Routing

- Topics:
  - Definition
  - Architecture for routing
    - data plane algorithm
  - Current routing algorithm
    - control plane algorithm
  - Optimal routing algorithm
    - known algorithms and implementation issues
    - new solution
  - Routing mis-behavior: selfish routing
    - price of anarchy or how bad can it be?

Routing

- Definition
  - The task of determining how data should travel from its source to destination in the network so that
    - overall delay is minimized (or user utility is maximized), and
    - network resource utilization is maximized
- An example
  - Use of "MapQuest.com" to find "driving directions"
    - data plane algorithm
- Ideally, MapQuest.com should be designed so that
  - Traffic directions are created such that overall congestion on roads is minimized
  - Overall social satisfaction by using such directions is maximized
    - control plane algorithm

Routing

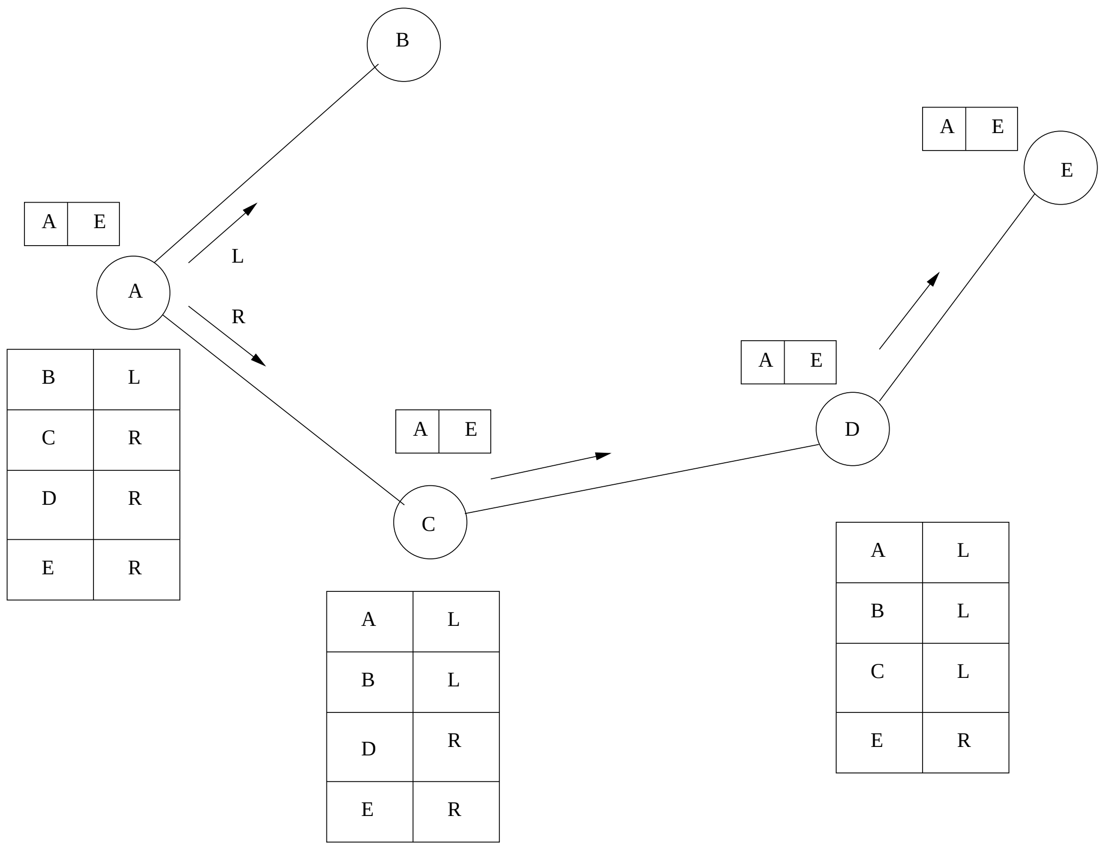
- Essential requirements for routing in a networks
    - Unique addressing mechanism in the network,
        - such as, street address
    - Network topology,
        - for example, road map
    - Or, at least local directional information,
        - e.g. information of the form: *left on Vassar St. from Main St. will connect it to Mass Ave*

- Next, we'll consider Internet

Routing: Internet

- Internet
  - All nodes have unique IP address
    - allocated according to certain criteria, such as
    - all MIT IP address are of the form 128.31.*.*
  - The topological information is stored in the form of "next-hop" information
    - routing-tables stored in local gateways
  - When data needs to be routed from source to destination,
    - "network layer" takes care of routing
    - converts web address info IP, e.g
    - www.yahoo.com → 142.226.51
    - appends destination IP to packet
    - determines next-hop using routing information of gateway
    - intermediate gateways forward packet using it's destination IP and their routing table

Example: Routing in Internet

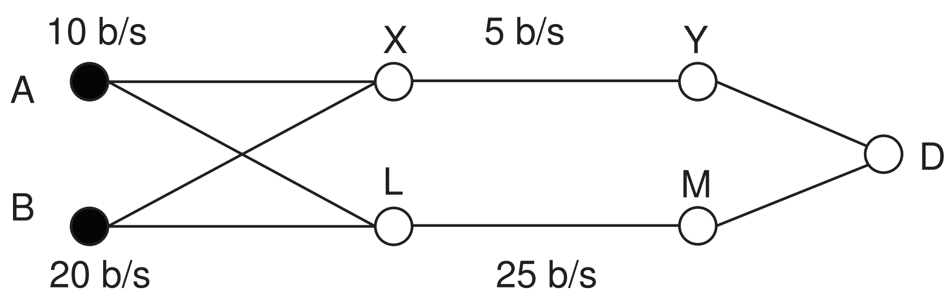


An example of data routing in Internet using routing table informtion

Q: How to configure these routing tables?

Routing: Control Algorithm

- The control part of routing is about configuring routing tables
- Given a network, traffic demands the routing determines
  - Network utilizatiton or throughput
  - Average delay or utility
- Example



Route 1:

A sends all data to D via X

B sends all data to D via L

Thput $= \frac{5+20}{10+20} = \frac{25}{30}$

Route 2:

A sends half-data to D via X
and other half via L

B sends all data to D via L

Thput $= 1$

Routing: Control Algorithm

- Formally, the problem of routing is as follows:
  - Each gateway needs to decide what fraction of data destined for certain destination needs to go through which of its outgoing link
  - So as to maximize overall network through put and minimized end-to-end delay
  - The parameters of this problem are:
    - network topology and link capacities
    - traffic demand
  - Constraints: decision of routing must be done in distributed manner and should be robust against few failures

- Next, we'll see how to model the problem of routing

  → It will lead to appropriate algorithm design

Routing: Problem Formulation

- Let nodes of network be numbered $1, \ldots, n$.
- Let $\mathcal{L} = \{(i,j) : 1 \leq i,j \leq n\}$ be set of all links
  - C_{ij} be capacity of link (i,j)
  - $C_{ij} = 0$, if link (i,j) is not present
- Let $r_i(j)$ be rate at which data is generated at node i for node j.
- Let $\phi_{\ell i}(j)$ be fraction of data arriving at node ℓ, destined for node j, that is routed to node i.
- Let $t_i(j)$ be net data arriving at node i destined for j. These satisfy the following relation

$$t_i(j) = r_i(j) + \sum_{(\ell,i)\in\mathcal{L}} t_\ell(j)\phi_{\ell i}(j) \; ; \quad \text{for all } i,j$$

Routing: Problem Formulation

- Let $F_{i\ell}$ rate at link $(i, \ell) \in \mathcal{L}$

$$F_{i\ell} = \sum_k t_k(k)\phi_{i\ell}(k)$$

- $D_{i\ell}(F_{i\ell})$: delay as func. of $F_{i\ell}$
  - Let it be convex, increasing, twice-differentiable
  - $D_{i\ell}(0) = 0 \ ; \quad D_{i\ell}(C_{i\ell}^{+}) = \infty$

- In general, the delay can be replaced by appropriate utility function

- Given this setup, next we describe the question of optimal routing

Optimal Routing

Route-opt:

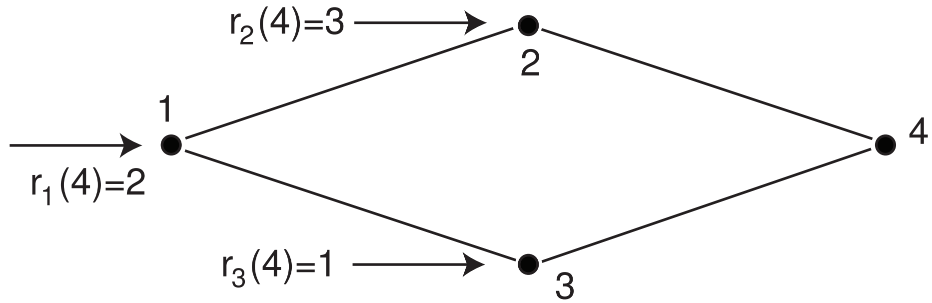
$$\min \sum_{(i,\ell)\in\mathcal{L}} D_{i\ell}(F_{i\ell})$$

subject to

$$\begin{aligned} F_{i\ell} &= \sum_k t_k(k)\phi_{i\ell}(k) \ ; \\ t_k(j) &= r_k(j) + \sum_\ell t_\ell(j)\phi_{\ell i}(j) \ ; \\ \phi_{\ell i}(j) &\geq 0 \ , \quad \text{for all } i, j, \ell \ ; \\ \sum_i \phi_{\ell i}(j) &= 1 \ , \quad \text{for all } \ell, j \ . \end{aligned}$$

- An implicit constraint is $F_{ij} < C_{ij}$, as otherwise optimal cost will be ∞.
  - If $r_i(j)$ are s.t. they can not be routed, then optimal cost will be ∞
- Before considering algorithms for Route-opt, let's look at an example.

Example: Route-opt



- Capacity of all links $= 5$
- Delay function for all links, $D(x) = \frac{x}{5-x}$
- A feasible routing:
  - $\phi_{12}(4) = 0.5$; $\phi_{13}(4) = 0.5$; $\phi_{34}(4) = 1$; $\phi_{24}(4) = 1$
  - $t_1(4) = 2$; $t_2(4) = 3 + 0.5 \times 2 = 4$; $t_3(4) = 1 + 0.5 \times 2 = 2$
  - $F_{12} = 1$; $F_{13} = 1$; $F_{24} = 4$; $F_{34} = 2$
  - Total delay

$$\frac{1}{5-1} + \frac{1}{5-1} + \frac{4}{5-4} + \frac{2}{5-2} \approx 5.17$$

- Next, some algorithms

A Natural Heuristic

- Consider following heuristic for Route-opt
  - Assign weights to edges, were weight reflects delay
  - An example, $\text{weight} = \frac{1}{\text{capacity}}$.
- Then, route with minimal delay between a pair of nodes corresponds to minimum weighted path (shortest path).
  - Routing algorithm is equivalent to finding shortest path between node-pairs
- Currently in the Internet
  - (A version of) Shortest path routing (OSPF) is used
  - Weights are based on certain heuristic utilizing observed link quality
- Next, we describe algorithms for finding shortest path

Dijkstra's Algorithm

- Main idea:
  - itreatively find shortest path between nodes
    - with paths of increasing length, starting with 1
- **Algorithm**: Find shortest path from node 1 to all nodes

1. Initially, $P = \{1\}$, $D_1 = 0$, $D_j = d_{ij}$, $j \neq 1$
   [d_{ij}: weight of edge (i, j); $d_{ij} = \infty$ if not connected]
2. Find next closest node: find $i \notin P$ s.t.
   $$D_i = \min_{j \notin P} D_j \ ;$$
   Set $P = P \cup \{i\}$. If P contains all nodes, then STOP.
3. Update: For all $j \notin P$, set
   $$D_j = \min[D_j, d_{ji} + D_i]$$
4. Go to (2)

Dijkstra's Algorithm

- The Dijkstra's algorithm is totally distributed
  - It can also be implemented in parallel and
  - Does not require synchronization

- In the algorithm
  - D_j can be thought of as estimate of shortest path length between 1 and j during the course of algorithm

- The algorithm is one of the earliest example of *graph algorithms*
  - Reference: Chapter 5.2, Bertsekas and Gallager

- Next, we present the proof of correctness of algorithm

Dijkstra's Algorithm

- We first state the following two properties of the Algorithm
  - **Claim 1.** $D_i \leq D_j \quad \forall i \in P \; ; \quad j \notin P$
  - **Claim 2.** D_j is, for each j, the shortest distance between j and 1, using paths whose nodes all belong to P (except, possibly, j)
- Given the above two properties
  - When algorithm stops, the shortest path lengths must be equal to D_j, for all j

  $\rightarrow$ That is, algorithm finds the shortest path as desired
- Next, we prove these two claims.
- **Proof of Claim 1**
  - The proof follows by simple Induction
    - initially Claim 1 is true, and
    - always remains true under the update rule

Dijkstra's Algorithm

- **Proof of Claim 2**
  - ◦ We will prove by Induction
  - ◦ Initally, $P = \{1\}$ and holds for $D_1 = 0$.
  - ◦ Induction hypothesis
    - – let it be true for all nodes till some interation
  - ◦ Next, node i is added to P
    - – let D_k be distances of nodes before i was added
    - – let D'_k be distances of nodes after i was added
  - ◦ Note that Claim 2 holds for all $j \in P$ from induction hypothesis as their $D_j = D'_j$.
  - ◦ For $j = i$, $D_i = D'_i$ satisfies the desired claim from induction hypothesis as well.

Dijkstra's Algorithm

- Let $j \notin P \cup \{i\}$. Let $\hat{D}_j$ be shortest distance from j to 1 along path containing nodes in $P \cup \{i\}$.
  - Let this path have arc (j, k); $k \in P \cup \{i\}$
  - Then,
  $$\begin{aligned} \hat{D}_j &= \min_{k \in P \cup \{i\}} [d_{jk} + D_k] \quad \text{[induction hypothesis]} \\ &= \min\left[\min_{k \in P}[d_{jk} + D_k], d_{ji} + D_i\right] \end{aligned}$$
  - By induction hypothesis, $D_j = \min_{k \in P}[d_{jk} + D_k]$
  - Hence,
  $$\begin{aligned} D_j &= \min[D_j, d_{ji} + D_i] \\ &= D'_j (\text{by update of algorithm}) \end{aligned}$$
- This completes the proof of Claim 2.

Other Algorithms

- Another popular algorithm
  - Bellman–Ford algorithm based on standard "Dynamic Programming"
- The Dijkstra's algorithm has very efficient distributed implementation

  → Hence, popular

- Current Internet protocols use some version of Dijkstra's algorithm
- Unfortunately, this algorithm does not perform very well
  - Easy to construct examples
  - Poor performance is often experienced in practice

Optimal Routing

- Why heuristic based on shortest path?
  - Poor in performance, but
  - Easy to implement, distributed
  - Robust against failures in network
  - Quickly adaptive
  - More importantly, allows heterogeneous networks (ISP) to operate without sharing "sensitive" information
- We'll look at the Route-opt problem
  - First, we'll see a non-implementable solution
  - Then, look for an implementable solution
    - very similar to Dijkstra's algorithm

Route-opt

- Convex optimization (minimization) problem
  - ◦ Convex cost function
  - ◦ Convex constraint set
  - $\rightarrow$ Known-standard methods to solve the problem iteratively

- Let's look at a simple method called *Descent* method
  - ◦ Essentially, it changes the solution iterative so that
  - ◦ Solution remains feasible and the cost decreases

- Later, we'll see a simple, distributed algorithm
  - ◦ An extension of descent method
    - – Subgradient method via dual decomposition

Descent Method

- Main steps:

  1. Start with any initial feasible solution
  2. Given current solution, find increment in solution
     - that will retain feasibility of solution and decrease cost.
  3. Repeat 2 until convergence.

- Questions:

  A. How to find feasible solution initially?
  B. What guarantees existence of a feasible "descent" direction?
  C. How to find feasible descent direction?
  D. What guarantees that convergent point is optional solution?

- Next, we answer these questions

A. Feasible Solution

- It is sufficient to know the method to route data destined for one node
  - Repeating it for all destinations gives a complete solution
- Parameters:
  - R_τ: residual graph at end of iteration τ
    - Initially, $R_0 = G$ (n nodes, $\mathcal{L}$ links, C_{ij} capacities)
    - R_τ changes over time only in capacities
  - Traffic demands $r_1(n), \ldots, r_{n-1}(n)$
- Find-Path$\Big(R_1, (i, n)\Big)$
  - Finds a path from $i \rightarrow n$ in R with some positive capacity
  - It identifies path and capacity that it has between $i \rightarrow n$
  - An important building block of method for finding feasible flow

A. Feasible Solution

(i) Initially: $\tau = 0$, $R_0 = G$, $i = 1$ and $r = r_1(n)$

(iii) Find-path$\Big(R_\tau, (i, n)\Big)$ returns path $(i, a_1, \ldots, a_{\ell_1}, n)$ with capacity c_τ

- $q = (r - c_\tau)^+$
- Update R_τ to obtain $R_{\tau+1}$ as follows:
  - Reduce capacities on edges $(i, a_1), (a_1, a_2), \ldots, (a_{\ell 1} n)$ by q
  - Increase link capacities on edges $(a_1, i), (a_2, a_1), (a_3, a_2), \ldots, (n_1 a_\ell)$ by a

(c) $\tau = \tau + 1$, $i = i + 1$

(iv) If $i > n$ then STOP; else set $r = r_i(n)$

(v) If $r > 0$, go to (iii); else set $i = i + 1$, go to (iv)

A′. Find-Path

- Essentially, probe all directions in given graph in an intelligent way!

- Find-Path$\Big(R, (i, n)\Big)$:

  (finds path in graph R from i to n with positive capacity)

  1. Initially $N = \{i\}$; $c(i) = \infty$; $P =$empty
  2. (Add new node)
     - Find $j \notin N$ s.t. $\exists\ k \in N$ with $c_{kj} > 0$
     - Add $P = P \cup \{(k_{ij})\}$; $N = N \cup \{j\}$
     - Set $c(j) = \min\{c(k), c_{kj}\}$
  3. If $n \in N$ then STOP
  4. Path found is of capacity $c(n)$ with edge in P (there is such a unique path)

B. Existence of Descent Direction

- Recall that a feasible routing is characterized by $\phi = (\phi_{k\ell})$
  - ◦ By definition, set of all feasible ϕ is convex
- **Claim.** ϕ is optimal $\Leftrightarrow$ No feasible descent direction.
- **Proof.**
  - ◦ First, direction $(\Rightarrow)$ which we prove by contradiction
    - – let ϕ be optimal and $\exists$ a feasible decent direction
    - – that is, there exists $\Delta\phi$ such that
    - – $\phi + \theta\Delta\phi$ is feasible and
    - – $D(\phi + \theta\Delta\phi) < D(\phi)$ for some $\theta > 0$
    - – this contradicts assumption of ϕ being optimal
  - ◦ Thus ϕ is optimal then no feasible descent direction

B. Existence of Descent Direction

- Next, we prove the other direction ($\Leftarrow$):
  - Equivalently, if ϕ is not optimal then
    - there exists a feasible descent direction
- Let ϕ not be optimal
  - That is, there is $\hat{\phi}$ feasible s.t. $D(\hat{\phi}) < D(\phi)$
- Let

$$\phi_\theta = \phi + \theta(\hat{\phi} - \phi) = (1-\theta)\phi + \theta\hat{\phi} \quad ; \quad \theta \in (0,1)$$

- Then, ϕ_θ is feasible due to convexity of feasible set.

$$D(\phi_\theta) \leq \theta D(\hat{\phi}) + (1-\theta)D(\phi) < D(\phi) \quad ; \quad \theta \in (0,1)$$

- Thus, $(\hat{\phi} - \phi)$ is a feasible descent direction.
- Thus, ϕ not opt. $\Rightarrow$ existence of feasible descent direction.
- This complete the proof of our claim $\Box$

C. Finding Descent Direction

- First, suppose we are in unconstrained set up
  - Let $\nabla D(\phi) = \left[\frac{\partial D(\phi)}{\partial \phi_{k\ell}}\right]_{k,\ell}^{T}$
  - Then
- **Claim.** $-\nabla D(\phi)$ is a descent direction.
- **Proof.**
  - D is assumed to be a strictly convex, twice differentialable function
    - that is, Hessian of D is strictly positive
  - Let, $\phi(t) = \phi - t\nabla D(\phi)$
  - By Taylor's expansion,

$$D\Big(\phi(t)\Big) = D(\phi) + t\Big(\phi(t) - \phi\Big)^T \nabla D(\phi) + \frac{t^2}{2}\Big(\phi(t) - \phi\Big)^T \nabla^2 D\Big(\phi(s)\Big)\Big(\phi(t) - \phi\Big)\ ;$$

for some $s \in (0, t)$.

C. Finding Descent Direction

- Let $\nabla^2 D\Big(\phi(s)\Big) \le MI \;; \quad s \in (0, t)$
- Then,

$$D\Big(\phi(t)\Big) \le D(\phi) - t||\nabla D(\phi)||_2^2 + \frac{t^2}{2} M ||\nabla D(\phi)||_2^2$$

- Then, for $t < 2/M$; $D(\phi(t)) < D(\phi)$, if

  $- \; ||\nabla D(\phi)||_2^2 \neq 0$.

□

Gradient Descent

- Algorithm
  - Start from some feasible ϕ.
  - Set $\nabla\phi = -\nabla D(\phi)$.
  - Find $t \in [0, 1]$ s.t. $D(\phi + t\nabla\phi)$ is minimum.
  - Set $\phi = \phi + t\nabla\phi$.
  - Repeat from #2 until $\nabla D(\phi) = 0$.
- Main Problem
  - Above works for unconstrained setup.
  - What about constrained situation?
    - "project" gradient descent into feasible space
    - we'll consider a modification of this specialized to Route-Opt setup

C. Finding Descent Direction

- Given ϕ, there exists $\Delta\phi$ such that
  - $\phi + \Delta\phi$ is feasible and
  - $D(\phi + \Delta\phi) < D(\phi)$
- Given this, for all $\theta \in (0, 1)$,

$$\phi(\theta) = \theta(\Delta\phi + \phi) + (1 - \theta)\phi, \quad \text{is feasible by convexity,}$$

$$D\Big(\phi(\theta)\Big) < D(\phi) \; ; \qquad \forall\theta \text{ by convexity.}$$

- For very small θ, $D\Big(\phi(\theta)\Big) \approx D(\phi) + \theta \cdot \nabla D(\phi)^T \Delta\phi$,
  - Then, $\nabla D(\phi)^T \Delta\phi < 0$
- Thus, one option is to look for $\Delta\phi$ such that
  - $\phi + \Delta\phi$ remains feasible and
  - $\nabla D(\phi)^T \Delta\phi < 0$
- We do this next.

C. Finding Feasible Descent Direction

- Let $\Delta\phi$ be such that $\phi + \Delta\phi$ is feasible, i.e.
  - Re-routing certain data for destination n from some i along other path
- Hence, descent direction means that for some (i, n) pair $\exists$ two paths

$$P_1 = (i, a_1, \ldots, a_\ell, n)$$
$$P_2 = (i, b_1, \ldots, b_k, n)$$

- Such that
  - ◦ P_1 has positive flow on P_1 from $i \to n$;
  - ◦ P_2 has some capacity left for $i \to n$ and
  - ◦ $\dfrac{\partial D(\phi)}{\partial \phi_{ia_1}} + \cdots + \dfrac{\partial D(\phi)}{\partial \phi_{a_\ell n}} > \dfrac{\partial D(\phi)}{\partial \phi_{\ell b_1}} + \cdots + \dfrac{\partial D(\phi)}{\partial \phi_{b_k n}}$.
- Next, we see a simple way to find such paths

C. Finding Feasible Descent Direction

- Given ϕ, create graph $R(\phi)$ as follows
  - If $\phi_{ij} > 0$, $F_{ij} < c_{ij}$, then
    - assign weight $\dfrac{\partial D(\phi)}{\partial \phi_{ij}}$ to (i, j)
  - If $\phi_{ij} > 0$, $F_{ij} > 0$, then
    - assign weight $-\dfrac{\partial D(\phi)}{\partial \phi_{ij}}$ to (j, i).
- Note that in $R(\phi)$, a feasible descent direction is
  - Equivalent to a negative cost cycle
- We'll look for min-cost path in $R(\phi)$
  - The Dijstra's algorithm does not work
    - due to negative weights
  - We will use Bellman–Ford algorithm to detect neg-cycles

D. Optimality at Convergence

- Convergent point ϕ^*
  - Exists because $D(\phi)$ is always decreasing
  - Further, ϕ^* is such that no descent direction.
  - This implies, ϕ^* is optimal.
- The described algorithm is
  - centralized

  $\rightarrow$ not implementable
  - However, it's very instructive for general optimization problems
- Next, we'll see a distributed algorithm
  - Based on "dual-decomposition" and "subgradient" methods.
  - Similar ideas will be used in the context of congestion-control.

Distributed Route-Opt

- Main method
  - Primal & Dual version
    - convexity implies: solving Primal = solving Dual
  - Interestingly,
    - Dual can be solved in distributed manner
    - using 'subgradient' method which is extension to gradient descent algorithm
- The algorithm is very similar in nature to Dijkstra's algorithm

Route-opt

Primal (P):

$$\min \sum_{(i,\ell)\in\mathcal{L}} D_{i\ell}(F_{i\ell}) \stackrel{\triangle}{=} \sum_{e\in\mathcal{L}} D_e(F_e)$$

subject to

$$\mathcal{C} = \left[\begin{array}{lcl} F_{i\ell} & = & \sum_k t_i(k)\phi_{i\ell}(k) \leq c_{i\ell} \\ \phi_{i\ell}(j) & \geq & 0\ , \quad \text{for all } i, j, \ell; \\ \sum_{\ell i}(j) & = & 1\ , \quad \text{for all } \ell, j. \end{array}\right.$$

$$\mathcal{O} = [t_i(j) = r_i(j) + \sum_{\ell} t_\ell(j)\phi_{\ell i}(j) \quad \text{for all } i, j;$$

- Reformulation: $x_{ik}(j) = t_i(j)\phi_{ik}(j)$
- Then,

$$\mathcal{C} \Leftrightarrow \{F_{i\ell} = \sum_k x_{i\ell}(k) \leq c_{i\ell}\ ; \quad x_{i\ell}(k) \geq 0\}$$

$$\mathcal{O} \Leftrightarrow \sum_{k:(ik)\in\mathcal{L}} x_{ik}(j) = r_i(j) + \sum_{\ell:(\ell,i)\in\mathcal{L}} x_{\ell i}(j)\ .$$

Dual of Route-Opt

- **Lagrangian:**

$$L(\mathbf{x};\boldsymbol{\gamma}) = \sum_{e\in\mathcal{L}} D_e(F_e) + \sum_{i,j}\gamma_i(j)\left[-\sum_k x_{ik}(j) + \sum_\ell x_{\ell i}(j) + r_i(j)\right]$$

$$= \sum_{e\in\mathcal{L}}\left[D_e(F_e) + \sum_{j=1}^{n}(\gamma_{e^+}(j) - \gamma_{e^-}(j))x_e(j)\right] + \boldsymbol{\gamma}^T\mathbf{r}$$

where $e = (e^-, e^+)$

- **Dual function:**

$$q(\boldsymbol{\gamma}) = \inf_{\mathbf{x}\in\mathcal{C}} L(\mathbf{x};\boldsymbol{\gamma})$$

$$= \boldsymbol{\gamma}^T\mathbf{r} + \sum_{e\in\mathcal{L}} \inf_{x_e=(x_e(j));x_e\in\mathcal{C}}\left[D_e(F_e) + \sum_{j=1}^{n} x_e(j)\nabla\gamma_e(j)\right]$$

where $\nabla\gamma_e(j) = \gamma_{e^+}(j) - \gamma_{e^-}(j)$.

- Thus, $q(\boldsymbol{\gamma})$ can be evaluated "locally", since
    - ○ $x_e \in \mathcal{C}$ is "locally" checkable

Dual of Route-Opt

- **Dual (D):**

$$\max_{\boldsymbol{\gamma}} q(\boldsymbol{\gamma}).$$

- Since there is no duality gap (**P** is convex minimization)
  - $\boldsymbol{\gamma}^*$ s.t. $q(\boldsymbol{\gamma}^*) = \max_{\boldsymbol{\gamma}} q(\boldsymbol{\gamma})$ gives $x^*(\boldsymbol{\gamma}^*) = L(x^*(\boldsymbol{\gamma}^*), \boldsymbol{\gamma}^*)$.

  $\rightarrow$ Sufficient to solve **D**

- In **D**, γ is "free" variable, hence
  - If we could evaluate $q(\boldsymbol{\gamma})$ locally, and
  - Use simple unconstrained optimization procedure
  - We'll obtain distributed solution
- Question: what unconstrained optimization procedure?
  - Can not used gradient descent as it gradient may not exists
  - Instead, we'll use subgradient method

Subgradient Method

- Similar to gradient, but used when gradient is non-unique

$$\max_{\boldsymbol{\gamma}} q(\gamma) \;=\; \min_{\boldsymbol{\gamma}} -q(\boldsymbol{\gamma}) \quad \text{: standard convex minimization}$$
$$-q(\boldsymbol{\gamma}) \;=\; -\inf_{x} L(x,\boldsymbol{\gamma}) = -L(x(\boldsymbol{\gamma}),\boldsymbol{\gamma}) \;.$$

- Then, gradient of $L(x(\boldsymbol{\gamma}),\boldsymbol{\gamma})$ is a subgradient for $(-q(\boldsymbol{\gamma}))$

$$-\nabla L(x(\boldsymbol{\gamma}),\boldsymbol{\gamma}) = \Big(-\frac{\partial q(\boldsymbol{\gamma})}{\partial \boldsymbol{\gamma}_i(j)} \Big)$$

where,

$$\frac{\partial q(\boldsymbol{\gamma})}{\partial \gamma_i(j)} \;=\; r_i(j)\Big[\sum_{e:e^+=i} x_e(j) - \sum_{e:e^-=i} x_e(j)\Big] + \Big[\sum_{e:e^+=i \text{ or } e^-=i} \frac{\partial D_e(x_e)}{\partial x_e}\Big\{\sum_{j=1}^{n}\frac{\partial x_e(j)}{\partial \gamma_i(j)}\Big\}\Big] + \sum_{e:e^+=i}\sum_{j=1}^{n}\frac{\partial x_e(j)}{\partial \gamma_i(j)} - \sum_{e:e^-=i}\sum_{j=1}^{n}\frac{\partial x_e(j)}{\partial \gamma_i(j)} \;.$$

Subgradient Method

- Hence, to compute subgradient of $(-q(\boldsymbol{\gamma}))$, we need

$$\left(\frac{\partial x_i(j)}{\partial \gamma_i(j)}\right)_{i,j} \text{ at } x = x(\boldsymbol{\gamma}) \text{ and } \frac{\partial D(x_e)}{\partial x_e} \quad \text{at} \quad x = x(\gamma)$$

  - Again, these are locally computable
  - Hence, subgradient components $\dfrac{\partial(-q(\gamma))}{\partial \gamma_i(j)}$ are computable at node i (using edge variables)

- Subgradient Algorithm

  1. Start with initial $\boldsymbol{\gamma}^0$. Set $t = 0$.
  2. Compute $q(\boldsymbol{\gamma}^t)$.
  3. Compute subgradient of $-q(\boldsymbol{\gamma}^t) \stackrel{\triangle}{=} (G_i^t(j))$.
  4. Update $\gamma^{t+1} = \gamma^t - \alpha_t G^t$.
  5. Set $t = t + 1$ and repeat from #2.

* Here, α_t is s.t. $\lim \alpha_t = 0$ but $\sum \alpha_t = \infty$.

Summary

- **Routing**
  - An essential network-layer task
  - Simple flow-based model to describe the setup formally
    - allows us to evaluate performance
  - Heuristics
    - simple, distributed and, hence, implemented
  - Optimal routing
    - at the final glance, solvable but difficult to implement
    - recent techniques can lead to simple, implementable solutions
    - will it be implemented ?

Summary

- Related results
  - Completely distributed primal algorithm
    - Algorithm by Gallager (1976)
  - Asynchronous algorithms
    - Algorithm by Tsitsiklis and Bertsekas (1985)
  - Effect of failure
  - Stability of algorithm (no oscillation)
- Broad impact
  - Led to development of distributed network algorithms
    - similar ideas in congestion control
  - Routing or job assignment tasks in other scenarios can benefit from these methods

References

1. Chapter 5, in book on Data Networks by Bertsekas-Gallager
2. Notes by Stephen Boyd (some posted on the class page)
   A. Link 1: `www.stanford.edu/course/ee3920/`
      – Subgradient: definition and properties
      – Subgradient algorithm: convergence and correctness

   B. Link 2: `www.stanford.edu/course/ee363/` [or Chapters 4 and 5, Convex Optimization by Boyd-Vandenberg]

   C. Notes on Decomposition Method
      – Again, see `www.stanford.edu/course/ee3920/`, or
      – Chapter 6, Nonlinear Programming by Betsekas
3. Miscellaneous
   – Some notes of current Internet routing practice will be posted
   – An excellent survey paper by Tsitsiklis and Bertsekas (1992) on Asynchronous algorithms