

Vectors

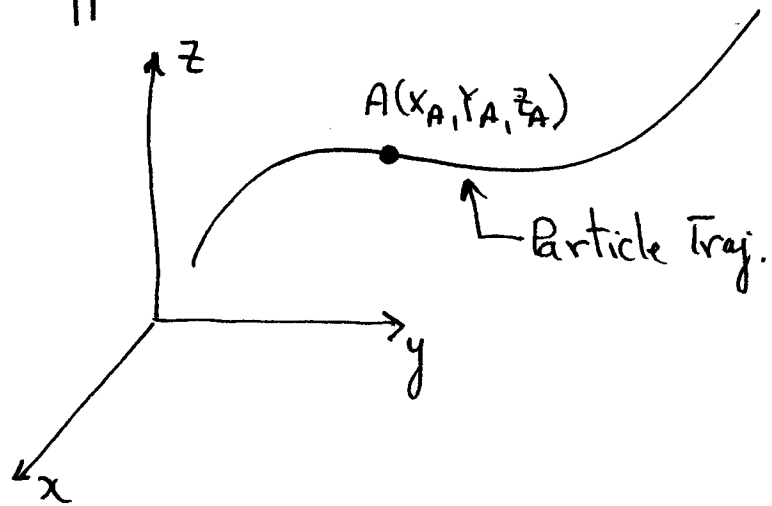
4-1

Vectors are mathematical quantities that are useful to describe physical quantities that have both a magnitude and direction associated with them.

We will review the operations of vector addition, subtraction and multiplication by a number. We will show two types of vector multiplication.

We will deal with vectors in two different ways:

- Geometric approach
- Algebraic approach

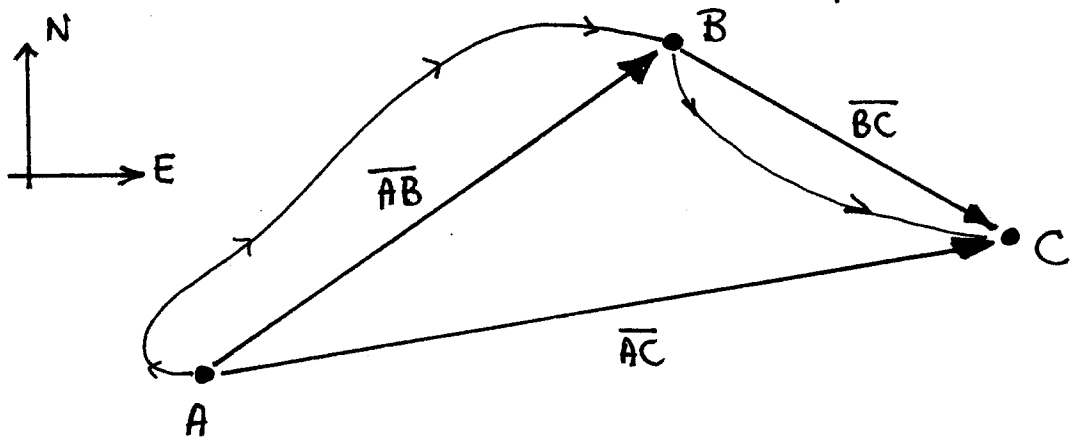


- Want to describe motion of an ideal particle
- Choose a coordinate system - e.g. Cartesian x, y, z
- Want to describe the motion in a general way.
- Note that the "laws of nature" must be invariant to the choice of an inertial (non-accelerated) coordinate system.

Displacement

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- Consider two successive displacements



- Start at A and move to B
- The directed line segment from A to B is called a displacement vector. Arrow indicates direction of motion.
- line does not represent actual path followed just the final result.
- Move from B \rightarrow C.
- \overline{AC} represents result of both moves. Net direction ; Net displacement.
- "Addition of Vectors"

Vector: Any quantity that has a magnitude and direction and behaves like the displacement vector.

e.g.

- displacement	- force
- velocity	- momentum
- acceleration	- torque

Scalar: Any quantity that has a magnitude but no direction.

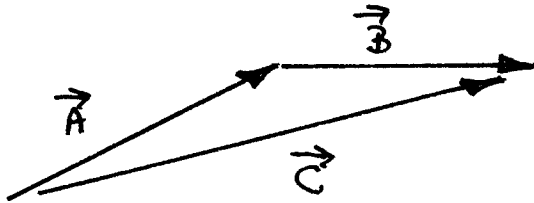
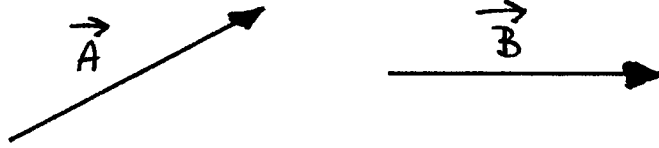
e.g.

- length	- density
- time	- energy
- mass	- temperature
- area	
- volume	

$$\vec{A} = \vec{B}$$

- only if $|A| = |B|$ magnitude
- only if directions are equal
- location and starting point do not matter.
- units must be the same.

Vector Addition



$$\vec{C} = \vec{A} + \vec{B}$$



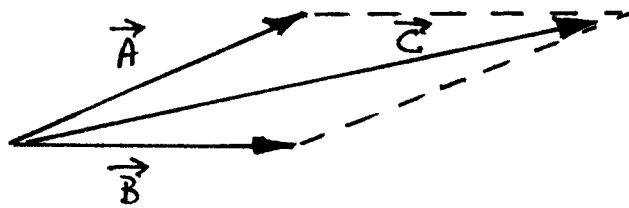
$$\vec{C} = \vec{B} + \vec{A}$$

$$\therefore \vec{C} = \vec{A} + \vec{B} = \vec{B} + \vec{A}$$

↑ Commutative Law

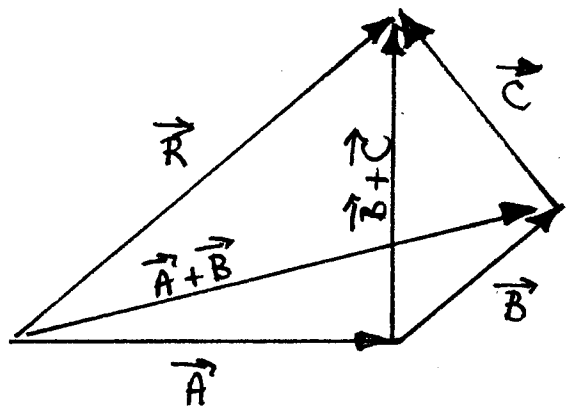
- order of addition does not matter!
- addition makes sense only for same kinds of vectors.

Parallelogram:



3-or More Vectors:

$$\begin{aligned} \vec{R} &= (\vec{A} + \vec{B}) + \vec{C} \\ &= \vec{A} + (\vec{B} + \vec{C}) \end{aligned}$$

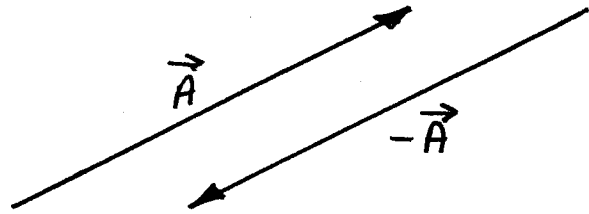


Negative of Vector

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- same magnitude
- opposite direction

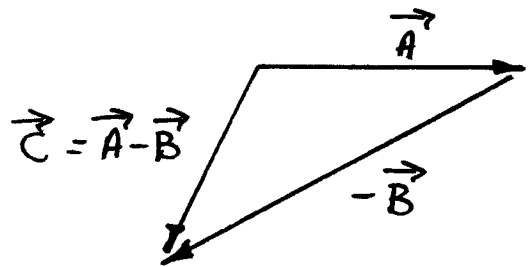
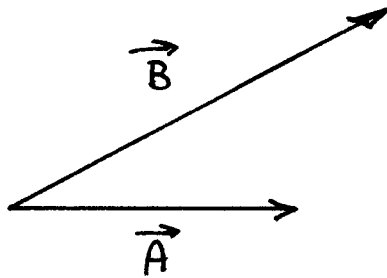
$$\vec{A} + (-\vec{A}) = 0$$



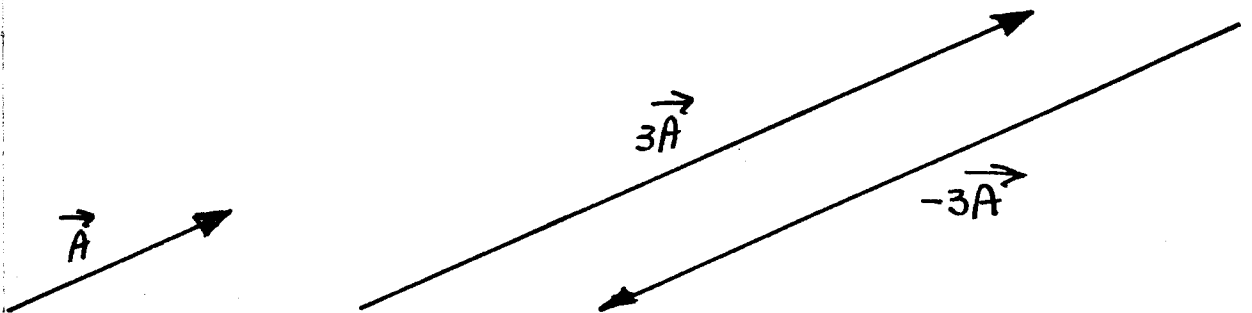
Subtraction

- The subtraction of two vectors \vec{A} and \vec{B} is defined as the sum of \vec{A} and $-\vec{B}$.

$$\vec{C} = \vec{A} - \vec{B} = \vec{A} + (-\vec{B})$$



scalar x Vector



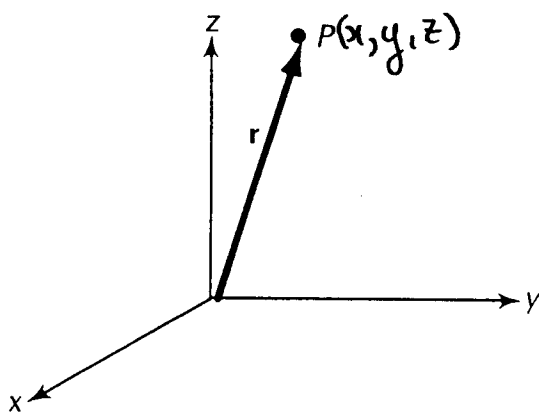
- magnitude multiplied by scalar.

ex: $\vec{F} = m\vec{a}$ same direction
different physical quantities: m has own units.

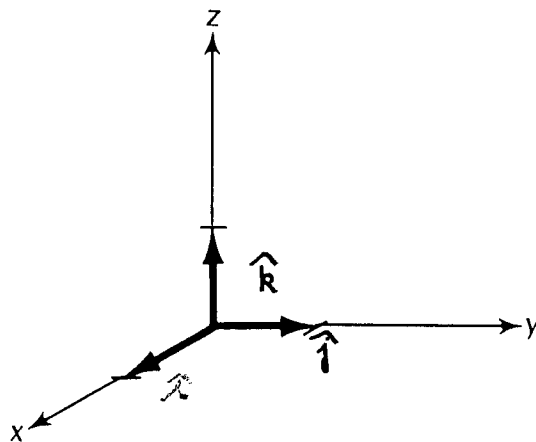
Vector Components

4-6

- A vector is completely described by its components.
- Useful for vector algebra
- choose a coordinate system
- choose an origin at foot-of-vector



The position vector \vec{r} of the point P .



The unit vectors

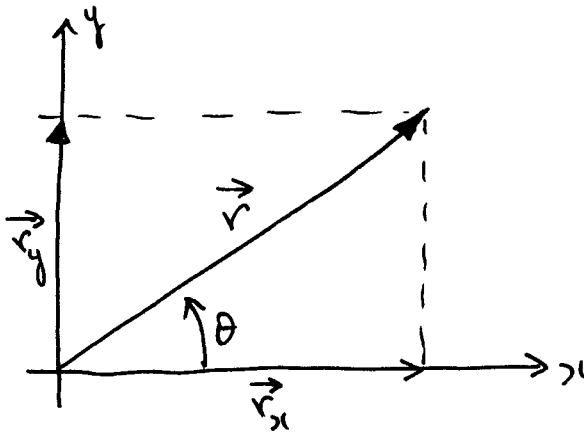
- $P(x, y, z)$ is an arbitrary point with coordinates x, y, z .
- \vec{r} is a position vector from the origin to the point x, y, z .

Unit Vectors

\hat{i} , \hat{j} and \hat{k} are three vectors of magnitude = 1 unit (dimensionless) pointing along the coordinate axes.

- Other vectors can be written in terms of them
- Called unit vectors.
- Carry no units

Components - 2 dimensions



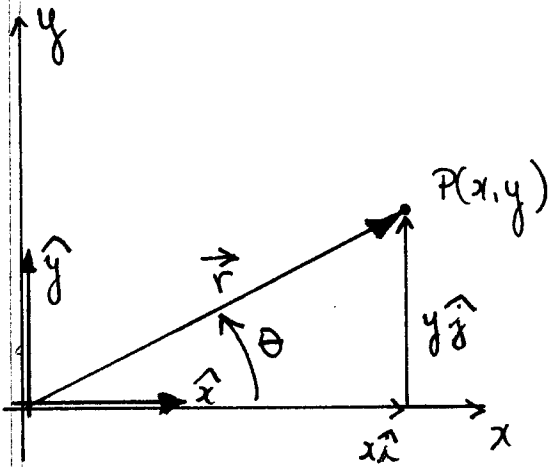
$$\vec{r} = \vec{r}_x + \vec{r}_y$$

$$r_x = r \cos \theta$$

$$r_y = r \sin \theta$$

-components can be used instead of vector itself.

4-6A



$$\vec{r} = x \hat{x} + y \hat{y}$$

$$r = \sqrt{x^2 + y^2}$$

$$x = r \cos \theta$$

$$y = r \sin \theta$$

$$\tan \theta = \frac{y}{x}$$

What is

$$x \hat{i} + y \hat{j} + z \hat{k} = ?$$

- Move a distance x -units along x -axis
 y -units \checkmark y -axis
 z -units \checkmark z -axis

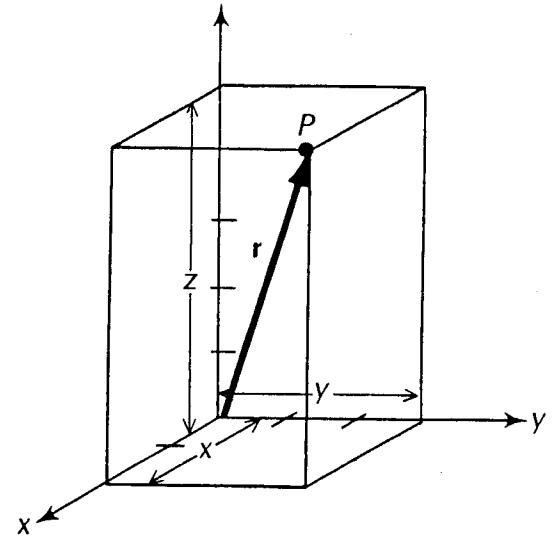
$$\vec{r} = x \hat{i} + y \hat{j} + z \hat{k}$$

Arbitrary Vector \vec{A}

- Coordinate system at the foot of vector \vec{A} .
- Drop perpendiculars from the tip to each of the coordinate axes.
- Intercepts give the components with values (numerical) A_x, A_y, A_z .

In terms of unit vectors

$$\vec{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k}$$



The components x, y, z of the position vector r .

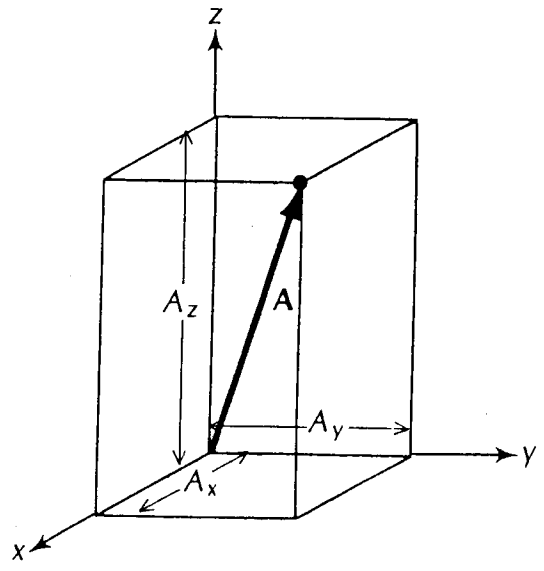


Fig. 3.17 The components A_x, A_y, A_z of an arbitrary vector A .

Example : 2-Dimensions

4-8

$$A_x = A \cos \theta_x$$

$$A_y = A \sin \theta_x$$

Magnitude of \vec{A} :

$$|\vec{A}| = A = \sqrt{A_x^2 + A_y^2}$$

$$\vec{A} = A_x \hat{i} + A_y \hat{j}$$

$$= A \cos \theta_x \hat{i} + A \sin \theta_x \hat{j}$$

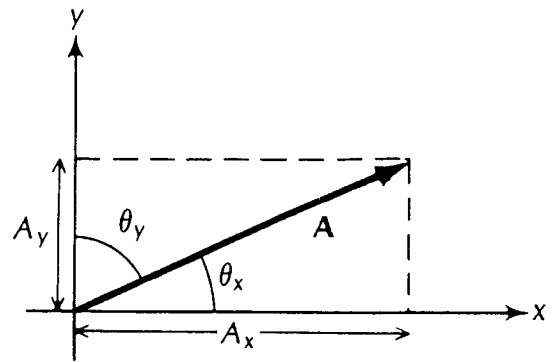
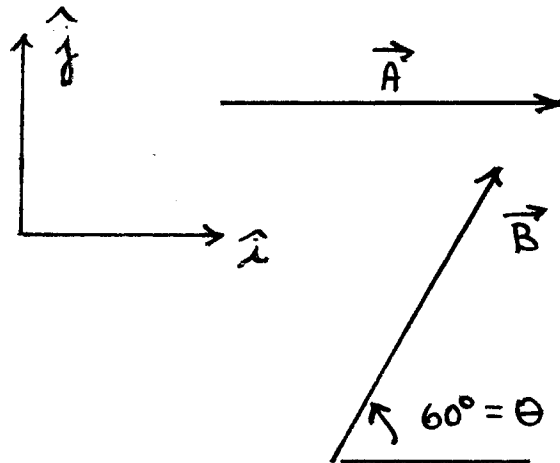


Fig. 3.18 Vector \vec{A} in two dimensions and its components.

Example



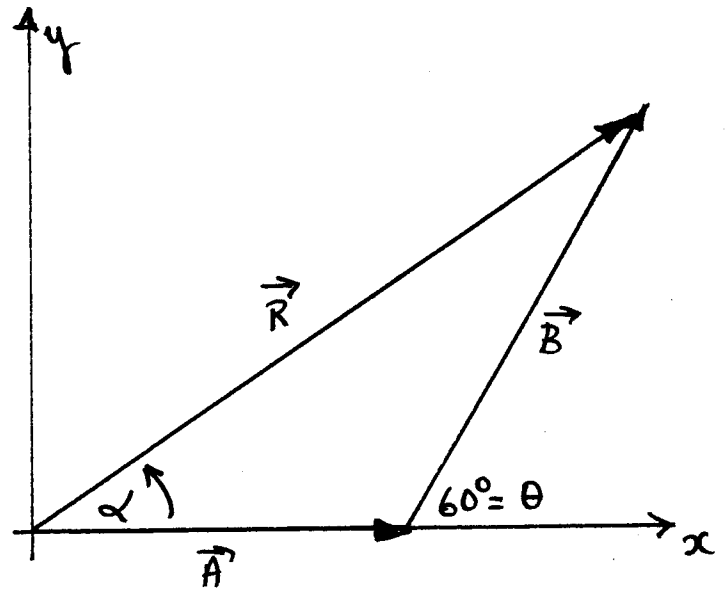
Want: $\vec{A} + \vec{B} = ?$

$$\left. \begin{array}{l} |\vec{A}| = 3 \\ |\vec{B}| = 4 \end{array} \right\} \text{Given magnitudes}$$

- Select most convenient coordinate system
- choose one axis along one of the vectors.

Want:

Magnitude of \vec{R}
 Direction of \vec{R}



$$\vec{A} = A_x \hat{i} + A_y \hat{j} = A \hat{i} + 0 \hat{j}$$

$$\vec{B} = B_x \hat{i} + B_y \hat{j} = B \cos 60^\circ \hat{i} + B \sin 60^\circ \hat{j}$$

$$\vec{R} = \vec{A} + \vec{B} = (A_x + B_x) \hat{i} + (A_y + B_y) \hat{j}$$

$$= (A + B \cos 60^\circ) \hat{i} + (0 + B \sin 60^\circ) \hat{j}$$

$$= (3 + 4 \cos 60^\circ) \hat{i} + (4 \sin 60^\circ) \hat{j}$$

$$\vec{R} = 5 \hat{i} + 3.46 \hat{j}$$

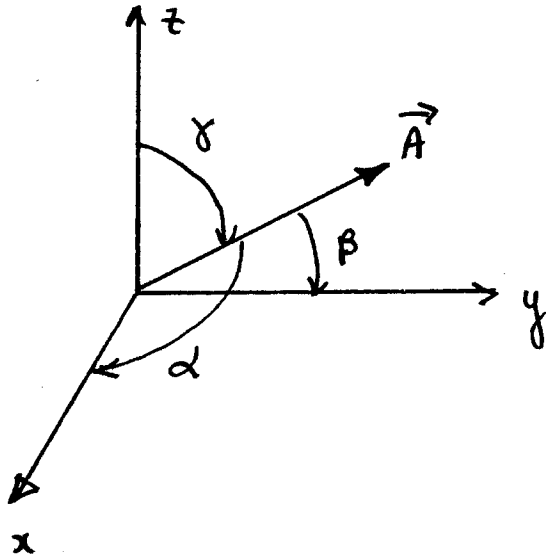
$$|\vec{R}| = \sqrt{R_x^2 + R_y^2} = \sqrt{5^2 + 3.46^2} = 6.08$$

$$\alpha = \tan^{-1} \frac{R_y}{R_x} = \tan^{-1} \frac{3.46}{5}$$

$$= 34.7^\circ$$

Example - General 3-D Vector

4-10



$$\begin{aligned}A_x &= A \cos \alpha \\A_y &= A \cos \beta \\A_z &= A \cos \gamma\end{aligned}$$

$\alpha, \beta, \gamma \Rightarrow$ direction cosines

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$$

(not all independent)

Magnitude: $A = \sqrt{A_x^2 + A_y^2 + A_z^2}$

$$\cos \alpha = \frac{A_x}{A}, \text{ etc.}$$

Position vector \vec{r}

$$\vec{r} = x \hat{i} + y \hat{j} + z \hat{k}$$

$$r = \sqrt{x^2 + y^2 + z^2}$$



distance from (x, y, z) to origin.

$$\vec{A} = A \cos \alpha \hat{i} + A \cos \beta \hat{j} + A \cos \gamma \hat{k}$$

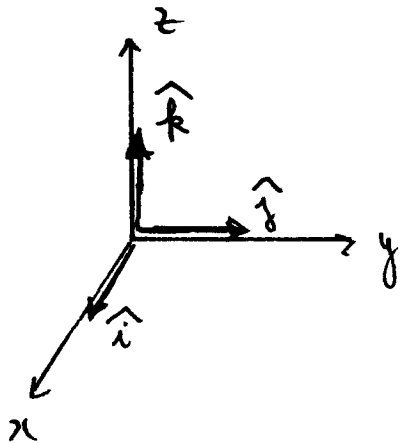
$$\text{Let } \vec{A} = A_x \hat{x} + A_y \hat{y} + A_z \hat{z}$$

$$\vec{B} = B_x \hat{x} + B_y \hat{y} + B_z \hat{z}$$

$$\vec{A} + \vec{B} = (A_x + B_x) \hat{x} + (A_y + B_y) \hat{y} + (A_z + B_z) \hat{z}$$

$$\vec{A} - \vec{B} = (A_x - B_x) \hat{x} + (A_y - B_y) \hat{y} + (A_z - B_z) \hat{z}$$

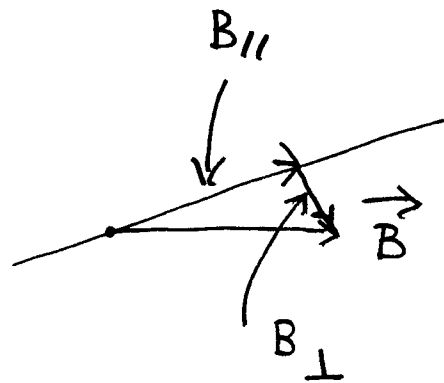
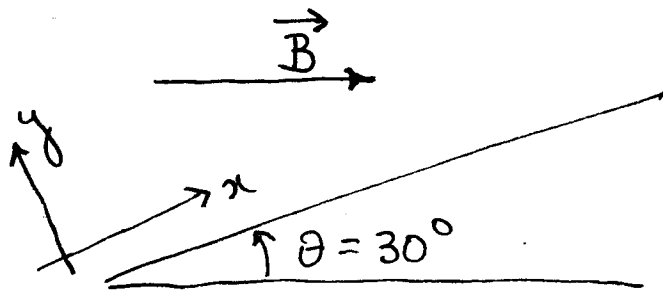
-Generalize to any number of vectors added or subtracted from each other.



$$\left. \begin{array}{l} \hat{x} \leftrightarrow \hat{i} \\ \hat{y} \leftrightarrow \hat{j} \\ \hat{z} \leftrightarrow \hat{k} \end{array} \right\} \text{Alternate Notation}$$

$$\vec{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k}$$

Inclined Plane



Decompose vector \vec{B} :

$B_{||}$: Parallel to x .

B_{\perp} : Perpendicular to x ; i.e. along $-y$.

$$\begin{aligned}\vec{B} &= \vec{B}_{||} + \vec{B}_{\perp} \\ &= B_{||} \hat{x} + B_{\perp} \hat{y}\end{aligned}$$

$$B_{||} = |\vec{B}| \cos 30^\circ$$

$$B_{\perp} = -|\vec{B}| \sin 30^\circ$$

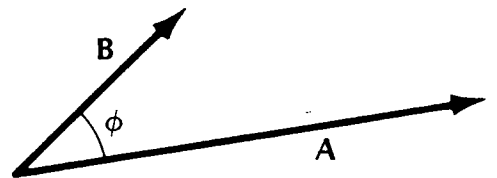
Vector Multiplication

4-12

- Several ways to multiply vectors
- Need to take into account magnitude and direction

1. Dot Product

- scalar product
- inner product



The vectors **A** and **B** and the angle between them.

Vectors: \vec{A} and \vec{B}

$$\vec{A} \cdot \vec{B} = AB \cos \phi \quad \phi \leq 180^\circ$$

$$\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A} \quad \text{— result is a scalar, number (commutative)}$$

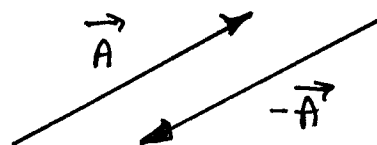
$$\vec{A} \cdot \vec{A} = AA \cos 0^\circ$$

$$= A^2 \quad \text{[parallel vectors]}$$
$$= (\text{magnitude})^2$$



$$\vec{A} \cdot (-\vec{A}) = -A^2$$

[antiparallel vectors]

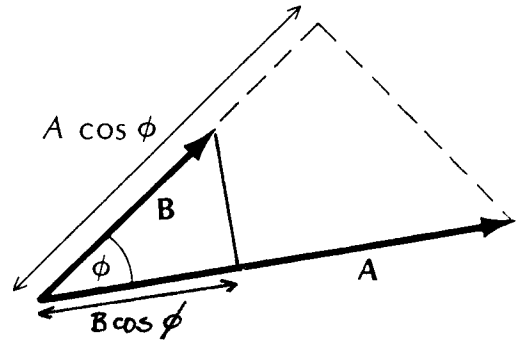


$$\vec{A} \cdot \vec{B} = A (B \cos \phi)$$

↑ Proj. of \vec{B} on \vec{A}

$$= B (A \cos \phi)$$

↑ Proj. of \vec{A} on \vec{B}



The component of A along B is $A \cos \phi$; the component of B along A is $B \cos \phi$.

$$0 < \phi < 90^\circ \quad (\vec{A} \cdot \vec{B}) > 0$$

$$90 < \phi < 180^\circ \quad (\vec{A} \cdot \vec{B}) < 0$$

$$\phi = 90^\circ \quad (\vec{A} \cdot \vec{B}) = 0$$

- \vec{A} and \vec{B} are \perp to each other
- A good test for perp vectors.

$$\left. \begin{array}{l} \hat{i} \cdot \hat{i} = 1 \\ \hat{j} \cdot \hat{j} = 1 \\ \hat{k} \cdot \hat{k} = 1 \end{array} \right\} \text{Parallel unit vectors}$$

$$\left. \begin{array}{l} \hat{i} \cdot \hat{j} = \hat{j} \cdot \hat{i} = 0 \\ \hat{i} \cdot \hat{k} = \hat{k} \cdot \hat{i} = 0 \\ \hat{j} \cdot \hat{k} = \hat{k} \cdot \hat{j} = 0 \end{array} \right\} \text{Perpendicular unit vectors}$$

Distributive Law

4-13A

To work out product of $\vec{A} \cdot \vec{B}$ we make use of the fact that vector multiplication obeys the 'distributive law':

$$(\vec{C} + \vec{D}) \cdot \vec{E} = \vec{C} \cdot \vec{E} + \vec{D} \cdot \vec{E}$$

Proof: Product is the magnitude of \vec{E} times component of $\vec{C} + \vec{D}$ along \vec{E} . But the component of $\vec{C} + \vec{D}$ along any direction is equal to the sum of the component of \vec{C} plus the component of \vec{D} .

$$\begin{aligned}
 \vec{A} \cdot \vec{B} &= (A_x \hat{i} + A_y \hat{j} + A_z \hat{k}) \cdot (B_x \hat{i} + B_y \hat{j} + B_z \hat{k}) \\
 &= A_x B_x \hat{i} \cdot \hat{i} + \cancel{A_x B_y \hat{i} \cdot \hat{j}} + \cancel{A_x B_z \hat{i} \cdot \hat{k}} \\
 &\quad + \cancel{A_y B_x \hat{j} \cdot \hat{i}} + A_y B_y \hat{j} \cdot \hat{j} + \cancel{A_y B_z \hat{j} \cdot \hat{k}} \\
 &\quad + \cancel{A_z B_x \hat{k} \cdot \hat{i}} + \cancel{A_z B_y \hat{k} \cdot \hat{j}} + A_z B_z \hat{k} \cdot \hat{k}
 \end{aligned}$$

$$\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z$$

Each term is a product of vectors which are either \parallel or \perp

What is

$$\hat{j} \cdot \vec{A} = \hat{j} \cdot [A_x \hat{i} + A_y \hat{j} + A_z \hat{k}]$$

$$= A_y !!$$

Component of A along y-axis.

First application of a scalar product will be the concept of work. The work done for a constant force \vec{F} acting on a body which is displaced an amount \vec{d} , is given by

$$W = \vec{F} \cdot \vec{d}$$

Example: [Dot Product]

4-15

$$\vec{A} = 3\hat{x} + 7\hat{k}$$

$$\vec{B} = -\hat{x} + 2\hat{y} + \hat{k}$$

$$A_x = 3 \quad B_x = -1$$

$$A_y = 0 \quad B_y = 2$$

$$A_z = 7 \quad B_z = 1$$

$$\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z$$

$$= (3)(-1) + (0)(2) + (7)(1)$$

$$= +4$$

$$\vec{A} \cdot \vec{B} = |\vec{A}| |\vec{B}| \cos \theta$$

$$\cos \theta = \frac{\vec{A} \cdot \vec{B}}{AB} = \frac{4}{\sqrt{3^2 + 0^2 + 7^2} \sqrt{(-1)^2 + 2^2 + 1^2}} = \frac{4}{\sqrt{58} \sqrt{6}}$$

$$\theta = 77.6^\circ \quad [\text{Angle between two vectors } \vec{A}, \vec{B}]$$

↑ Easy Way !!