Chapter 1

Control of Linear Finite Dimensional Differential Systems Revisited

1 Introduction

This part serves the purpose of an introduction to Volume II of Representation and Control of Infinite Dimensional Systems which is mainly concerned with the quadratic cost optimal control problem for distributed parameter systems and systems with time delay, both over a finite and and infinite time interval. For problems over a finite time interval, the main tool used is Dynamic Programming which leads to a Hamilton-Jacobi equation for the value function. For the class of control problems considered, the Hamilton–Jacobi equation can be explicitly solved via the study of an operator Riccati equation. The study of the operator Riccati equation when control is exercised through the boundary in the case of distributed parameter systems or when delays are present in the control in the case of systems with time delay poses additional technical difficulties. The results of Volume I are needed to overcome these difficulties. For problems over an infinite time interval the concepts of controllability and observability (and the weaker concepts of stabilizability and detectability) play an essential role in the development of the theory.

2 Controllability and observability of finite dimensional linear systems

In this section we present the theory of controllability and observability for finite dimensional linear systems in a manner which has implications in the study of controllability and observability for infinite dimensional systems. In particular, we show that if the system is controllable then the transfer from the zero-state to any other state can be carried out by means of minimum-energy controls. It is the same idea, which is later used in the theory of exact controllability of hyperbolic (second order) systems. The mathematical techniques needed to accomplish it however are far more difficult. Throughout this book **R** and **C** denote the respective fields of real and complex numbers.

Consider the linear finite-dimensional control system in the interval $[0,T]$:

$$\begin{cases} \frac{dx}{dt} = Ax(t) + Bu(t) \\ y(t) = Cx(t) \end{cases} \tag{2.1}$$

where the state $x(t) \in \mathbf{R}^n$, the control functions $(t \mapsto u(t)) \in L^2(0,T;\mathbf{R}^m)$, the output $y(t) \in \mathbf{R}^p$ and $A \in \mathcal{L}(\mathbf{R}^n;\mathbf{R}^n)$, $B \in \mathcal{L}(\mathbf{R}^m;\mathbf{R}^n)$ and $C \in \mathcal{L}(\mathbf{R}^n;\mathbf{R}^p)$. In this chapter we use $|\cdot|$ to denote the norm in various Hilbert spaces, $(\cdot,\cdot)$ to denote scalar product and * denotes adjoint of an operator (and also the dual space).

We assume that the initial state $x(0) = 0$.

The solution of the linear differential equation (2.1) can be written as

$$\begin{aligned} x(t;u) &= \int_0^t e^{A(t-s)} Bu(s)\, ds \\ &\triangleq (L_t u), \end{aligned} \tag{2.2}$$

where L_t is a linear bounded transformation from $L^2(0,t;\mathbf{R}^m)$ into $\mathbf{R}^n$.

2.1 Controllability

Definition 2.1 *We say that the system is* controllable *(from $0 \in \mathbf{R}^n$) in $[0,T]$ if given any $\bar{x} \in \mathbf{R}^n$, there exists a control function $u(\cdot) \in L^2(0,T;\mathbf{R}^m)$ such that*

$$x(T;u) = \bar{x}.$$

Controllability in $[0,T]$ is therefore equivalent to the surjectivity of the map $u \mapsto L_T u$. We now give necessary and sufficient conditions for controllability using elementary facts from the theory of linear transformations on Hilbert spaces.

Let H and K be Hilbert spaces and let $A \in \mathcal{L}(H;K)$. Let $A^* \in \mathcal{L}(K^*;H^*)$ be the adjoint operator. In the sequel we identify H and H^* and K and K^*. Let $\mathcal{R}(A)$ denote the range of the operator A and $\mathcal{N}(A)$ denote the null space of A. $\perp$ denotes the orthogonal complement of a closed subspace.

Proposition 2.1 *The following relations are true*

(i) $\mathcal{R}(A) \subseteq \mathcal{N}(A^*)^\perp$ *and* $\mathcal{R}(A) = \mathcal{N}(A^*)^\perp \Leftrightarrow \mathcal{R}(A)$ *is closed.*

(ii) $\mathcal{R}(A^*) \subseteq \mathcal{N}(A)^\perp$ *and* $\mathcal{R}(A^*) = \mathcal{N}(A)^\perp \Leftrightarrow \mathcal{R}(A)$ *is closed.*

($\mathcal{R}(A)$ *closed* $\Leftrightarrow \mathcal{R}(A^*)$ *is closed.*)

(iii) $\mathcal{N}(A) = \mathcal{R}(A^*)^\perp$

(iv) $\mathcal{N}(A^*) = \mathcal{R}(A)^\perp$

(v) Furthermore

$\mathcal{R}(A)$ *is closed* $\Leftrightarrow \exists c > 0$ *such that* $||h|| \le c||Ah||$, $\forall\, h \in H$.

Using the above, we can prove

Proposition 2.2 *The following are true:*

(i) $\mathcal{R}(A)$ *is dense in* $K \Leftrightarrow \mathcal{N}(A^*) = \{0\}$

(ii) $\mathcal{N}(A) = \{0\} \Leftrightarrow \mathcal{R}(A^*)$ *is dense in* H

(iii) $\mathcal{R}(A)$ *is dense in* $K \Leftrightarrow AA^* : K \to K$ *satisfies* $AA^* > 0$.

(iv) $\mathcal{N}(A) = \{0\} \Leftrightarrow A^*A : H \to H$ *satisfies* $A^*A > 0$

(v) $A \in \mathcal{L}(H; K)$ *is invertible* $\Leftrightarrow \mathcal{R}(A) = K, \mathcal{N}(A) = \{0\}$

$\Leftrightarrow \exists c > 0,$ *such that* $||h|| \le c||Ah||, \forall\, h \in H$

$\Leftrightarrow \exists c > 0,$ *such that* $||k|| \le c||AA^*k||, \forall\, k \in K$.

Remark 2.1 Much of the above extends to operators A which are densely defined and closed to spaces H and K, which are Banach spaces. For proofs of these facts, see, for example M. SCHECHTER [1]. For a use of these ideas in a systems context, see M.C. DELFOUR and S.K. MITTER [1]. ■

We can immediately use the above to get a necessary and sufficient condition for controllability. Using (2.2) and noting that

$$L_T^* \;:\; \mathbf{R}^n \to L^2(0,T;\mathbf{R}^m)$$
$$\;:\; y \mapsto B^*e^{(T-\cdot)}y$$

we get

$$W(0,T) \triangleq L_T L_T^* = \int_0^T e^{A(T-s)}BB^*e^{A^*(T-s)}\,ds. \tag{2.3}$$

Hence we have proved.

Theorem 2.1 *The system* (2.1) *is controllable in* $[0,T]$ *if and only if* $W(0,T) > 0$ *(positive definite).*

We can transform this criteria of controllability to an algebraic criterion by noting that

$$\mathcal{N}(W(0,T)) = \mathcal{N}(\mathcal{C}) \tag{2.4}$$

where

$$\mathcal{C} = [B\vdots AB\vdots \cdots \vdots A^{n-1}B]\,[B\vdots AB\vdots \cdots \vdots A^{n-1}B]^* \tag{2.5}$$

This can be proved as follows.

Let $x \in \mathcal{N}(W(0,T))$. Then

$$0 = (x, W(0,T)x) = \int_0^T |B^*e^{A^*(T-t)}x|^2\,dt,$$

and hence $B^*e^{A^*(T-t)}x = 0,\ \forall\, t \in [0,T]$.

Differentiating $(n-1)$-times with respect to t and setting $t = 0$, we get

$$B^*x = 0, B^*A^*x = 0, \ldots, B^*A^{*n-1}x = 0,$$

and hence

$$x \in \mathcal{N}(\mathcal{C}).$$

Conversely, suppose $x \in \mathcal{N}(\mathcal{C})$. By the Cayley-Hamilton theorem,

$$e^{A(T-t)} = \sum_{i=0}^{n-1} \alpha_i(T-t)A^i.$$

Hence

$$x^*W(0,T) = \int_0^T \left[\sum_{i=0}^{n-1} \alpha_i(T-t)x^*A^iB\right] B^*e^{A^*(T-t)}\,dt.$$

Now $x^*A^iB = 0,\ i = 0, \cdots, n-1$, since $x \in \mathcal{N}(\mathcal{C})$. Therefore $x^*W(0,T) = 0$ and since $W(0,T)$ is symmetric,

$$x \in \mathcal{N}(W(0,T)).$$

Hence using the fact that $W(0,T)$ is a symmetric matrix

$$\mathcal{R}(W(0,T)) = \mathcal{R}(\mathcal{C}). \tag{2.6}$$

But

$$\mathcal{R}(\mathcal{C}) = \mathcal{R}([B\vdots AB\vdots \cdots \vdots A^{n-1}B]), \tag{2.7}$$

and hence we have proved the following theorem.

Theorem 2.2 *The following conditions are equivalent:*

(i) System (2.1) *is controllable in* $[0,T]$

(ii) $W(0,T) > 0$ *(positive–definite)*

(iii) *rank* $[B \vdots AB \vdots \cdots \vdots A^{n-1}B] = n$.

When the system is controllable we shall refer to (A, B) as a *controllable pair*.

It is of some interest to establish that if the system is controllable then we can transfer the state x_0 at time t_0 to the state x_1 at time t_1 by using a control u^* which has least energy

$$\int_{t_0}^{t_1} |u^*(s)|^2 \, ds$$

amongst all controls that transfer the phase (x_0, t_0) to the phase (x_1, t_1).

To establish this, let us use the notation

$$\Phi(t, s) = e^{A(t-s)}.$$

Introduce the adjoint system

$$\begin{cases} \dfrac{dp}{dt} = -A^* p(t), \quad t \in [t_0, t_1[\,. \\ p(t_1) = \eta, \end{cases} \tag{2.8}$$

where η will be specified later. The solution of (2.8) is given by

$$p(t) = \Phi^*(t_1, t)\eta. \tag{2.9}$$

Now choose the control

$$\begin{aligned} u^*(t) &= -B^* p(t) \\ &= -B^* \Phi^*(t_1, t)\eta. \end{aligned} \tag{2.10}$$

If the control is to transfer (x_0, t_0) to (x_1, t_1), we must have

$$x_1 = x(t_1) = \Phi(t_1, t_0)x_0 - \int_{t_0}^{t_1} \Phi(t_1, t) B B^* \Phi^*(t_1, t) \, dt \, \eta$$

and hence if the system is controllable and we choose

$$\eta = W^{-1}(t_0, t_1)[\Phi(t_1, t_0)x_0 - x_1]$$

the desired transfer will be effected. Let us now prove that $u^*(t)$ is the minimum energy control.

Let $\bar{u}(t)$ be some other control which effects the desired transfer. Then

$$\begin{aligned} x_1 &= \Phi(t_1, t_0)x_0 + \int_{t_0}^{t_1} \Phi(t_1, t) B \bar{u}(t) \, dt \\ &= \Phi(t_1, t_0)x_0 + \int_{t_0}^{t_1} \Phi(t_1, t) B u^*(t) \, dt. \end{aligned}$$

Therefore

$$\int_{t_0}^{t_1} \Phi(t_1, t) B[\bar{u}(t) - u^*(t)] \, dt = 0,$$

and hence

$$\int_{t_0}^{t_1} ([\bar{u}(t) - u^*(t)], -B^* \Phi^*(t_1, t)\eta) \, dt = 0$$

and hence

$$\int_{t_0}^{t_1} (\bar{u}(t) - u^*(t), u^*(t)) \, dt = 0.$$

Therefore

$$\begin{aligned} \int_{t_0}^{t_1} |\bar{u}(t)|^2 \, dt &= \int_{t_0}^{t_1} |\bar{u}(t) - u^*(t) + u^*(t)|^2 \, dt \\ &= \int_{t_0}^{t_1} |u^*(t)|^2 \, dt + \int_{t_0}^{t_1} |\bar{u}(t) - u^*(t)|^2 \, dt \\ &> \int_{t_0}^{t_1} |u^*(t)|^2 \, dt \quad \text{if } \bar{u} \neq u^*. \end{aligned}$$

Suppose now that U and X are infinite dimensional Hilbert spaces, $A : X \to X$ an unbounded linear operator with domain $D(A)$ and $B : U \to X$ a bounded linear operator. Then there is a difficulty with carrying through the arguments given earlier (for example, $\mathcal{R}(L_T)$ need not be a closed subspace of X). Nevertheless, $\mathcal{R}(L_T)$ can be given the structure of a Hilbert space with continuous injection in X (Proposition 2.1 in Chapter 2 of Part I). Furthermore, there is a need to make a distinction between approximate controllability ($\mathcal{R}(L_T)$ is dense in X) and exact controllability (whose definition is somewhat subtle, see Definition 2.2 in Chapter 2 of Part I); indeed $\mathcal{R}(L_T) = X$

will usually not hold). However, an abstract criterion for exact controllability is related to an estimate on $L_T L_T^*$ (see Proposition 2.2; this chapter). It is also of interest to note that the minimum energy control viewpoint of verifying exact controllability as outlined in this section extends to infinite dimensional situations, but of course at the cost of using much more elaborate technical machinery. Part I, Chapter 2 gives a treatment of exact controllability both for parabolic and hyperbolic partial differential equations when control is exercised in a distributed manner and also when control is exercised through the boundary.

2.2 Observability

We now discuss the dual concept of observability. For this purpose we may assume that the input is known and since we are dealing with linear time-invariant finite dimensional systems, we may take $u(t) \equiv 0$, $t \in [t_0, \infty[$. Consider therefore the system

$$\begin{cases} \dfrac{dx}{dt} = Ax(t) \\ y(t) = Cx(t), \end{cases} \tag{2.11}$$

and let the initial state $x(t_0) = x_0$, for some arbitrary x_0.

The solution of (2.11) can be written as

$$y(t) = Ce^{A(t-t_0)}x_0. \tag{2.12}$$

Definition 2.2 *The system* (2.1) *is said to be* observable *in* $[t_0, t_1]$ *if the map*

$$\mathcal{O} : \mathbf{R}^n \to L^2(t_0, t_1; \mathbf{R}^p) : x_0 \to y(\cdot) = Ce^{A(\cdot - t_0)}x_0$$

is injective.

The definition expresses the fact that we can recover uniquely the initial state from a knowledge of the output $y(\cdot)$ in the time interval $[t_0, t_1]$. The interval $[t_0, t_1]$ can in fact be taken to be arbitrarily small.

Now,

$$\begin{aligned} \mathcal{O} \text{ is injective} \quad &\Leftrightarrow \quad \mathcal{N}(\mathcal{O}) = \{0\} \\ &\Leftrightarrow \quad \mathcal{N}(\mathcal{O}^*\mathcal{O}) = \{0\}. \end{aligned}$$

An easy computation shows that

$$\Sigma(t_0, t_1) \triangleq \mathcal{O}^*\mathcal{O} = \int_{t_0}^{t_1} e^{(t_1-t)A^*} C^* C e^{(t_1-t)A}\, dt,$$

and clearly

$$\mathcal{N}(\mathcal{O}^*\mathcal{O}) = \{0\} \Leftrightarrow \Sigma(t_0, t_1) > 0 \text{ (positive-definite)}.$$

It is also easy to show that,

$$\begin{aligned} \mathcal{N}(\mathcal{O}^*\mathcal{O}) &= \mathcal{N}([C^* \vdots A^*C^* \vdots \cdots \vdots A^{*n-1}C^*]^* [C^* \vdots A^*C^* \vdots \cdots \vdots A^{*n-1}C^*]), \\ &= \mathcal{N}([C^* \vdots A^*C^* \cdots A^{*n-1}C^*]) \end{aligned}$$

and hence we have proved:

Theorem 2.3 *The system* (2.1) *is observable if and only if*

$$\textit{rank}\ [C^* \vdots A^*C^* \vdots \cdots \vdots A^{*n-1}C^*] = n.$$

When the system is observable we shall refer to (A, C) as an observable pair.

2.3 Duality

There is a formal duality between the concepts of controllability and observability. For this purpose we introduce the dual system

$$\begin{cases} \dfrac{d\xi}{dt} = -A^*\xi(t) - C^*\gamma(t) \\ \eta(t) = -B^*\xi(t), \end{cases} \tag{2.1'}$$

which evolves backward in time.

Then mathematically,
The system (2.1) is observable $\Leftrightarrow$ The system (2.1') is controllable.

The system (2.1') is observable $\Leftrightarrow$ The system (2.1) is controllable.

In this book we do not study observability of partial differential equations but exploiting the above duality ideas much of the theory of controllability developed in Part I can be used to develop a theory of observability for partial differential equations (even with unbounded observation operators).

Conceptually, controllability of a system permits the choice of feedback controls resulting in certain desirable (e.g. stability) properties of closed-loop systems while observability of a system permits the design of state estimators with desirable properties. We shall discuss this in a later section.

2.4 Canonical structure for linear systems

In this section we wish to obtain a decomposition (structure) theorem for the system (2.1) which is not necessarily controllable and observable. Let us denote by

$$V := \mathbf{R}(\mathcal{C}) = \mathcal{R}([B \vdots AB \vdots \cdots \vdots A^{n-1}B])$$

V is the smallest A-invariant subspace containing the image of B.

Let us also denote by

$$W := \mathcal{N}(\mathcal{O}) = \mathcal{N}([C^* \vdots A^*C^* \cdots \vdots A^{*n-1}C^*])$$

and clearly W is the smallest A^*-invariant subspace containing the image of C^*.

Now we may write the state space $X := \mathbf{R}^n$ as

$$X = (V \cap W^\perp) \oplus (V^\perp \cap W^\perp) \oplus (V \cap W) \oplus (V^\perp \cap W).$$

Then there exists a basis in which we may write the system as

$$\begin{bmatrix} \frac{d\bar{x}_1}{dt} \\ \frac{d\bar{x}_2}{dt} \\ \frac{d\bar{x}_3}{dt} \\ \frac{d\bar{x}_4}{dt} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} & 0 \\ 0 & A_{22} & A_{23} & A_{24} \\ 0 & 0 & A_{33} & 0 \\ 0 & 0 & A_{43} & A_{44} \end{bmatrix} \begin{bmatrix} \bar{x}_1(t) \\ \bar{x}_2(t) \\ \bar{x}_3(t) \\ \bar{x}_4(t) \end{bmatrix} + \begin{bmatrix} B_1u_1(t) \\ 0 \\ B_3u_3(t) \\ 0 \end{bmatrix} \tag{2.13}$$

$$y(t) = [C_1 \quad C_2 \quad 0 \quad 0] \begin{bmatrix} \bar{x}_1(t) \\ \bar{x}_2(t) \\ \bar{x}_3(t) \\ \bar{x}_4(t) \end{bmatrix} \tag{2.14}$$

In the above, $(A_{11}, B_{11}), (A_{33}, B_3)$ are controllable pairs and (A_{11}, C_1), (A_{22}, C_2) are observable pairs.

2.5 The pole-assigment theorem

An important consequence of a linear system possessing the controllability property is that by means of linear state feedback control the poles of the corresponding closed-loop system can be placed in arbitrary locations.

By state feedback control we mean that the control law is of the form

$$u(t) = Kx(t) + v(t),$$

where $K : \mathbf{R}^n \to \mathbf{R}^m$ and $v(\cdot)$ is a reference input.

Theorem 2.4 *Let $\Sigma = \{\lambda_i \in \mathbf{C}, i = 1, 2, \ldots, n\}$ with the proviso that if λ_i is complex then its conjugate $\bar{\lambda}_i \in \Sigma$. Then the system is controllable iff there exists a feedback control $u(t) = Kx(t)$ with spectrum $(A + BK) = \Sigma$.*

Proof. We start by noting two facts:

(i) Controllability is invariant under non-singular linear transformations of state space. In case $m = 1$, that is we are dealing with the system:

$$\frac{dx}{dt} = Ax(t) + bu(t),\ u(t) \in \mathbf{R} \text{ and } b \in \mathbf{R}^n,$$

then assuming (A, b) is a controllable pair, there exists a non-singular linear transformation $T : \mathbf{R}^n \to \mathbf{R}^n$ such that the above system transforms to

$$\tilde{A} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ & & \ddots & & \\ \vdots & & \ddots & \ddots & \\ & & & \ddots & 1 \\ \alpha_1 & \alpha_2 & \dots & & \alpha_n \end{bmatrix}, \quad \tilde{b} = \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ \vdots \\ 1 \end{bmatrix}.$$

It is then easy to see that if the characteristic polynomial corresponding to the desired spectrum Σ is

$$\chi = s^n - \beta_n s^{n-1} \dots - \beta_2 s - \beta_1,$$

then the feedback law

$$\tilde{u}(t) = K\tilde{x}(t) + v(t), \quad K = \text{diag}\,(\beta_1 - \alpha_1, \dots, \beta_n - \alpha_n)$$

transforms

$$\frac{d\tilde{x}}{dt} = \tilde{A}\tilde{x}(t) + \tilde{b}\tilde{u}(t)$$

to

$$\frac{d\tilde{x}}{dt} = (\tilde{A} + \tilde{b}k)\tilde{x}(t) + \tilde{b}v(t)$$

such that the spectrum of $(\tilde{A} + \tilde{b}K)$ in Σ.

(ii) Controllability is not destroyed by state feedback. That is

$$\mathcal{R}[B \vdots (A + BK)B \vdots \cdots \vdots (A + BK)^{n-1}B] = \mathcal{R}[B \vdots AB \vdots \cdots \vdots A^{n-n}B]$$

Now the general case is reduced to the case $m = 1$ by virtue of the following lemma. ∎

Lemma 2.1 *Let $0 \neq b \in \mathcal{R}(B)$. If (A, B) is a controllable pair, there exists a $K \in \mathcal{L}(\mathbf{R}^m; \mathbf{R}^n)$ such that $(A + BK, b)$ is a controllable pair.*

Let $b_1 = b$ and let $n_1 =$ dimension of the cyclic subspace $\mathcal{X}_1$ generated by (A, b_1). Let $x_1 = b_1$ and define $x_j = Ax_{j-1} + b_1 (j = 2, \cdots, n_1)$. Then the x_j, $j = 1, \cdots, n_1$, forms a basis for the cyclic subspace generated by (A, b_1). If $n_1 < n$, choose $b_2 \in \mathcal{R}(B)$ such that $b_2 \notin \mathcal{X}_1$. Since (A, B) is controllable such a b_2 exists. Let n_2 be the largest integer such that

$$x_1, x_2, \cdots, x_{n_1}, b_2, Ab_2, \cdots, A^{n_2 - 1}b_2$$

are independent.

Define

$$x_{n_1 + i} = Ax_{n_1 + i - 1} + b_2,\ i = 1, 2, \cdots, n_2.$$

Then $x_1, x_2, \cdots, x_{n_1 + n_2}$ is a basis for the cyclic subspace generated by $(A, b_1 + b_2)$. Continuing in this way, we eventually get an independent set of vectors $x_1, \cdots, x_n$, and

$$x_{i+1} = Ax_i + \tilde{b}_i,\ i = 1, \cdots, n - 1,$$

and $\tilde{b}_i \in \mathcal{R}(B)$.

Since $\tilde{b}_i = Bu_i$ for some $u_i \in \mathbf{R}^m$ (the control space) and the $x_1, \cdots, x_n$ are independent, there exists a K_1 such that

$$BK_1 x_i = \tilde{b}_i,\ i = 1, \cdots, n,$$

where $\tilde{b}_i \in \mathcal{R}(B)$ is arbitrary. Therefore

$$(A + BK_1)x_i = x_{i+1},\ i = 1, 2, \cdots, n - 1,$$

and hence

$$x_i = (A + BK_1)^{i-1}b,\ i = 1, \cdots, n.$$

Therefore $(A + BK_1, b)$ is a controllable pair.

Hence the problem has been reduced to the case of $n = 1$ and the control law $K = K_1 + bK^*$ achieves the desired property.

To prove the theorem in the other direction, let λ_i, $i = 1, \cdots, n$, be real and distinct with $\lambda_i \notin$ spectrum of A. Choose $K \in \mathcal{L}(\mathbf{R}^m; \mathbf{R}^n)$ such that spectrum $(A + BK) = \{\lambda_1, \cdots, \lambda_n\}$. Let x_i, $i = 1, \cdots, n$, be the corresponding eigenvectors. Hence

$$x_i = (\lambda_i I - A)^{-1} BK x_i, \quad i = 1, 2, \cdots, n.$$

Now

$$(\lambda I - A)^{-1} = \sum_{j=1}^{n} \rho_j(\lambda) A^{j-1},$$

where $\rho_j(\lambda)$ are rational functions defined on the complement $\mathbf{C} \backslash \sigma(A)$ of the spectrum $\sigma(A)$ of A. Hence

$$x_i = \sum_{j=1}^{n} \rho_j(\lambda_i) A^{j-1} BF x_i \in \mathcal{R}([B \vdots AB \vdots \cdots \vdots A^{n-1}B]).$$

Now the x_i's span $\mathbf{R}^n$ and hence $\mathcal{R}([B \vdots AB \vdots \cdots \vdots A^{n-1}B]) = n$.

No satisfactory result similar to the Pole-Assignment theorem is known in infinite dimensions. It would be interesting to prove a result of this kind for delay systems

$$\begin{cases} \dfrac{dx}{dt} = A_1 x(t) + A_2 x(t-h) + Bu(t) \\ x(0) = \phi_0 \in \mathbf{R}^n \\ x(0) = \phi_1(\theta) \text{ a.e. } \theta \in [-h, 0), \quad \phi_1 \in L^2(-h, 0; \mathbf{R}^n). \end{cases}$$

Corollary 2.1 *If the system is controllable then there exists a feedback control $u(t) = Kx(t)$ such that the closed-loop system*

$$\frac{dx}{dt} = (A + BK)x(t)$$

is asymptotically stable.

2.6 Stabilizability and detectability

The structure theorem of linear systems and the pole-assigment theorem motivates the introduction of the concepts of stabilizability and detectability. Consider the input-state part of the control system (2.1):

$$\frac{dx}{dt} = Ax(t) + Bu(t).$$

Then from the structure theorem, we know that there is a coordinate system in which the above system has a representation

$$\begin{bmatrix} \dfrac{d\bar{x}_1}{dt} \\ \dfrac{d\bar{x}_2}{dt} \end{bmatrix} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ 0 & \bar{A}_{22} \end{bmatrix} \begin{bmatrix} \bar{x}_1(t) \\ \bar{x}_2(t) \end{bmatrix} + \begin{bmatrix} \bar{B}_{11} \\ 0 \end{bmatrix} u(t) \qquad (2.15)$$

and $(\bar{A}_{11}, \bar{B}_{11})$ is a controllable pair. Hence from the corollary of the pole-assignment system we can stabilize the system by means of the control $u(t) = K\bar{x}_1(t)$, provided

$$\frac{d\bar{x}_2}{dt} = \bar{A}_{22}\bar{x}_2(t)$$

is asymptotically stable.

Definition 2.3 *(i) We say that the system* (2.1) *is* open-loop stabilizable *if*

$$\forall x_0 \in \mathbf{R}^n, \quad \exists u \in L^2(0, \infty; \mathbf{R}^m)$$

such that the solution $x(t; x_0, u)$ of (2.1) *satisfies*

$$\int_0^\infty |x(t; x_0, u)|^2 \, dt < \infty \qquad (2.16)$$

(ii) We say that the system (2.1) *is* stabilizable by feedback *if there exists a feedback matrix K such that*

$$u(t) = Kx(t)$$

and

$$\frac{dx}{dt} = (A + BK)x(t)$$

is asymptotically stable.

Remark 2.2 For finite-dimensional linear systems if the system is open-loop stabilizable then it is stabilizable by feedback and conversely. Note that by the previous discussion, stabilizability is a weaker concept than controllability, since stabilizability only requires that the uncontrollable part of the system be asymptotically stable. ∎

The dual of the concept of stabilizability is that of detectability. This is best discussed by considering the dual system

$$\frac{d\xi}{dt} = -A^*\xi(t) - C^*y(t), \tag{2.17}$$

which involves only the state-output part of the system.

Definition 2.4 *The system* (2.1) *is said to be* detectable *if the dual system* (2.17) *is stabilizable by feedback.*

Remark 2.3 Detectability requires that the unobservable part of the system be asymptotically stable and hence is a weaker concept than observability. ∎

A matrix test for stabilizability and detectability can be given.

Theorem 2.5 *The following properties are equivalent:*
(i) (A, B) *is a controllable pair*

(ii) rank $[\lambda I - A\vdots B] = n, \quad \forall\, \lambda \in \mathbf{C}$

(iii) rank $[\lambda I - A\vdots B] = n$, *for each eigenvalue* λ *of* A.

Proof. Note that (ii) and (iii) are equivalent since the first $n \times n$ block of the $n \times (n+m)$ matrix $[\lambda I - A\vdots B]$ has full rank whenever λ is not an eigenvalue of A.

We now prove that (i) $\Rightarrow$ (ii). For this purpose, assume that rank $[\lambda I - A\vdots B] < n$ for some $\lambda \in \mathbf{C}$. Hence the space spanned by the rows of $[\lambda I - A\vdots B]$ has dimension less than n and therefore there exists some vector $z^* \in \mathbf{R}^n$ and some λ such that $z^*[\lambda I - A\vdots B] = 0$. Hence

$$z^*A = \lambda z^* \text{ and } z^*B = 0,$$

and therefore $z^*A^kB = \lambda^k z^*B = 0, \forall\, K$, and hence

$$z^*[B\vdots AB\vdots\cdots\vdots A^{n-1}B] = 0$$

which contradicts controllability.

To prove (i) $\Rightarrow$ (ii) assume (A, B) is not a controllable pair. Then from the Structure Theorem, there exists a non-singular linear transformation $T : \mathbf{R}^n \to \mathbf{R}^n$ such that

$$(A, B) \mapsto \left[\bar{A} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}\right]$$

where A_{11}, B_1 is a controllable pair and A_{11} is $r \times r$ with $r < n$. Let λ be an eigenvalue of A_{22}^* with the corresponding eigenvector v so that

$$v^*(\lambda I - A_{22}) = 0. \text{ Then the } n\text{-vector } (\neq 0)$$

$$w = \begin{bmatrix} 0 \\ v \end{bmatrix} \text{ is such that}$$

$w^*\bar{A} = \lambda w^*$ and $w^*\bar{B} = 0$. Hence $z = (T^*)^{-1}w \neq 0$ satisfies $z^*(\lambda I - A)T^{-1}\vdots B) = 0$ and therefore $z^*(\lambda I - A\vdots B) = 0$ contradicting (ii). ∎

Corollary 2.2 *(i)* (A, B) *is* stabilizable *if and only if*

$$rank\ [\lambda I - A\vdots B] = n, \ \forall\, \lambda \in \mathbf{C}, \ Re\,\lambda \geq 0.$$

(ii) Dually (A, C) *is detectable if and only if*

$$rank \begin{bmatrix} \lambda I - A^* \\ C^* \end{bmatrix} = n, \ \forall\, \lambda \in \mathbf{C} \textit{ such that } Re\,\lambda \geq 0.$$

Corollary 2.2 is generalized to an infinite dimensional setting when A is the generator of a C_0-semigroup on a Hilbert space H with e^{tA} compact for $t > 0$ (and indeed for more general situations) and $B \in \mathcal{L}(U, H)$ and $C \in \mathcal{L}(H; Y)$ and U and Y are Hilbert spaces (Proposition 3.3, Part III, Chapter 1).

2.7 Applications of controllability and observability

The concepts of controllability, observability and the slightly weaker notions of stabilizability and detectability have important applications in diverse areas of systems theory and optimal control. In this section we present two such applications. The first application is to theory of stability and tests for stability via the Lyapunov equations. Our second application is to regulator theory where we show that if a system is controllable and observable then we can build a compensator to regulate the system in a satisfactory way.

2.7.1 Stability

Our first application is to the study of stability of linear differential equations via the Lyapunov method.

Consider the stability problem for the equation

$$\frac{dx}{dt} = Ax(t), \quad A : \mathbf{R}^n \to \mathbf{R}^n. \tag{2.18}$$

It is well-known (cf. Chapter 1 in Volume I) that the following are equivalent:

(i) (2.18) is asymptotically stable

(ii) (2.18) is exponentially stable

(iii) $\forall x_0 \in \mathbf{R}^n, \quad \int_0^\infty |e^{At}x_0|^2 \, dt < \infty.$

To test for stability of (2.18), we look for a Lyapunov function of the form

$$V(x) = (x, Px), \quad P^* = P > 0 \text{ positive definite}$$

Computing the derivative of V along trajectories of (2.18) we get

$$\frac{dV}{dt} = (x, A^*P + PAx)$$

and if it is to be negative we require that

$$A^*P + PA = -Q, \quad Q > 0 \text{ positive definite.} \tag{2.19}$$

It is well-know that if A is a stability matrix (all its eigenvalues have strictly negative real part), then the solution of (2.19) can be written as

$$P = \int_0^\infty e^{A^*t} Q e^{At} \, dt, \tag{2.20}$$

and conversely if (2.19) has a positive definite solution then A is a stability matrix. A generalization of these results in infinite dimensional spaces has been extensively discussed in Volume I, Chapter 1.

Now suppose that $Q \geq 0$ and factorize Q as $Q = C^*C$ $C \in \mathcal{L}(\mathbf{R}^p; \mathbf{R}^n)$. Introduce the state-output system:

$$\begin{cases} \dfrac{dx}{dt} = Ax(t) \\ y(t) = Cx(t). \end{cases} \tag{2.21}$$

Definition 2.5 *The state-output system is said to be* asymptotically output stable *if*

$$\forall x \in \mathbf{R}^n, \quad \int_0^\infty |Ce^{At}x|^2 \, dt < \infty.$$

Now it is easy to see that if (A, C) is an observable pair and A is asymptotically stable then the equation

$$A^*P + PA = -C^*C$$

has a solution and

$$P = \int_0^\infty e^{A^*t} C^* C e^{At} \, dt > 0 \text{ (positive-definite)},$$

the positive-definiteness being a consequence of observability. Conversely if the state-output system is asymptotically output stable and if (A, C) is an observable pair then A is asymptotically stable. We can now weaken this further.

Theorem 2.6 *If the state-output system is asymptotically output stable (by some feedback matrix K) and if (A, C) is a detectable pair then the system*

$$\frac{dx}{dt} = (A + BK)\, x(t)$$

is asymptotically stable.

Proof. By hypothesis there exists a $K^* \in \mathcal{L}(\mathbf{R}^n;\mathbf{R}^p)$ such that $(A^*+C^*K^*)$ is a stability matrix and hence $(A+KC)$ is a stability matrix. Now write

$$\frac{dx}{dt} = Ax(t)$$

as

$$\frac{dx}{dt} = (A+KC)x(t) - KCx(t)$$

and we can write its solution as

$$x(t) = e^{(A+KC)t}x + \int_0^t e^{(A+KC)(t-s)}KCx(s)\,ds. \qquad (2.22)$$

Hence

$$\begin{aligned}&[\int_0^\infty |x(t)|^2\,dt]^{\frac{1}{2}}\\ &\le [\int_0^\infty |e^{(A+KC)t}x|^2\,dt]^{\frac{1}{2}}\\ &\quad +[\int_0^\infty |\int_0^t e^{(A+KC)(t-s)}KCx(s)\,ds|^2\,dt]^{\frac{1}{2}}.\end{aligned} \qquad (2.23)$$

Furthermore, since $(A+KC)$ is a stability matrix, there exists $\alpha > 0$ and $M \ge 1$ such that

$$\forall\, x \in \mathbf{R}^n, \quad |e^{(A+KC)t}x| \le M\,e^{-\alpha t}|x|,$$

and the second term of the right-hand side of inequality (2.23) can be majorized by

$$\int_0^\infty [\int_0^t M\,e^{-\alpha(t-s)}\,||K||\,|Cx(s)|\,ds]^2\,dt$$

Now introduce the function

$$f(s) = \begin{cases} M||K||e^{-\alpha s}, & s \ge 0\\ 0, & \text{otherwise}\end{cases}$$

and

$$g(s) = \begin{cases} |Cx(s)|, & s \ge 0\\ 0, & \text{otherwise.}\end{cases}$$

Now $f \in L^1(-\infty,\infty;\mathbf{R})$ and $g \in L^2(-\infty,\infty;\mathbf{R})$. Hence by Young's inequality

$$||f*g||_{L^2} \le ||f||_{L^1}.||g||_{L^2},$$

where

$$(f*g)(t) = \int_{-\infty}^{\infty} f(t-s)g(s)\,ds = \int_0^t M\,||K||\,e^{-\alpha(t-s)}\,|Hx(s)|\,ds.$$

This proves the theorem. ■

Remark 2.4 This theorem and its proof generalize to certain infinite dimensional Hilbert space situations and have implications in the study of the algebraic Riccati equation. See §3.4, Part III, Chapter 1. ■

2.7.2 Compensators for linear systems

Consider the linear control system

$$\begin{cases} \dfrac{dx}{dt} = Ax(t) + Bu(t)\\ y(t) = Cx(t).\end{cases} \qquad (2.24)$$

We have seen previously that if (A,B) is a controllable pair then the spectrum of the closed-loop system can be made arbitrary by state-feedback. We now wish to discuss the situation where full-state feedback is not available and the control has to be of the form

$$u(t) = \int_0^t \Lambda(t-s)y(s)\,ds + v(t) \qquad (2.25)$$

where the kernel Λ corresponds to some finite-dimensional system and $v(t)$ is a reference input. We would like to investigate the properties of the corresponding closed-loop system. We shall construct such a control by

(i) constructing a state estimator $\hat{x}(\cdot)$ from the observations $y(\cdot)$.

(ii) and by constructing an appropriate control as a linear function of the estimate $\hat{x}(t)$.

We first construct such an estimator in the form

$$\begin{cases} \dfrac{d\hat{x}}{dt} = A\hat{x}(t) + M\nu(t) + Bu(t), \\ \nu(t) = y(t) - C\hat{x}(t), \\ \text{and } M \text{ is to be chosen later.} \end{cases} \tag{2.26}$$

The vector $\nu(\cdot)$ has the interpretation as an "innovation" function, that is, $\nu(t)$ is the new information in the output $y(t)$ at time t not contained in $C\hat{x}(t)$. Let $e(t) = x(t) - \hat{x}(t)$ be the error between the state and its estimate. We see that

$$\frac{de}{dt} = (A - MC)e(t). \tag{2.27}$$

Now if we assume that (A, C) is an observable pair then the spectrum of $(A - MC)$ can be made arbitrary by suitable choice of M. In particular, by suitable choice of M, $(A - MC)$ can be made a stability matrix. Note that for the requirement of stability of the $(A - MC)$ it is enough to assume that (A, C) is a detectable pair.

Now consider a feedback control of the form

$$u(t) = -K\hat{x}(t), \tag{2.28}$$

giving us a closed-loop system,

$$\begin{aligned} \frac{dx}{dt} &= Ax(t) - BK\hat{x}(t) \\ &= (A - BK)x(t) + BKe(t). \end{aligned} \tag{2.29}$$

By the pole-assignment theorem, the spectrum of $(A - BK)$ can be made arbitrary by suitable choice of K, provided (A, B) is a controllable pair (stabilizability is enough).

Consider the pair of differential equations:

$$\begin{bmatrix} \dfrac{dx}{dt} \\ \\ \dfrac{de}{dt} \end{bmatrix} = \begin{bmatrix} A - BK & BK \\ \\ 0 & A - MC \end{bmatrix} \begin{bmatrix} x(t) \\ \\ e(t) \end{bmatrix}. \tag{2.30}$$

It is clear that the spectrum of the above block matrix is determined by that of $A - BK$ and $A - MC$ and hence can be made arbitrary by proper choice of K and M and in particular can be made a stability matrix. This is one of the fundamental results of linear feedback control.

We now turn to a discussion of quadratic-cost optimal control and show how this theory gives rise to optimal feedback controllers which render the closed-loop system asymptotically stable. It turns out that the theory of controllability and observability plays an important role in the study of optimal control over an infinite time horizon.

3 Optimal control

Consider the *linear control system*

$$\begin{cases} \dfrac{dx}{dt} = Ax(t) + Bu(t) \\ y(t) = Cx(t) \end{cases} \tag{3.1}$$

on the time interval $[s, T]$, where $0 < s < T$ with the initial condition $x(s) = x$; where x is arbitrary. The notation and terminology is the same as the ones defined in the section on controllability and observability.

We shall be primarily concerned with the following *optimal control problem*: to choose a control $\hat{u}(\cdot) \in L^2(s, T; \mathbf{R}^m)$ which minimizes the cost functional

$$J(u; s, x) = \frac{1}{2}\int_s^T [(u(t), Ru(t)) + (x(t), Qx(t)]\, dt + \frac{1}{2}(x(T), Sx(T)) \tag{3.2}$$

where $R = R^* > 0$, $Q = Q^* = C^*C \geq 0$ and $S = S^* \geq 0$. A control $\hat{u}(\cdot)$ minimizing $J(u; x)$ will be called an *optimal control*. We shall not be concerned with the question of existence and uniqueness of solutions but with the characterization of the optimal control $\hat{u}(\cdot)$ and the corresponding optimal trajectory $\hat{x}(\cdot)$.

Even though we are interested in solving the optimal control problem over a fixed interval $[0, T]$ and for the fixed initial condition x_0, it will turn out to be conceptually important to solve the problem for all initial points (s, x), $0 \leq s < T$.

We shall also be interested in the infinite time problem: find a control $\hat{u}(\cdot) \in L^2(s,\infty;\mathbf{R}^m)$ which minimizes:

$$J(u;s,x) = \frac{1}{2}\int_s^\infty [(u(t),Ru(t)) + (x(t),Qx(t))]\,dt \tag{3.3}$$

3.1 Optimal control in a finite time interval

We first discuss the finite-time situation. We are interested in obtaining the optimal control $\hat{u}(\cdot)$ in feedback form $\hat{u}(t) = K(t)x(t)$, where $K(t) \in \mathcal{L}(\mathbf{R}^n;\mathbf{R}^m)$. There are three basic approaches to the solution of this problem. The first is via the Calculus of Variations or equivalently the Maximum Principle and makes use of the adjoint equation. It can be shown that the optimal control is characterized by the solution of the following system:

$$\begin{cases} \dfrac{d\hat{x}}{dt} = A\hat{x}(t) + B\hat{u}(t) \\ \hat{x}(s) = x \end{cases} \tag{3.4}$$

$$\begin{cases} \dfrac{dp}{dt} = -A^*p(t) - Q\hat{x}(t) \\ p(T) = S\hat{x}(T) \end{cases} \tag{3.5}$$

$$R\hat{u}(t) + B^*p(t) = 0 \quad s \le t \le T. \tag{3.6}$$

We recognize this as a *two-point boundary value problem.* The system (3.4)–(3.6) can be decoupled as follows. It is here the idea of invariant embedding, namely, the idea of considering the optimal control problem for all initial (s,x) where $0 < s < T$ and $x \in \mathbf{R}^n$ arbitrary becomes important. One first shows that the system of equations

$$\begin{cases} \dfrac{d\xi}{dt} = A\xi(t) - BR^{-1}B^*\eta(t) \\ \xi(s) = x. \end{cases} \tag{3.7}$$

$$\begin{cases} \dfrac{d\eta}{dt} = -A^*\eta(t) - Q\xi(t) \\ \eta(T) = S\xi(T) \end{cases} \tag{3.8}$$

admits a unique solution. We can then prove that the mapping

$$x \mapsto (\xi(\cdot),\eta(\cdot)) : \mathbf{R}^n \to W(s,T;\mathbf{R}^n) \times W(s,T;\mathbf{R}^n) \tag{3.9}$$

where

$$W(s,T;\mathbf{R}^n) = \{z \,:\, z \in L^2(s,T;\mathbf{R}^n),\ \frac{dz}{dt} \in L^2(s,T;\mathbf{R}^n)\}$$

is an affine continuous map and finally that the mapping

$$x \mapsto \eta(s) : \mathbf{R}^n \to \mathbf{R}^n \text{ is an affine mapping.} \tag{3.10}$$

From this it follows that we can write

$$\eta(s) = P(s)x + r(s), \tag{3.11}$$

where

$$P(s) \in \mathcal{L}(\mathbf{R}^n;\mathbf{R}^n) \text{ and } r(s) \in \mathbf{R}^n. \tag{3.12}$$

Let $(x(\cdot),p(\cdot))$ be a solution of (3.4)–(3.6) in the interval $[0,T]$. Then

$$p(t) = P(t)x(t) + r(t), \quad \forall\, t \in [0,T],$$

and a calculation shows that $P(t)$ and $r(t)$ satisfy:

$$\begin{cases} \dfrac{dP}{dt} + A^*P(t) + P(t)A - P(t)BR^{-1}B^*P(t) + Q = 0 \\ P(T) = S \end{cases} \tag{3.13}$$

$$\begin{cases} \dfrac{dr}{dt} + A^*r(t) - P(t)BR^{-1}B^*r(t) = 0 \\ r(T) = 0. \end{cases} \tag{3.14}$$

Clearly $r(t)$ is identically zero and standard results of ordinary differential equations prove that (3.13) has a unique global solution.

Therefore we have obtained the optimal control in feedback from:

$$\hat{u}(t) = -R^{-1}B^*P(t)\hat{x}(t).$$

The same result can be obtained using the method of Dynamic Programming. Conceptually, the Calculus of Variations gives necessary conditions of optimality while the method of Dynamic Programming gives sufficient conditions of optimality. But for the strictly convex quadratic cost problem both methods give necessary and sufficient conditions of optimality.

Let

$$V(s,x) = \inf_u J(u;s,x).$$

Then an application of the Principle of Optimality leads to the following partial differential equation for V:

$$\begin{cases} \dfrac{\partial V}{\partial s}(s,x) + \min\limits_u \left[\dfrac{1}{2}(u(s), Ru(s)) + \dfrac{1}{2}(x(s), Qx(s)) \right. \\ \qquad\qquad \left. + (\nabla_x V(x,s), Ax(s) + Bu(s))\right] = 0 \\ V(T,x) = (Sx, x), \end{cases} \tag{3.15}$$

where $\nabla_x V(x,s)$ is the gradient of $V(x,s)$ with respect to the vector x.

Carrying out the minimization, we obtain

$$u(s) = -R^{-1}B^*\nabla_x V(x,s), \quad \forall\, s \in [0,T]. \tag{3.16}$$

Hence we obtain the so-called Bellman-Hamilton-Jacobi equation

$$\begin{aligned} &\frac{\partial V}{\partial s}(s,x) + \frac{1}{2}(\nabla_x V(x,s), BR^{-1}B^*\nabla_x V(x,s)) \\ &+\frac{1}{2}(x, Qx) + (A^*\nabla_x V(s,x), x) = 0 \end{aligned} \tag{3.17}$$

Now, equation (3.17) can be solved by looking for a solution of the form:

$$V(s,x) = \frac{1}{2}(x, P(s)x) + (x, r(s)). \tag{3.18}$$

We can check that $P(s)$ and $r(s)$ satisfies equations (3.13) and (3.14) previously obtained, and we recover the same results as obtained by Calculus of Variation arguments.

There is a third approach, in some sense related to the Dynamic Programming approach to the solution of quadratic cost optimal control problems. This is the method of completing the squares. The main idea here is to observe that if

$$\frac{dx}{dt} = Ax(t) + Bu(t),$$

and if $P(t) = P^*(t)$ be such that

$$\frac{dP}{dt}$$

exists in the interval $[s,T]$, then defining

$$z(t) = \begin{bmatrix} u(t) \\ x(t) \end{bmatrix} \text{ and } L(t) = \begin{bmatrix} 0 & R^{-1}B^*P(t) \\ P(t)B & \dfrac{dP}{dt} + A^*P(t) + P(t)A \end{bmatrix}$$

we get

$$\int_s^T (z(t), L(t)z(t))dt - (x(T), P(T)x(T)) + (x(s), P(s)x(s)) = 0 \tag{3.19}$$

Now let $P(t)$ be the global solution on the interval $[s,T]$ of the equation

$$\begin{cases} \dfrac{dP}{dt} + A^*P(t) + P(t)A - P(t)BR^{-1}B^*P(t) + Q = 0 \\ P(T) = S. \end{cases} \tag{3.20}$$

Then adding the identity (3.20) to $J(u;s,x)$ and doing some algebraic manipulations, we obtain:

$$J(u;s,x) = \int_s^T |u(t) + R^{-1}B^*P(t)x(t)|^2 dt + (x(s), P(s)x(s)). \tag{3.21}$$

Hence the optimal control is given by

$$\hat{u}(t) = -R^{-1}B^*P(t)x(t), \tag{3.22}$$

and the optimal cost $V(s,x)$ by

$$V(s,x) = (x, P(s)x). \tag{3.23}$$

In this book, we use the latter two methods in infinite dimensional situations to obtain results on optimal control for the quadratic cost problem in a finite-time interval. Apparently, one obtains more general results using this methodology.

Part II, Chapter 1 to Chapter 3 develop the optimal control of parabolic, hyperbolic and delay equations over a finite time interval. The observation operator is usually considered to be also bounded but the case of unbounded observation operators is also treated. Chapter 2 deals with parabolic equations with unbounded control operators while Chapter 3 deals with hyperbolic equations with unbounded control operators. The latter two chapters are the most technical and the full power of the theory developed in Volume I needs to be used.

3.2 Optimal control over an infinite time interval

The study of the quadratic cost problem over a finite and an infinite time interval is essentially a study of the Riccati differential equation over the finite time interval $[s,T]$

$$\frac{dP}{dt} + A^*P(t) + P(t)A - P(t)BR^{-1}B^*P(t) + C^*C = 0 \qquad (3.24)$$

subject to the terminal condition

$$P(T) = P_T \geq 0, \qquad (3.25)$$

its asymptotic behaviour and the corresponding matrix quadratic equation

$$A^*P + PA - PBR^{-1}B^*P + C^*C = 0. \qquad (3.26)$$

The following theorem is proved (in certain infinite dimensional situations) in Part II, Chapter 1, §2 and Part III, Chapter 1, §2 to 4.

Theorem 3.1 *There exists a matrix $P(t)$ satisfying the following:*
(i) $P(\cdot)$ is defined and belongs to $C^1(s,T;\mathcal{L}(\mathbf{R}^n))$ and satisfies (3.24) *and* (3.25).
(ii) $P(t) \geq 0, s \leq t \leq T$ and is the unique solution of (3.24)–(3.25).
(iii) Let $\tilde{K}(t)$ be a continuous function on $[s,T]$ and let $\tilde{P}(t)$ be the solution of the linear differential equation

$$\frac{d\tilde{P}}{dt} + (A - B\tilde{K}(t))^*\tilde{P}(t) + \tilde{P}(t)(A - B\tilde{K}(t)) + C^*C + \tilde{K}(t)^*R\tilde{K}(t) = 0 \qquad (3.27)$$

$$\tilde{P}(T) = P_T \geq 0. \qquad (3.28)$$

If $P(t)$ is a solution (3.24) *and* (3.25), *then*

$$P(t) \leq \tilde{P}(t),\ s \leq t \leq T.$$

(iv) Consider the equation (3.24) *with $s \to -\infty$ and $T = 0$. If (A,B) is stabilizable, then $P(t)$ is bounded on $]-\infty,0]$. If (A,C) is detectable then*

$$P_\infty = \lim_{t\to-\infty} P(t)$$

exists and is positive semi-definite.

In that case P_∞ is the unique positive semi-definite solution of (3.26) and

$$A - BR^{-1}B^*P_\infty$$

is a stability matrix.

The optimal control over the infinite time interval is then given by

$$\hat{u}(t) = -R^{-1}B^*P_\infty\hat{x}(t). \qquad (3.29)$$

The study of these control problems when control is exercised through the boundary is most complicated for hyperbolic equations and is taken up in Chapter 3 of Part III. The study of exact controllability comes into the picture here to ensure that the space of admissible controls is nonempty.

4 A glimpse into H^∞-theory: state feedback case

4.1 Introduction

In many control problems the quadratic criterion is not the most appropriate. We consider a *disturbance attenuation problem* when the full state vector can be measured and introduce the so-called $\mathcal{H}^\infty$-optimal control problem. There is now a vast literature on this topic. For two textbook presentations the reader is referred to T. Basar and P. Bernhard [1] and B.A. Francis [1].

Consider the finite-dimensional linear time-invariant system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Lw(t) + Bu(t) \\ z(t) &= Cx(t) + Du(t), \end{aligned} \tag{4.1}$$

where u is interpreted as the controls and z as controlled outputs which are to be made small in the presence of exogenous disturbances w. The matrices and vectors have appropriate dimensions.

Let $S(A, B)$ denote the set of all constant gain stabilizing state feedback matrices. That is

$$S(A, B) = \{K : \operatorname{Re}[\lambda_i(A + BK)] < 0\}.$$

Then for $u \in S(A, B)$, the closed loop dynamics take the form

$$\dot{x}(t) = (A + BK)x(t) + Lw(t) \tag{4.2}$$
$$z(t) = (C + DK)x(t). \tag{4.3}$$

or in transfer function form

$$z(s) = T_K(s)w(s)$$

where

$$T_K(s) = (C + DK)(sI - (A + BK))^{-1}L.$$

The problem of $\mathcal{H}^\infty$-optimal state feedback is now stated as

$$\inf_{K \in S(A,B)} ||T_K||_{\mathcal{H}^\infty},$$

where

$$||T_K||_{\mathcal{H}_\infty} = \sup_{\omega} \sigma_{\max} ||T(j\omega)||$$

and $\sigma_{\max}$ denotes the maximum singular value of the matrix $T(j\omega)$.

This $\mathcal{H}^\infty$-optimal state feedback problem represents a special case of the more general $\mathcal{H}^\infty$- optimal output feedback disturbance rejection problem.

4.2 Main results

The following assumptions are made on (4.1)
(A1) The pair (A, B) is stabilizable,
(A2) The pair (A, C) is observable, and
(A3) $D^*[C\ D] = [0\ I]$.

(A1) is necessary for the stabilization of (4.1). Assumption (A2) is a technical assumption which guarantees the invertibility of certain algebraic Riccati equation solutions. The orthogonality assumption (A3) is analogous to no cross weighting between the state and control in the standard LQ problem. In case (A3) is violated, the change of control variables from u to v given by

$$u(t) = (D^*D)^{-\frac{1}{2}}v(t) - (D^*D)^{-1}Cx(t)$$

yields a new system which satisfies (A3). Furthermore, it can be shown that a controlled output of the form

$$z(t) = Cx(t) + Du(t) + Eu(t)$$

may be transformed to the form of (4.1). Thus, (A3) is made with minimal loss of generality.

The main result is now stated.

Theorem 4.1 *Consider the linear system* (4.1) *under assumptions* (A1) *to* (A3). *Under these conditions, there exists a* $K \in S(A, B)$ *such that* $||T_K||_{\mathcal{H}^\infty} < \gamma$ *if and only if there exists an* $X = X^* > 0$ *which satisfies*

$$XA + A^*X + C^*C + X(\frac{1}{\gamma^2}LL^* - BB^*)X = 0 \tag{4.4}$$

with $A + (\frac{1}{\gamma^2}LL^* - BB^*)X$ *stable (i.e. all eigenvalues in the open left half complex plane).*

The remainder of this section is devoted to the proof of Theorem 4.1 for $\gamma = 1$. By linearity, this simplification is made without loss of generality.

First, some preliminary results from linear quadratic optimization theory are stated for the linear system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \quad x(0) = x_0 \\ y(t) &= Cx(t) \end{aligned}$$

Lemma 4.1 *Let (A, B) be stabilizable and (A, C) detectable (resp. observable). Then there exists a unique $P = P^* \geq 0$ (resp. > 0) which satisfies*

$$PA + A^*P + C^*C - PBB^*P = 0$$

with $A - BB^*P$ stable. Furthermore, the state feedback control law $u(t) = -B^*Px(t)$ is a minimizing solution for the problem

$$\inf_u \frac{1}{2}\int_0^\infty (|y(t)|^2 + |u(t)|^2)\, dt.$$

This is Theorem 3.1 restated.

Lemma 4.2 *Let $G(s) = C(sI-A)^{-1}B$ with A stable. Then $||G||_\infty < 1$ if and only if there exists an $X = X^* \geq 0$ which satisfies*

$$XA + A^*X + C^*C + XBB^*X = 0$$

with $A + BB^*X$ stable. Furthermore, the state feedback $u(t) = BB^*Xx(t)$ solves

$$\sup_u \frac{1}{2}\int_0^\infty (|y(t)|^2 - |u(t)|^2)\, dt.$$

For a proof of the above lemma see J.C. Willems [1].

The proof of Theorem 4.1 is now presented. To prove the "if" direction, let $X = X^* > 0$ satisfies (4.4) with $A + (LL^* - BB^*)X$ stable. A manipulation of (4.4) yields

$$\begin{aligned} &X(A - BB^*X) + (A - BB^*X)^*X \\ &+(C - DB^*X)^*(C - DB^*X) + XLL^*X = 0. \end{aligned} \tag{4.5}$$

Now assumptions (A2) to (A3) imply that the pair $(A - BB^*X, C - DB^*X)$ is observable. This observability, (4.5), and $X > 0$ together imply that the matrix $A - BB^*X$ is stable via standard Lyapunov stability theory. Thus, $K = -B^*X \in S(A, B)$. Finally, using (4.5) and the stability of $A + (LL^* - BB^*)X$ in Lemma 4.2 implies that $||T_K||_{\mathcal{H}^\infty} < 1$, which is the desired result.

To prove the "only if" direction requires the following preliminary result:

Lemma 4.3 *There exists an $X = X^* > 0$ which satisfies*

$$XA + A^*X + C^*C + X(LL^* - BB^*)X = 0 \tag{4.6}$$

with $A+(LL^*-BB^*)X$ stable if and only if there exists a $P = P^* > 0$ which satisfies

$$AP + PA^* + PC^*CP + LL^* - BB^* = 0 \tag{4.7}$$

with $-A^* - C^*CP$ stable.

Proof. Let $P = X^{-1}$. This establishes an equivalence of the existence of positive definite solutions to either (4.6) or (4.7). To show equivalence of the stability conditions, a manipulation of either (4.6) or (4.7) yields the similarity condition

$$A + (LL^* - BB^*)X = P(-A^* - C^*CP)P^{-1}.$$

■

Thus to prove the "only if" portion of Theorem 4.1, it suffices to prove the equivalent condition associated with (4.7).

Towards this end, let $K \in S(A, B)$ be such that $||T_K||_{\mathcal{H}^\infty} < 1$. Then from Lemma 4.2, there exists an $\tilde{X} = \tilde{X}^* \geq 0$ which satisfies

$$\tilde{X}(A + BK) + (A + BK)^*\tilde{X} + (C + DK)^*(C + DK) + \tilde{X}LL^*\tilde{X} = 0 \tag{4.8}$$

with $A + BK + LL^*\tilde{X}$ stable. A straightforward manipulation of (4.8) yields

$$\begin{aligned} &\tilde{X}A + A^*\tilde{X} + C^*C + \tilde{X}(LL^* - BB^*)\tilde{X} \\ &= -(K + B^*\tilde{X})^*(K + B^*\tilde{X}) \end{aligned} \tag{4.9}$$

Let $\tilde{P} = \tilde{X}^{-1}$. Note that assumptions (A2)–(A3) imply observability of the pair $(A + BK, C + DK)$ which in turn implies $\tilde{X} > 0$ thus allowing the inversion of $\tilde{X}$. In terms of $\tilde{P}$, (4.9) now takes the form

$$A\tilde{P} + \tilde{P}A^* + \tilde{P}C^*C\tilde{P} + LL^* - BB^* = -\tilde{V}^*\tilde{V} \tag{4.10}$$

where

$$\tilde{V} = K\tilde{P} + B^*.$$

Now (4.10) is not quite in the desired form of (4.7). Thus, as in J.C. WILLEMS [1], let $\Delta P = P - \tilde{P}$. Then subtracting (4.10) from (4.7) yields (after some straightforward manipulations)

$$(-\tilde{A})^*\Delta P + \Delta P(-\tilde{A}) + \tilde{V}^*\tilde{V} - \Delta P C^* C \Delta P = 0 \qquad (4.11)$$

where $\tilde{A} = A^* + C^*C\tilde{P}$. Clearly, if one can find a symmetric positive semidefinite solution, ΔP, to (4.11), then $P = \tilde{P} + \Delta P$ satisfies the desired (4.7). However, (4.7) is of the form of a standard LQ Riccati equation as in Lemma 4.1. The requisite stabilizability and detectability in Lemma 4.1 is now shown as follows:

(1) The matrix pair $(-\tilde{A}, C^*)$ is controllable. This follows directly from assumption (A2) and that $-\tilde{A} = -A^* - C^*C\tilde{P}$.

(2) The matrix pair $(-\tilde{A}, \tilde{V})$ is detectable. To see this, a straightforward manipulation of (4.10) yields the similarity condition

$$-\tilde{A} - K^*\tilde{V} = \tilde{P}(A + BK + LL^*\tilde{X})\tilde{P}^{-1}.$$

However, recall that $\tilde{X}$ is such that $A + BK + LL^*\tilde{X}$ is stable, thus proving the desired detectability.

These observations along with Lemma 4.1 guarantees the existence and uniqueness of a positive definite ΔP which satisfies (4.11). Thus, $P = \tilde{P} + \Delta P > 0$ and satisfies (4.7).

It then remains to be shown that $-A^* - C^*CP$ is stable. However, from Lemma 4.1, one has that $-\tilde{A} - C^*C\Delta P$ is stable. But

$$-\tilde{A} - C^*C\Delta P = -A^* - C^*CP$$

which, via Lemma 4.3, completes the proof.

It is worthwhile remarking that the present proof exploits the fact that the condition $||T_K|| \leq 1$ is a characterization of dissipativity in terms of scattering variables. It is well known that the dissipative property in turn can be characterized in terms of a quadratic Lyapunov function obtained from an indefinite quadratic cost problem (cf. Lemma 4.2). The exposition here essentially exploits these ideas to obtain the desired result.

It should be remarked that the state space theory of $\mathcal{H}^\infty$ should only be considered as a computational device and not as a substitute for the operator theory solution of $\mathcal{H}^\infty$-problems. The important questions of approximation, and robustness should be treated from the input-output viewpoint and it is unclear whether it is meaningful in the state-space description. Finally, the characterization of optimality in the state-space framework is apparently an unsolved problem.

5 Final remarks

If one were to write a Volume III it should be concerned with a generalization of the following finite dimensional problem to infinite dimensions:

$$\inf_u \int_s^T w(x(t), u(t))\, dt \qquad (5.1)$$

where $w(x,u) = \frac{1}{2}[(x, Qx) + (u, Ru) + 2(u, \mathbf{C}x)]$ and $Q = Q^*$ but without any definiteness condition, $R = R^* \geq 0$, subject to the dynamical constraint

$$\begin{cases} \dfrac{dx}{dt} = Ax(t) + Bu(t) \\ x(s) = x \end{cases} \qquad (5.2)$$

and the related infinite time problem

$$\inf_u \int_s^\infty w(x(t), u(t))\, dt \qquad (5.3)$$

with the dynamical constraint (5.2) and final value conditions $x(\infty) = 0$ or $x(\infty) = \text{free}$.

The paper of J.C. WILLEMS [1] is the subject of this problem in finite dimensions. A variety of problems such as linear quadratic differential games, the KALMAN–YACUBOVICH–POPOV lemma of stability theory and H^∞-theory (in state space form) is related to this topic. Undoubtedly a large part of Willem's results can be generalized to infinite dimensions using the ideas presented in this work.

Notes

The results of this chapter are for the most part classical. The concepts of controllability and observability were introduced by R.E. KALMAN [1], [2] where the criteria of controllability and observability and the structure theorem were developed. In this connection see also the early paper of E. GILBERT[1]. The minimum energy control for transfer from the origin to a final state was first discussed in R.E. KALMAN, Y.C. HO and K.S. NARENDRA [1]. For a text-book study of these questions as well as a treatment of stability theory, see R.W. BROCKETT [1]. The concepts of stabilizability and detectability were introduced by Wonham in his study of the Algebraic Riccati equation (cf. (W.M. WONHAM [1]). The criteria for stabilizability and detectabiity are due to M.L.J. HAUTUS [1]. The pole-assignment theorem in the real case was proved by W.M. WONHAM [2] and independently by J.D. SIMON and S.K. MITTER [1]. The proof presented here follows W.M. WONHAM [3]. Theorem 2.6 is due to J. ZABCZYK [1]. The idea of a state estimator was first introduced by D.G. LUENBERGER [1]. For the treatment of compensators for linear systems as estimator-controller and the use of the pole-assigment theorem in this context, see J.D. SIMON and S.K. MITTER [1]. For a discussion of the quadratic cost optimal control problem see the book of R.W. BROCKETT [1] where the completing the square argument is developed. The idea of invariant embedding used in §3.1 is due to R.E. BELLMAN [1]. Its use in decoupling the Hamiltonian equations in the form described here is due to J.-L. LIONS [4]. It was formally used in the same way by S.K. MITTER [1]. Theorem 3.1 is due to W.M. WONHAM [2] (under the slightly stronger hypothesis of observability). The results of §4 follows an unpublished manuscript of S.K. MITTER and J.S. SHAMMA [1] but are essentially an application of the results given in the paper of J.C. WILLEMS [1]. There is now a vast literature on the so-called H^∞-approach to the control of linear systems. For two textbook presentations see T. BASAR and P. BERNHARD [1] and B.A. FRANCIS [1].

Chapter 2

Controllability and Observability for a Class of Infinite Dimensional Systems

1 Introduction

In §2.1 and §2.2 of Chapter 1 we have discussed criteria for controllability and observability for finite dimensional systems and have also shown that when the system is controllable we can transfer the state $z_0 \in H$ at time t_0 to the state $z_1 \in H$ at time t_1 using minimum energy controls. These results were obtained by considering the controllability operator

$$\begin{aligned} L_T &: L^2(0,T;U) \to H \\ &: u \longmapsto \int_0^T e^{(T-s)A} Bu(s)ds, \end{aligned}$$

and its adjoint

$$\begin{aligned} L_T^* &: H \to L^2(0,T;U) \\ &: y \longmapsto B^* e^{(T-\cdot)A^*} y, \end{aligned}$$

and studying the relation between the ranges and null spaces of these two operators and by showing that controllability is equiva-