems are more sophisticated than arithmetic ones. When a problem is solved in multiple ways, it serves as a tie connecting several pieces of mathematical knowledge. How the Chinese teachers view the four basic arithmetical operations shows how they manage to unify the whole field of elementary mathematics.

### Relationships Among the Four Basic Operations: The "Road System" Connecting the Field of Elementary Mathematics

Arithmetic, "the art of calculation," consists of numerical operations. The U.S. teachers and the Chinese teachers, however, seemed to view these operations differently. The U.S. teachers tended to focus on the particular algorithm associated with an operation, for example, the algorithm for subtraction with regrouping, the algorithm for multidigit multiplication, and the algorithm for division by fractions. The Chinese teachers, on the other hand, were more interested in the operations themselves and their relationships. In particular, they were interested in faster and easier ways to do



a given computation, how the meanings of the four operations are connected, and how the meaning and the relationships of the operations are represented across subsets of numbers—whole numbers, fractions, and decimals.

When they teach subtraction with decomposing a higher value unit, Chinese teachers start from addition with composing a higher value unit. When they discussed the "lining-up rule" in multidigit multiplication, they compared it with the lining-up rule in multidigit addition. In representing the meaning of division they described how division models are derived from the meaning of multiplication. The teachers also noted how the introduction of a new set of numbers—fractions—brings new features to arithmetical operations that had previously been restricted to whole numbers. In their discussions of the relationship between the perimeter and area of a rectangle, the Chinese teachers again connected the interview topic with arithmetic operations.

In the Chinese teachers' discussions two kinds of relationships that connect the four basic operations were apparent. One might be called "derived operation." For example, multiplication is an operation derived from the operation of addition. It solves certain kinds of complicated addition problems in a easier way.[3] The other relationship is inverse operation. The term "inverse operation" was never mentioned by the U.S. teachers, but was very often used by the Chinese teachers. Subtraction is the inverse of addition, and division is the inverse of multiplication. These two kinds of relationships tightly connect the four operations. Because all the topics of elementary mathematics are related to the four operations, understanding of the relationships among the four operations, then, becomes a road system that connects all of elementary mathematics.[4] With this road system, one can go anywhere in the domain.

## KNOWLEDGE PACKAGES AND THEIR KEY PIECES: UNDERSTANDING LONGITUDINAL COHERENCE IN LEARNING

Another feature of Chinese teachers' knowledge not found among U.S. teachers is their well-developed "knowledge packages." The four features discussed above concern teachers' understanding of the field of elementary mathematics. In contrast, the knowledge packages reveal the teachers'

[3] Although the four interview questions did not provide room for discussion of the relationship between addition and multiplication, Chinese teachers actually consider it a very important concept in their everyday teaching.

[4] The two kinds of relationships among the four basic operations, indeed, apply to all advanced operations in the discipline of mathematics as well. The "road system" of elementary mathematics, therefore, epitomizes the "road system" of the whole discipline.

understanding of the longitudinal process of opening up and cultivating such a field in students' minds. Arithmetic, as an intellectual field, was created and cultivated by human beings. Teaching and learning arithmetic, creating conditions in which young humans can rebuild this field in their minds, is the concern of elementary mathematics teachers. Psychologists have devoted themselves to study how students learn mathematics. Mathematics teachers have their own theory about learning mathematics.

The three knowledge package models derived from the Chinese teachers' discussion of subtraction with regrouping, multidigit multiplication, and division by fractions share a similar structure. They all have a sequence in the center, and a "circle" of linked topics connected to the topics in the sequence. The sequence in the subtraction package goes from the topic of addition and subtraction within 10, to addition and subtraction within 20, to subtraction with regrouping of numbers between 20 and 100, then to subtraction of large numbers with regrouping. The sequence in the multiplication package includes multiplication by one-digit numbers, multiplication by two-digit numbers, and multiplication by three-digit numbers. The sequence in the package of the meaning of division by fractions goes from meaning of addition, to meaning of multiplication with whole numbers, to meaning of multiplication with fractions, to meaning of division with fractions. The teachers believe that these sequences are the main paths through which knowledge and skill about the three topics develop.

Such linear sequences, however, do not develop alone, but are supported by other topics. In the subtraction package, for example, "addition and subtraction within 10" is related to three other topics: the composition of 10, composing and decomposing a higher value unit, and addition and subtraction as inverse operations. "Subtraction with regrouping of numbers between 20 and 100," the topic raised in interviews, was also supported by five items: composition of numbers within 10, the rate of composing a higher value unit, composing and decomposing a higher value unit, addition and subtraction as inverse operations, and subtraction without regrouping. At the same time, an item in the circle may be related to several pieces in the package. For example, "composing and decomposing a higher value unit" and "addition and subtraction as inverse operations" are both related to four other pieces. With the support from these topics, the development of the central sequences becomes more mathematically significant and conceptually enriched.

The teachers do not consider all of the items to have the same status. Each package contains "key" pieces that "weigh" more than other members. Some of the key pieces are located in the linear sequence and some are in the "circle." The teachers gave several reasons why they considered a certain piece of knowledge to be a "key" piece. They pay particular attention to the first occasion when a concept or skill is introduced. For example, the topic of "addition and subtraction within 20" is considered to be such



a case for learning subtraction with regrouping. The topic of "multiplication by two-digit numbers" was considered an important step in learning multidigit multiplication. The Chinese teachers believe that if students learn a concept thoroughly the first time it is introduced, one "will get twice the result with half the effort in later learning." Otherwise, one "will get half the result with twice the effort."

Another kind of key piece in a knowledge package is a "concept knot." For example, in addressing the meaning of division by fractions, the Chinese teachers referred to the meaning of multiplication with fractions. They think it ties together five important concepts related to the meaning of division by fractions: meaning of multiplication, models of division by whole numbers, concept of a fraction, concept of a whole, and the meaning of multiplication with whole numbers. A thorough understanding of the meaning of multiplication with fractions, then, will allow students to easily reach an understanding of the meaning of division by fractions. On the other hand, the teachers also believe that exploring the meaning of division by fractions is a good opportunity for revisiting, and deepening understanding of these five concepts.

In the knowledge packages, procedural topics and conceptual topics were interwoven. The teachers who had a conceptual understanding of the topic and intended to promote students' conceptual learning did not ignore procedural knowledge at all. In fact, from their perspective, a conceptual understanding is never separate from the corresponding procedures where understanding "lives."

The Chinese teachers also think that it is very important for a teacher to know the entire field of elementary mathematics as well as the whole process of learning it. Tr. Mao said:

> *As a mathematics teacher one needs to know the location of each piece of knowledge in the whole mathematical system, its relation with previous knowledge.* For example, this year I am teaching fourth graders. When I open the textbook I should know how the topics in it are connected to the knowledge taught in the first, second, and third grades. When I teach three-digit multiplication I know that my students have learned the multiplication table, one-digit multiplication within 100, and multiplication with a two-digit multiplier. Since they have learned how to multiply with a two-digit multiplier, when teaching multiplication with a three-digit multiplier I just let them explore on their own. I first give them several problems with a two-digit multiplier. Then I present a problem with a three-digit multiplier, and have students think about how to solve it. We have multiplied by a digit at the ones place and a digit at the tens place, now we are going to multiply by a digit at the hundreds place, what can we do, where are we going to put the product, and why? Let them think about it. Then the problem will be solved easily. I will have them, instead of myself, explain the rationale. *On the other hand, I have to know what knowledge will be built on what I am teaching today* (italics added).

## ELEMENTARY MATHEMATICS AS FUNDAMENTAL MATHEMATICS

The Chinese teachers' discussion presented a sophisticated and coherent picture of elementary mathematics. It showed that elementary mathematics is not a simple collection of disconnected number facts and calculational algorithms. Rather, it is an intellectually demanding, challenging, and exciting field—a foundation on which much can be built. Elementary mathematics is fundamental mathematics. The term *fundamental* has three related meanings: foundational, primary, and elementary. Mathematics is an area of science that concerns spatial and numerical relationships in which reasoning is based on these relationships. Historically, arithmetic and geometry were the two main branches of the discipline of mathematics. Today, although the number of branches of the discipline has increased and the field of the discipline has been expanded, the foundational status of arithmetic and geometry in mathematics is still unchanged. None of the new branches, whether pure or applied, operates without the basic mathematical rules and computational skills established in arithmetic and geometry. Elementary school mathematics, composed of arithmetic and primary geometry, is therefore the foundation of the discipline on which advanced branches are constructed.

The term *primary* refers to another feature of elementary mathematics. Elementary mathematics contains the rudiments of many important concepts in more advanced branches of the discipline. For instance, algebra is a way of arranging knowns and unknowns in equations so that the unknowns can be made knowable. As we have seen in the previous chapters, the three basic laws with which these equations are solved—commutative, distributive, and associative—are naturally rooted in arithmetic. Ideas of set, one-to-one correspondence, and order are implicit in counting. Set-theoretic operations, like union and Cartesian product, are related to the meaning of whole number addition and multiplication. Basic ideas of calculus are implicit in the rationale of the calculation of area of a circle in elementary geometry.[5]

The foundational and primary features of mathematics, however, are presented in an elementary format. It is elementary because it is at the

[5]When teaching the formula for the area of a circle, Chinese teachers bring a paper disc to class. Half of the disc has one color and half has another color. The disc is first cut into two halves. Then the two halves are cut into thin pie-shaped pieces with the edges connected. The two half circles are opened and fit together to form a rectangle-like region: . Teachers inspire students to imagine subdividing the disc into more slices so that the region more closely approximates a rectangle. Then, drawing on the formula for the area of a rectangle, students learn the rationale for the formula for the area of a circle. This method of approximating the area of a circle was known in the 17th century (see Smith & Mikami, 1914, p. 131).



beginning of students' learning of mathematics. Therefore it appears straightforward and easy. The seemingly simple ideas embedded in students' minds at this stage will last for the duration of their mathematics learning. For example, in their later learning students will never erase their conceptions of equation learned from "1 + 1 = 2," although they will be changed and enriched.

From a perspective of attaining mathematical competence, teaching elementary mathematics does not mean bringing students merely to the end of arithmetic or to the beginning of "pre-algebra." Rather, it means providing them with a groundwork on which to build future mathematics learning.

U.S. scholars have claimed that advanced concepts can be presented in an intellectually honest way to elementary students. Three decades ago, Bruner claimed that ideas of advanced mathematics such as topology, projective geometry, probability theory, and set theory could be introduced to elementary school students (Bruner, 1960/1977). His proposal was raised again recently by Hirsch (1996). Kaput, Steen, and their colleagues have suggested a "strand-oriented organization" of school mathematics (Kaput & Nemirovsky, 1995; Steen, 1990). They criticized the traditional "layer-cake" organization of school mathematics because it "picks very few strands (e.g., arithmetic, geometry, and algebra) and arranges them horizontally to form the curriculum" (Steen, p. 4). Instead, they propose a longitudinal structure "with greater vertical continuity, to connect the roots of mathematics to the branches of mathematics in the educational experience of children" (Steen, p. 4) illustrated by a tree with roots that represent strands such as "dimension," "space," "change and variation," etc. (Kaput & Nemirovsky, p. 21).

The elementary teachers with conceptual understanding in this study, however, may not be as radical as Kaput and Steen. As shown in the teachers' interviews, elementary mathematics, constituted of arithmetic and primary geometry, already contains important mathematical ideas. For these teachers, a "horizontally arranged curriculum" may also possess "vertical continuity." Arithmetic can also have "multiple representations," "serious mathematics," and "genuine mathematical conversations."[6] I consider the metaphor that Chinese teachers use to illustrate school mathematics to be more accurate. They believe that elementary mathematics is the foundation for their students' future mathematical learning, and will contribute to their students' future life. Students' later mathematical learning is like a multistoried building. The foundation may be invisible from the

[6]"Multiple representations," "genuine mathematical conversations," and "qualitative understanding of mathematical models" are features of mathematical teaching advocated by Kaput and his colleagues (Kaput & Nemirovsky, 1995).

upper stories, but it is the foundation that supports them and makes all the stories (branches) cohere. The appearance and development of new mathematics should not be regarded as a denial of fundamental mathematics. In contrast, it should lead us to an ever better understanding of elementary mathematics, of its powerful potentiality, as well as of the conceptual seeds for the advanced branches.

## PROFOUND UNDERSTANDING OF FUNDAMENTAL MATHEMATICS

Indeed, it is the mathematical substance of elementary mathematics that allows a coherent understanding of it. However, the understanding of elementary mathematics is not always coherent. From a procedural perspective, arithmetic algorithms have little or no connection with other topics, and are isolated from one another. Taking the four topics studied as an example, subtraction with regrouping has nothing to do with multidigit multiplication, nor with division by fractions, nor with area and perimeter of a rectangle.

Figure 5.1 illustrates a typical procedural understanding of the four topics. The letters S, M, D, and G represent the four topics: subtraction with regrouping, multidigit multiplication, division with fractions, and the geometry topic (calculation of perimeter and area). The rectangles represent procedural knowledge of these topics. The ovals represent other procedural knowledge related to these topics. The trapezoids underneath the rectangles represent pseudoconceptual understanding of each topic. The dotted outlines represent missing items. Note that the understandings of the different topics are not connected.

In Fig. 5.1 the four topics are essentially independent and few elements are included in each knowledge package.[7] Pseudoconceptual explanations for algorithms are a feature of understanding that is only procedural. Some teachers invented arbitrary explanations. Some simply verbalized the algorithm. Yet even inventing or citing a pseudoconceptual explanation requires familiarity with the algorithm. Teachers who could barely carry out an algorithm tended not to be able to explain it or connect it with other procedures, as seen in some responses to the division by fractions and geometry topics. With isolated and underdeveloped knowledge packages,

[7]Given a topic, a teacher tends to see other topics related to its learning. If it is procedural, a teacher may see an explanation for it. If it is conceptual, a teacher may see a related procedure or concept. This tendency initiates organization of a well-developed "knowledge package." So I use the term "knowledge package" here for the group of topics that teachers tend to see around the topic they are teaching.



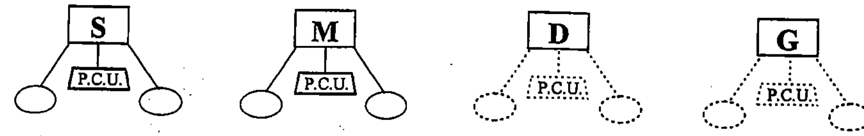


FIG. 5.1. Teachers' procedural knowledge of the four topics.

the mathematical understanding of a teacher with a procedural perspective is fragmentary.

From a conceptual perspective, however, the four topics are connected, related by the mathematical concepts they share. For example, the concept of place value underlies the algorithms for subtraction with regrouping and multidigit multiplication. The concept of place value, then, becomes a connection between the two topics. The concept of inverse operations contributes to the rationale for subtraction with regrouping as well as to the explanation of the meaning of division by fractions. Thus the concept of inverse operations connects subtraction with regrouping and division by fractions. Some concepts, such as the meaning of multiplication, are shared by three of the four topics. Some, such as the three basic laws, are shared by all four topics. Figure 5.2 illustrates how mathematical topics are related from a conceptual perspective.

Although not all the concepts shared by the four topics are included, Fig. 5.2 illustrates how relations among the four topics make them into a network. Some items are not directly related to all four topics. However, their diverse associations overlap and interlace. The three basic laws appeared in the Chinese teachers' discussions of all four topics.

In contrast to the procedural view of the four topics illustrated in Fig. 5.1, Fig. 5.3 illustrates a conceptual understanding of the four topics. The four rectangles at the top of Fig. 5.3 represent the four topics. The ellipses represent the knowledge pieces in the knowledge packages. White ellipses represent procedural topics, light gray ones represent conceptual topics,

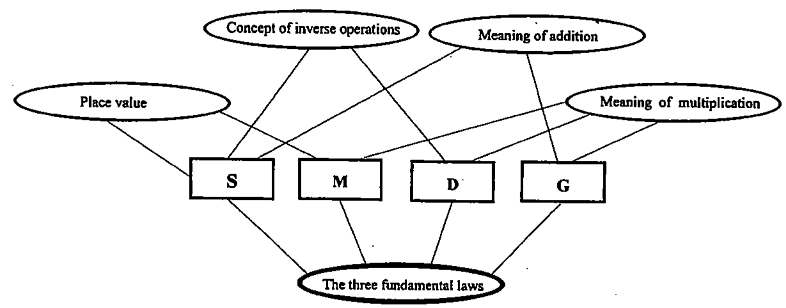


FIG. 5.2. A few shared concepts connect the four topics.

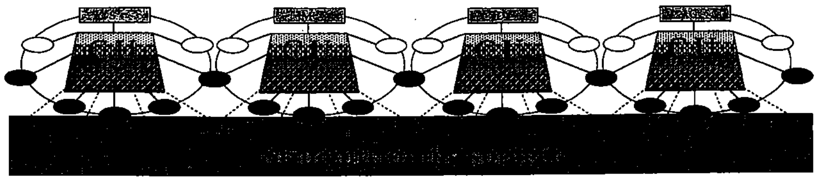

FIG. 5.3. Teachers' conceptual knowledge of the four topics.

dark gray ones represent the basic principles, and ones with dotted outlines represent general attitudes toward mathematics.

When it is composed of well-developed, interconnected knowledge packages, mathematical knowledge forms a network solidly supported by the structure of the subject. Figure 5.3 extends the model of a conceptual understanding of a particular topic given in Fig. 1.4 and illustrates the breadth, depth, connectedness, and thoroughness of a teacher's conceptual understanding of mathematics. Because the four topics are located at various subareas of elementary mathematics, this model also serves as a miniature of a teacher's conceptual understanding of the field of elementary mathematics.

The ellipses with dotted outlines, general attitudes toward mathematics, are usually not included in teachers' knowledge packages for particular topics. However, they contribute significantly to the coherence and consistency of a teacher's mathematical knowledge. Basic attitudes of a subject may be even more penetrating than its basic principles. A basic principle may not support all topics, but a basic attitude may be present with regard to every topic. Basic attitudes toward mathematics mentioned by teachers during interviews, such as "to justify a claim with a mathematical argument," "to know how as well as to know why," "to keep the consistency of an idea in various contexts," and "to approach a topic in multiple ways" pertain to all topics in elementary mathematics.[8]

I call the subject matter knowledge illustrated in Fig. 5.3 profound understanding of fundamental mathematics (PUFM). By profound understanding I mean an understanding of the terrain of fundamental mathematics that is deep, broad, and thorough. Although the term *profound* is often considered to mean intellectual depth, its three connotations, *deep*, *vast*, and *thorough*, are interconnected.

[8]Both dimensions of the structure—basic principles and basic attitudes (Bruner, 1960/1977)—are very powerful in making connections. Unfortunately Fig. 5.3 is too simple to well illustrate the one-to-many relationships between general principles or attitudes and mathematical concepts or topics.



Duckworth, a former student and colleague of Jean Piaget, believes we should keep learning of elementary mathematics and science "deep" and "complex" (1987, 1991). Inspired by Piaget's concern for how *far*, instead of how *fast*, learning would go, she proposed the notion of "learning with depth and breadth" (1979). After a comparison between building a tower "with one brick on top of another" and "on a broad base or a deep foundation," Duckworth said:

> What is the intellectual equivalent of building in breadth and depth? I think it is a matter of making connections: breadth could be thought of as the widely different spheres of experience that can be related to one another; depth can be thought of as the many different kinds of connections that can be made among different facets of our experience. I am not sure whether or not intellectual breadth and depth can be separated from each other, except in talking about them. (p. 7)

I agree with Duckworth that intellectual breadth and depth "is a matter of making connections," and that the two are interwoven. However, her definition of intellectual breadth and depth is too general for use in discussing mathematical learning.[9] Moreover, she does not explain what their relationship is.

Based on my research, I define *understanding a topic with depth* as connecting it with more conceptually powerful ideas of the subject. The closer an idea is to the structure of the discipline, the more powerful it will be, consequently, the more topics it will be able to support. *Understanding a topic with breadth*, on the other hand, is to connect it with those of similar or less conceptual power. For example, consider the knowledge package for subtraction with regrouping. To connect subtraction with regrouping with the topics of addition with carrying, subtraction without regrouping, and addition without carrying is a matter of breadth. To connect it with concepts such as the rate of composing or decomposing a higher value unit or the concept that addition and subtraction are inverse operations is a matter of depth. Depth and breadth, however, depend on thoroughness—the capability to "pass through" all parts of the field—to weave them together. Indeed, it is this thoroughness which "glues" knowledge of mathematics into a coherent whole.

[9]For educational researchers, the depth of teachers' subject matter knowledge seems to be subtle and intriguing. On one hand, most would agree that teachers' understanding should be deep (Ball, 1989; Grossman, Wilson, & Shulman, 1989; Marks, 1987; Steinberg, Marks, & Haymore, 1985; Wilson, 1988). On the other hand, because the term *depth* is "vague" and "elusive in its definition and measurements" (Ball, 1989; Wilson, 1988), progress in understanding it has been slow. Ball (1989) proposed three "specific criteria" for teachers' substantive knowledge: correctness, meaning, and connectedness to avoid the term *deep*, which she considered a vague descriptor of teachers' subject matter knowledge.

Of course, the reason that a profound understanding of elementary mathematics is possible is that first of all, elementary mathematics is a field of depth, breadth, and thoroughness. Teachers with this deep, vast, and thorough understanding do not invent connections between and among mathematical ideas, but reveal and represent them in terms of mathematics teaching and learning. Such teaching and learning tends to have the following four properties:

***Connectedness.*** A teacher with PUFM has a general intention to make connections among mathematical concepts and procedures, from simple and superficial connections between individual pieces of knowledge to complicated and underlying connections among different mathematical operations and subdomains. When reflected in teaching, this intention will prevent students' learning from being fragmented. Instead of learning isolated topics, students will learn a unified body of knowledge.

***Multiple Perspectives.*** Those who have achieved PUFM appreciate different facets of an idea and various approaches to a solution, as well as their advantages and disadvantages. In addition, they are able to provide mathematical explanations of these various facets and approaches. In this way, teachers can lead their students to a flexible understanding of the discipline.

***Basic Ideas.*** Teachers with PUFM display mathematical attitudes and are particularly aware of the "simple but powerful basic concepts and principles of mathematics" (e.g., the idea of an equation). They tend to revisit and reinforce these basic ideas. By focusing on these basic ideas, students are not merely *encouraged* to approach problems, but are *guided* to conduct real mathematical activity.

***Longitudinal Coherence.***[10] Teachers with PUFM are not limited to the knowledge that should be taught in a certain grade; rather, they have achieved a fundamental understanding of the whole elementary mathematics curriculum. With PUFM, teachers are ready at any time to exploit an opportunity to review crucial concepts that students have studied previously. They also know what students are going to learn later, and take opportunities to lay the proper foundation for it.

These four properties are interrelated. While the first property, connectedness, is a general feature of the mathematics teaching of one with PUFM, the other three—multiple perspectives, basic ideas, and longitudinal coherence—are the kinds of connections that lead to different aspects of meaningful understanding of mathematics—breadth, depth, and thoroughness.

[10]Kaput (1994) used this term to describe curricula, here I use it to describe the corresponding property for teacher knowledge. This property is related to an aspect of what Shulman (1986) called curricular knowledge.



Unfortunately, a static model like Fig. 5.3 cannot depict the dynamics of these connections. When they teach, teachers organize their knowledge packages according to teaching context. Connections among topics change with the teaching flow. A central piece in a knowledge package for one topic may become a marginal piece in the knowledge package for another, and vice versa.

Conducting interviews for my study made me think of how people know the town or city they live in. People know the town where they live in different ways. Some people—for example, newcomers—only know the place where their home is located. Some people know their neighborhoods quite well, but rarely go farther away. Some people may know how to get to a few places in the town—for example, the place they work, certain stores where they do their shopping, or the cinemas where they go for a movie. Yet they may only know one way to get to these places, and never bother to explore alternative routes. However some people, for example, taxi drivers, know all the roads in their town very well. They are very flexible and confident when going from one place to another and know several alternative routes. If you are a new visitor, they can take the route that best shows the town. If you are in a rush, at any given time of day they know the route that will get you to your destination fastest. They can even find a place without a complete address. In talking with teachers, I noticed parallels between a certain way of knowing school mathematics and a certain way of knowing roads in a town. The way those teachers with PUFM knew school mathematics in some sense seemed to me very like the way a proficient taxi driver knows a town. There may also be a map in development of the town in a taxi driver's mind as well. Yet a teacher's map of school mathematics must be more complicated and flexible.

## SUMMARY

This chapter contrasted the Chinese and U.S. teachers' overall understanding of the four topics discussed in the previous chapters. The responses of the two groups of teachers suggest that elementary mathematics is construed very differently in China and in the United States. Although the U.S. teachers were concerned with teaching for conceptual understanding, their responses reflected a view common in the United States—that elementary mathematics is "basic," an arbitrary collection of facts and rules in which doing mathematics means following set procedures step-by-step to arrive at answers (Ball, 1991). The Chinese teachers were concerned with knowing why algorithms make sense as well as knowing how to carry them out. Their attitudes were similar to those of practicing mathematicians. They tended to justify an explanation with a symbolic derivation, give multiple solutions for a problem, and discuss relationships among the four basic operations of arithmetic.

For each of the three interview topics that they taught, the Chinese teachers described a "knowledge package," a network of procedural and conceptual topics supporting or supported by the learning of the topic in question. Items in a knowledge package differed in status; the first occasions when a particular concept was introduced were considered "key pieces" and given more emphasis in teaching. For instance, "addition and subtraction within 20" is considered a key piece of the knowledge package for subtraction with regrouping because it is the first occasion when the concept of composing and decomposing a ten is used.

Elementary mathematics can be viewed as "basic" mathematics—a collection of procedures—or as fundamental mathematics. Fundamental mathematics is elementary, foundational, and primary. It is elementary because it is at the beginning of mathematics learning. It is primary because it contains the rudiments of more advanced mathematical concepts. It is foundational because it provides a foundation for students' further mathematics learning.

Profound understanding of fundamental mathematics (PUFM) is more than a sound conceptual understanding of elementary mathematics—it is the awareness of the conceptual structure and basic attitudes of mathematics inherent in elementary mathematics and the ability to provide a foundation for that conceptual structure and instill those basic attitudes in students. A profound understanding of mathematics has breadth, depth, and thoroughness. Breadth of understanding is the capacity to connect a topic with topics of similar or less conceptual power. Depth of understanding is the capacity to connect a topic with those of greater conceptual power. Thoroughness is the capacity to connect all topics.

The teaching of a teacher with PUFM has connectedness, promotes multiple approaches to solving a given problem, revisits and reinforces basic ideas, and has longitudinal coherence. A teacher with PUFM is able to reveal and represent connections among mathematical concepts and procedures to students. He or she appreciates different facets of an idea and various approaches to a solution, as well as their advantages and disadvantages—and is able to provide explanations for students of these various facets and approaches. A teacher with PUFM is aware of the "simple but powerful" basic ideas of mathematics and tends to revisit and reinforce them. He or she has a fundamental understanding of the whole elementary mathematics curriculum, thus is ready to exploit an opportunity to review concepts that students have previously studied or to lay the groundwork for a concept to be studied later.