6.976/ESD.937 Quantitative Foundations of Engineering Systems

Homework 4

Date Handed Out: April 24, 2006 *Due Date: Before Class, May 3, 2006*

1. (Stability) [65 pts] Consider the differential equation

$$\dot{x} = (\sin \log t + \cos \log t - a)x, t \geq 1 \quad (1)$$

Here, a is a constant. Note that $x = 0$ is an equilibrium point and it's general solution for $t \geq t_0 \geq 1$ is

$$x(t) = x(t, t_0, x_0) = x_0 \exp[t(\sin \log t - a) - t_0(\sin \log t_0 - a)] \quad (2)$$

Show that

(i) If $a < 1$, $x = 0$ is unstable.

(ii) If $a = 1$, $x = 0$ is stable but not uniformly stable. Also, in this case, show that it is not asymptotically stable.

(iii) If $1 < a < \sqrt{2}$, $x = 0$ is asymptotically stable but not uniformly stable or uniformly asymptotically stable.

(iv) If $a = \sqrt{2}$, $x = 0$ is asymptotically stable. Though it is uniformly stable, it is not uniformly asymptotically stable.

(v) If $a > \sqrt{2}$, $x = 0$ is uniformly asymptotically stable.

2. (Physical Lyapunov Function) [35 pts] Lyapunov functions naturally arise as some *physical* characterization of system. This question will help understand this via two examples.

(a). Consider a system whose state is $x(t) \in \mathbf{R}$. The system dynamics is given by the following equation:

$$\frac{d^2x(t)}{dt^2} = -x(t). \quad (3)$$

Answer the following questions.

(i) Write this system in the form $\frac{dy(t)}{dt} = Ay$, where A is a 2×2 matrix. (*Hint:* Use $y(t) = \frac{dx(t)}{dt}$.)

(ii) Interpret this system as a physical system in either of the following two ways: (1) a mass-spring system, or (2) a capacitor-inductor-resistor circuit. (*Hint:* Do you remember Newtonian Mechanichs? Or, Basic Electrical Circuits?) Also, there might be other ways too, of making a physical system, and if you know of any such way, you can use that too.

(iii) Consider the energy function, and prove that it is a Lyapunov function. Use this Lyapunov function to identify the stability properties of the system, i.e. among stability, asymptotic stability, uniform stability and uniform asymptotic stability, which does it satisfy, and which does it not satisfy?

(iv) How can you infer the stability/instability properties of the system from matrix A? (*Hint:* Eigenvalues ?)

(b). Next, consider another similar system with state $x(t) \in \mathbf{R}$.

$$\frac{d^2x(t)}{dt^2} + 2\frac{dx(t)}{dt} + x(t) = 0. \quad (4)$$

Now answer the following questions.

(v) As in (i), write this system in the form $\frac{dy(t)}{dt} = Ay(t)$, where A is a 2×2 matrix.

(vi) Interpret this system as a physical system. Two possible ways in which this can be done is by considering a mass-spring-damper system or a capacitor-inductor-damper circuit.

(vii) Again, consider the energy function, and prove that it is a Lyapunov function. What kind of stability properties does this system have ?

(viii) How can you infer the stability/instability properties of the system from matrix A?