6.976/ESD.937 Quantitative Foundations of Engineering Systems

Homework 3

Date Handed Out: April 12, 2006 *Due Date: Before Class, April 24, 2006*

1. (Max-Min Fair Allocation)[50 pts] Consider a link of capacity $C = 100$ units, which is shared by n flows with demands $d_1, \ldots, d_n$. An allocation of capacity to flows, $\mathbf{f} = (f_1, \ldots, f_n)$ is called *feasible* if

$$\begin{aligned} 0 &\leq f_i \leq d_i, \text{ for } i = 1, \ldots n, \\ \sum_{i=1}^{n} f_i &\leq C. \end{aligned}$$

We wish to allocate the link capacity in a *Max-Min Fair* manner. Recall that a flow allocation $\mathbf{f}^* = (f_1^*, \ldots, f_n^*)$ is called Max-Min Fair if the following condition is satisfied: for any feasible flow allocation $\mathbf{f} = (f_1, \ldots, f_n)$ such that $\mathbf{f} \neq \mathbf{f}^*$ if there exists m such that $f_m > f_m^*$ then there must exist p such that $f_p < f_p^* \leq f_m^*$.

Given the above definition, consider the following heuristic to find *Max-Min Fair* allocation starting from any feasible flow allocation.

1. Let $\mathbf{f}(0)$ be initial feasible flow allocation and $k = 0$. First, do the following:
   (a) If $\sum_{i=1}^{n} d_i \leq C$, set $\mathbf{f}(1) = (d_1, \ldots, d_n)$, $k = 1$ and Go to (4).
   (b) Else, increase the components of $\mathbf{f}(0)$ (in any manner) to obtain $\mathbf{f}(1)$, which is feasible as well as $\sum_{i=1}^{n} \mathbf{f}(1)_i = C$. Go to (3).
2. In iteration k do the following:
   (a) If there exists any pair (m, p) such that $\mathbf{f}(k)_m > \mathbf{f}(k)_p$ and $\mathbf{f}(k)_p < d_p$ then set
   $$\mathbf{f}(k+1)_p = \min\left(d_p, \frac{\mathbf{f}(k)_m + \mathbf{f}(k)_p}{2}\right);\ \mathbf{f}(k+1)_m = \mathbf{f}(k)_m - (\mathbf{f}(k+1)_p - \mathbf{f}(k)_p)\,.$$
   (b) Else, Go to (4).
3. Set $k = k + 1$ and repeat from (2).
4. Declare the current assignment $\mathbf{f}(k)$ as Max-Min Fair allocation.

Answer the following questions.

(i) Let $n = 3$, $d_1 = 10, d_2 = 20, d_3 = 110$, $C = 100$ and $\mathbf{f}(0) = (0, 0, 100)$. Prove or disprove that the above described heuristic converges to the Max-Min Fair allocation.

(ii) Do you think the above described heuristic always finds Max-Min Fair allocation? Explain your answer.

2. (Optimization and Network Algorithms)[50 pts] Consider a single link accessed by n. Let $x_i(t), 1 \leq i \leq n$, be the rate of user i at time t. Let $p(t)$ be the price charged by network to use the link. Suppose the users and network update their rates and price respectively as follows:

$$\frac{dx_i(t)}{dt} = \left(\frac{1}{x_i(t)} - p(t) \right),$$

$$\frac{dp(t)}{dt} = \begin{cases} \max(0, \sum_{i=1}^{n} x_i(t) - C) & \text{if } p(t) = 0 \\ \sum_{i=1}^{n} x_i(t) - C & \text{Otherwise} \end{cases}$$

Answer the following questions.

(o) Initially, let $t = 0$, $x_i(0) = 0$ and $p(0) = 0$. Do you think $\sum_i x_i(t)$ will ever become equal to C for any t. If yes, what is the first time when $\sum_i x_i(t) = C$?

(i) Identify the above update rules with appropriate Primal and Dual algorithm of an optimization problem. Explain the optimization problem as resource allocation problem.

(ii) Can you suggest simple window-based rate-control mechanism and queue-drop mechanism that correspond to the above update rules?