

2. Cross-Product

4-16

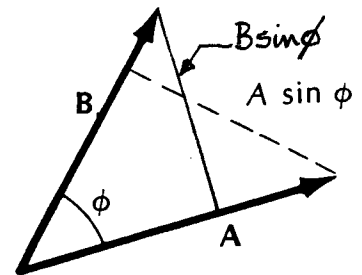
- vector product
- outer product
- result is another vector

$$\vec{C} = \vec{A} \times \vec{B}$$

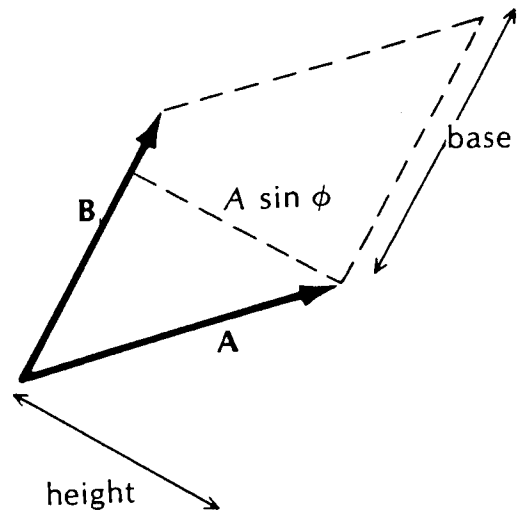
$$C = AB \sin \phi \quad (\text{magnitude})$$

↑ component of B
⊥ A.

$$0 < \phi < \pi$$

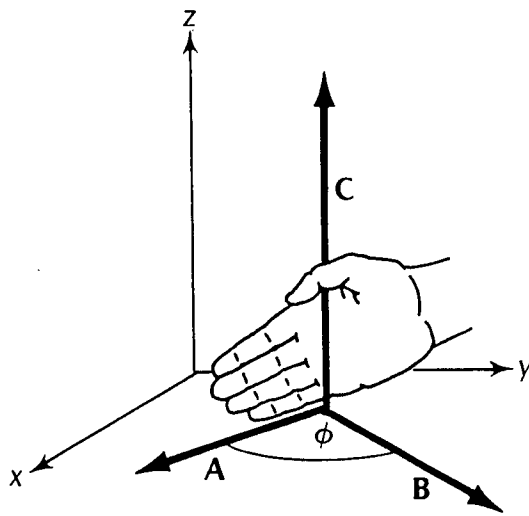


(a)

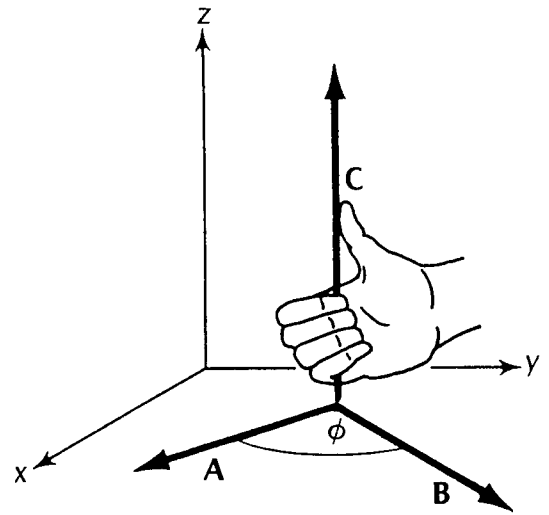


(b)

(a) $A \sin \phi$ is the component of **A** perpendicular to **B**. (b) $AB \sin \phi$ is the area of the parallelogram.



(a)



(b)

Right Hand Rule

- Direction of \vec{C} is \perp to plane formed by \vec{A} and \vec{B}

- $\therefore \vec{C}$ is \perp to both \vec{A} and \vec{B} !!

$$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$$

\vec{A} and \vec{B} define a plane
 \vec{C} is along normal to this plane } RHR

If vectors are parallel:

$$\begin{aligned} \vec{A} \times \vec{A} &= AA \sin 0^\circ \\ &= 0 \end{aligned}$$

↑ Cross-product is a useful test for vectors which are parallel

$$\left. \begin{aligned} \hat{i} \times \hat{i} &= 0 \\ \hat{j} \times \hat{j} &= 0 \\ \hat{k} \times \hat{k} &= 0 \end{aligned} \right\} \text{Parallel unit vectors}$$

$$\left. \begin{aligned} \hat{i} \times \hat{j} &= -\hat{j} \times \hat{i} = \hat{k} \\ \hat{k} \times \hat{i} &= -\hat{i} \times \hat{k} = \hat{j} \\ \hat{j} \times \hat{k} &= -\hat{k} \times \hat{j} = \hat{i} \end{aligned} \right\} \text{Right-Handed Coordinate System} \\ \text{[Definition]}$$

$$\begin{aligned} \vec{A} \times \vec{B} &= (A_x \hat{i} + A_y \hat{j} + A_z \hat{k}) \times (B_x \hat{i} + B_y \hat{j} + B_z \hat{k}) \\ &= (A_y B_z - A_z B_y) \hat{i} + (A_z B_x - A_x B_z) \hat{j} \\ &\quad + (A_x B_y - A_y B_x) \hat{k} \end{aligned}$$

[Result following multiplication
term-by-term and elimination
of many vanishing terms]

$$\vec{A} \times \vec{B} \equiv \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

↓
// - vanish!
⊥ - survive!

Determinants

4-19

a) Order - 2

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1$$

Example: $\begin{vmatrix} 3 & -2 \\ 4 & 5 \end{vmatrix} = 3(5) - 4(-2) = 23$

b) Order - 3

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

Example:

$$\begin{vmatrix} 3 & 2 & -1 \\ 4 & 3 & 3 \\ -2 & 7 & 1 \end{vmatrix} = 3 \begin{vmatrix} 3 & 3 \\ 7 & 1 \end{vmatrix} - 2 \begin{vmatrix} 4 & 3 \\ -2 & 1 \end{vmatrix} + (-1) \begin{vmatrix} 4 & 3 \\ -2 & 7 \end{vmatrix}$$

$$= 3[3(1) - 7(3)] - 2[4(1) - (-2)(3)] + (-1)[4(7) - (-2)(3)]$$

$$= -54 - 20 - 34$$

$$= \underline{\underline{-108}}$$

Applications of the cross-product will include angular momentum, \vec{L} , and torque, $\vec{\tau}$.

$$\vec{L} = \vec{r} \times \vec{p}$$

$$\vec{\tau} = \vec{r} \times \vec{F}$$

Example - Cross Product

4-20

$$\vec{A} = 3\hat{i} + 7\hat{j} - \hat{k}$$

$$\vec{B} = \hat{i} - \hat{j}$$

$$\vec{C} = \vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ +3 & +7 & -1 \\ +1 & -1 & 0 \end{vmatrix}$$

$$= \hat{i} [7(0) - (-1)(-1)] - \hat{j} [3(0) - (1)(-1)] + \hat{k} [3(-1) - (1)(7)]$$

$$= -\hat{i} - \hat{j} - 10\hat{k}$$

$$\vec{C} \cdot \vec{A} = (-\hat{i} - \hat{j} - 10\hat{k}) \cdot (3\hat{i} + 7\hat{j} - \hat{k}) = -3 - 7 + 10 = 0$$

$$\vec{C} \cdot \vec{B} = (-\hat{i} - \hat{j} - 10\hat{k}) \cdot (\hat{i} - \hat{j}) = -1 + 1 = 0$$

\vec{C} is \perp to both \vec{A} and \vec{B}

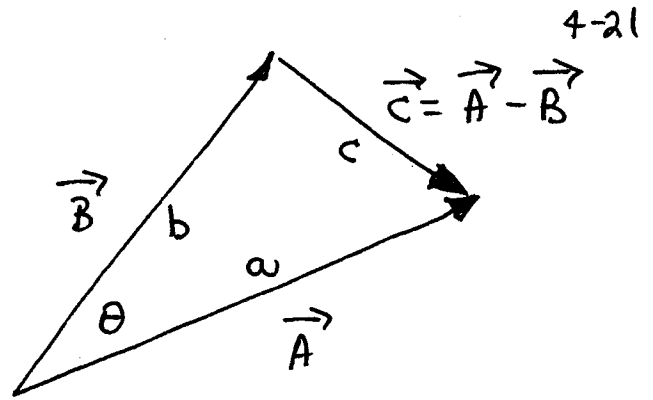
$$C = AB \sin \theta \quad [\text{magnitudes}]$$

$$\sin \theta = \frac{C}{AB} = \frac{\sqrt{(-1)^2 + (-1)^2 + (-10)^2}}{\sqrt{3^2 + 7^2 + (-1)^2} \sqrt{1^2 + (-1)^2}} = \frac{\sqrt{102}}{\sqrt{59} \sqrt{2}}$$

$$= \sqrt{\frac{102}{118}}$$

$$\theta = \sin^{-1} \sqrt{\frac{102}{118}} = 70.8^\circ$$

Example



$$\vec{C} = \vec{A} - \vec{B} \quad [\text{From } \Delta]$$

$$\vec{C} \cdot \vec{C} = (\vec{A} - \vec{B}) \cdot (\vec{A} - \vec{B})$$

$$= \vec{A} \cdot \vec{A} - 2\vec{A} \cdot \vec{B} + \vec{B} \cdot \vec{B}$$

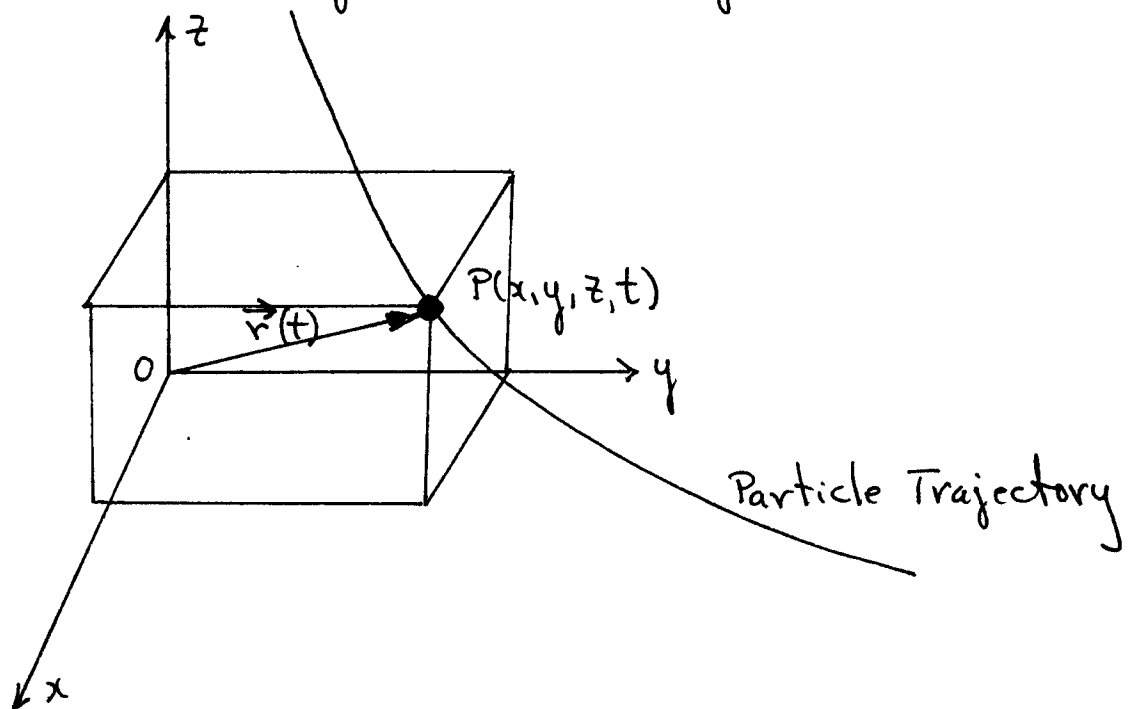
$$c^2 = a^2 - 2ab\cos\theta + b^2$$

Law of Cosines

Kinematics in Three Dimensions

5-1

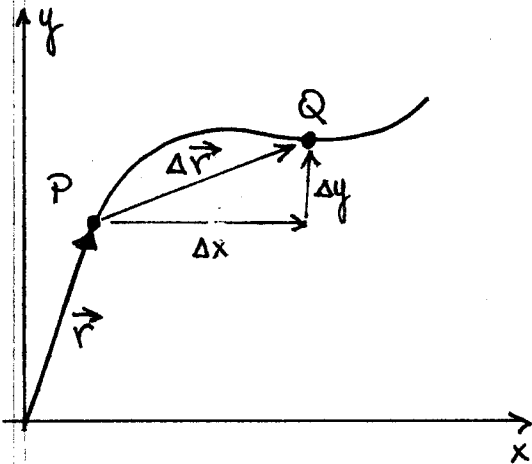
- extension of our description of 1-dimensional motion
- 3-dimensional motion is basically 3 one dimensional motions occurring simultaneously.



$\vec{r}(t) \equiv$ instantaneous position vector of the particle.

$$\vec{r}(t) = x(t) \hat{i} + y(t) \hat{j} + z(t) \hat{k}$$

Average velocity



5-2

Consider a particle in a plane moving from P to Q. Position vector at P is \vec{r} . The change in position vector is $\Delta\vec{r}$.

$$\Delta\vec{r} = \Delta x \hat{i} + \Delta y \hat{j}$$

Let Δt be the time interval for the motion from P to Q. The average velocity of the particle is then defined as the vector quantity equal to the displacement divided by the time interval:

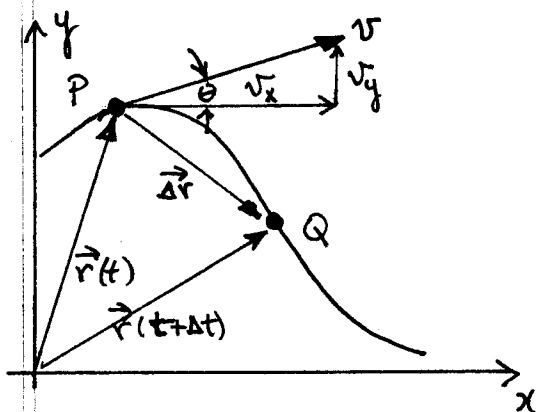
$$\vec{v} = \frac{\Delta\vec{r}}{\Delta t} = \frac{\Delta x}{\Delta t} \hat{i} + \frac{\Delta y}{\Delta t} \hat{j} + \frac{\Delta z}{\Delta t} \hat{k} \quad [3-D]$$

- direction of vector $\Delta\vec{r}$
- magnitude $\Delta r / \Delta t$

$$\text{Let } \bar{v}_x = \frac{\Delta x}{\Delta t}, \quad \bar{v}_y = \frac{\Delta y}{\Delta t}, \quad \bar{v}_z = \frac{\Delta z}{\Delta t}$$

$$\vec{v} = \bar{v}_x \hat{i} + \bar{v}_y \hat{j} + \bar{v}_z \hat{k}$$

Instantaneous velocity, \vec{v} , at the point P is defined in magnitude and direction as the limit approached by the average velocity when point Q is taken to be closer and closer to P (as $\Delta t \rightarrow 0$).



$$\vec{v} = \lim_{\Delta t \rightarrow 0} \frac{\Delta \vec{r}}{\Delta t} = \frac{d\vec{r}}{dt}$$

- direction of \vec{v} is tangent to path of particle at P.
- equal to time rate of change of position vector.
- describes motion at a particular point and particular instant of time

$$\vec{v} = \frac{d}{dt} [x \hat{i} + y \hat{j} + z \hat{k}]$$

$$= \frac{dx}{dt} \hat{i} + \frac{dy}{dt} \hat{j} + \frac{dz}{dt} \hat{k}$$

since $\frac{d\hat{i}}{dt} = 0$, etc.

$$\vec{v} = v_x \hat{i} + v_y \hat{j} + v_z \hat{k}$$

Magnitude of $|\vec{v}| = \sqrt{v_x^2 + v_y^2 + v_z^2}$

[rect. coord.]

$$\tan \theta = v_y / v_x$$

[2-D]

Represent \vec{v} : components, or magnitude and direction

speed

5-4

$$[\text{speed}] = v = \left| \frac{d\vec{r}}{dt} \right| = \sqrt{v_x^2 + v_y^2 + v_z^2}$$

- instantaneous speed is magnitude of velocity vector.
- velocity has direction and magnitude.

$$\text{average speed} = \frac{\text{distance travelled}}{\text{time}}$$

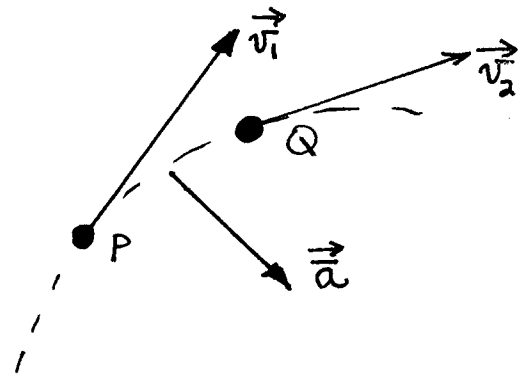
Acceleration

5-5

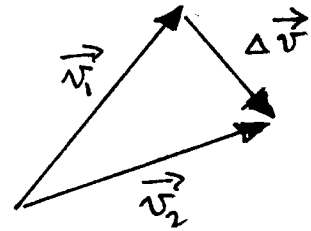
The average and instantaneous accelerations in 3-dimensions are straight forward generalizations of one-dimensional motion.

\vec{v}_1 and \vec{v}_2 are velocities for particle moving in a curved path at P and Q.

- different magnitudes
- different direction



$$\Delta \vec{v} = \vec{v}_2 - \vec{v}_1$$



The average acceleration, \vec{a} , of the particle as it moves from P to Q is defined as the vector change in velocity, $\Delta \vec{v}$, divided by the time interval Δt .

$$\vec{a} = \frac{\Delta \vec{v}}{\Delta t} = \frac{\vec{v}_2 - \vec{v}_1}{\Delta t} = \frac{\vec{v}(t+\Delta t) - \vec{v}(t)}{\Delta t}$$

$$= \frac{\Delta v_x}{\Delta t} \hat{i} + \frac{\Delta v_y}{\Delta t} \hat{j} + \frac{\Delta v_z}{\Delta t} \hat{k}$$

$$= \bar{a}_x \hat{i} + \bar{a}_y \hat{j} + \bar{a}_z \hat{k}$$

$\vec{a} \neq 0$ if velocity changes direction between P+Q or if velocity changes in magnitude.

The instantaneous acceleration, \vec{a} , at point P is defined in magnitude and direction as the limit approached by the average acceleration when point Q approaches point P and Δv and Δt both approach zero

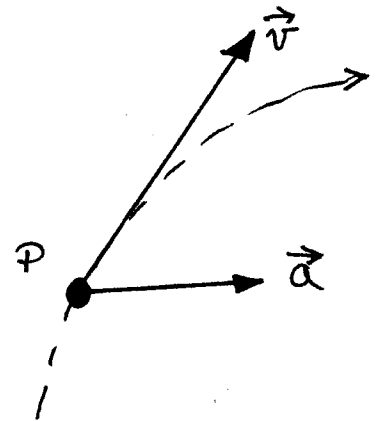
$$\vec{a} = \lim_{\Delta t \rightarrow 0} \frac{\Delta \vec{v}}{\Delta t} = \frac{d\vec{v}}{dt} = \frac{d^2 \vec{r}}{dt^2}$$

$$= a_x \hat{i} + a_y \hat{j} + a_z \hat{k}$$

$$a_x = \frac{dv_x}{dt} = \frac{d^2 x}{dt^2}$$

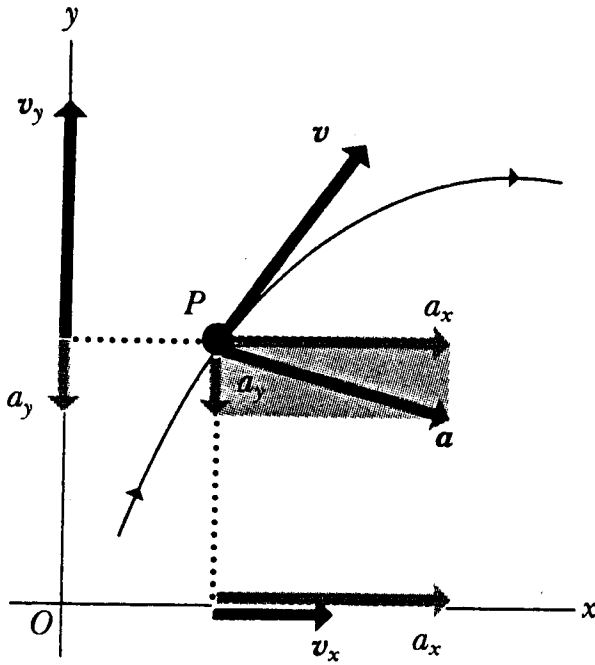
$$a_y = \frac{dv_y}{dt} = \frac{d^2 y}{dt^2}$$

$$a_z = \frac{dv_z}{dt} = \frac{d^2 z}{dt^2}$$



- Equal to time rate of change of velocity
- $\neq 0$ if velocity changes in magnitude or direction
- It does not have same direction as velocity vector.
- Acceleration vector lies on concave side of curved path.

- average acceleration refers to finite time interval during which velocity changes.
- instantaneous acceleration is the rate of change of velocity at a specific point at a specific time.



The acceleration a is resolved into its components a_x and a_y .

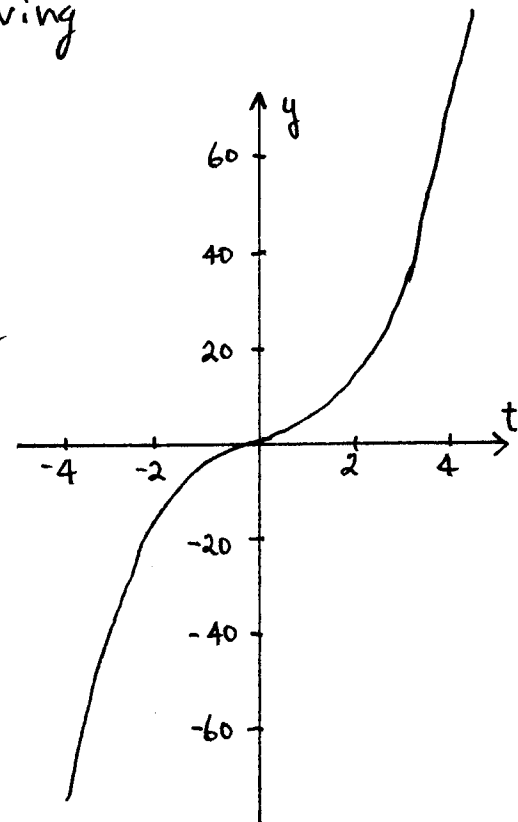
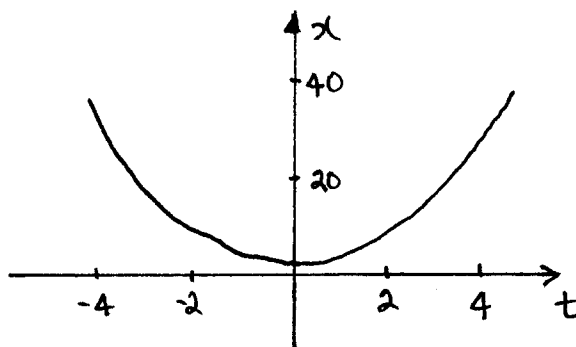
Example

The coordinates of a particle moving in the xy -plane are given by

$$x = 1 + 2t^2 \quad (\text{m})$$

$$y = 2t + t^3 \quad (\text{m})$$

Find the particle's position, velocity and acceleration at time $t = 2\text{s}$.



Position

$$x = 1 + 2(2)^2 = 9 \text{ m}$$

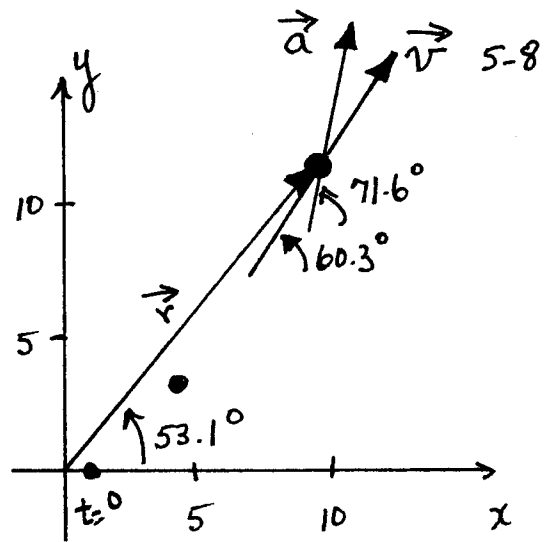
$$y = 2(2) + 1(2)^3 = 12 \text{ m}$$

$$\vec{r} = 9\hat{i} + 12\hat{j}$$

Distance from origin

$$r = \sqrt{x^2 + y^2} = \sqrt{9^2 + 12^2} = 15 \text{ m}$$

$$\theta = \tan^{-1} \frac{y}{x} = \tan^{-1} \frac{12}{9} = 53.1^\circ$$



Velocity

$$v_x = \frac{dx}{dt} = 4t \quad (\text{m/s})$$

$$v_y = \frac{dy}{dt} = 2 + 3t^2 \quad (\text{m/s})$$

At $t=2\text{s}$

$$v_x(2) = 8 \text{ m/s}$$

$$v_y(2) = 14 \text{ m/s}$$

$$\vec{v}(t=2) = 8\hat{i} + 14\hat{j}$$

$$v = |\vec{v}| = \sqrt{v_x^2 + v_y^2} = \sqrt{8^2 + 14^2} = 16.1 \text{ m/s}$$

$$\theta_v = \tan^{-1} \frac{v_y}{v_x} = \tan^{-1} \frac{14}{8} = 60.3^\circ$$

Acceleration

$$a_x = \frac{dv_x}{dt} = \frac{d^2x}{dt^2} = 4 \quad (\text{m/s}^2)$$

$$a_y = \frac{dv_y}{dt} = \frac{d^2y}{dt^2} = 6t \quad (\text{m/s}^2)$$

At $t=2\text{s}$

$$a_x = 4 \text{ m/s}^2 \quad a_y = 12 \text{ m/s}^2$$

$$\vec{a} = 4\hat{i} + 12\hat{j}$$

$$a = |\vec{a}| = \sqrt{a_x^2 + a_y^2} = \sqrt{4^2 + 12^2} = 12.6 \text{ m/s}^2$$

$$\theta_a = \tan^{-1} \frac{a_y}{a_x} = \tan^{-1} \frac{12}{4} = 71.6^\circ$$

Direction of $\vec{a} \neq \vec{v}$
and not tangent to path

Acceleration: a_{\perp} and a_{\parallel}

- Instructive to represent the acceleration of a particle moving along a curved path in terms of rectangular components:

a_{\perp} - normal to path

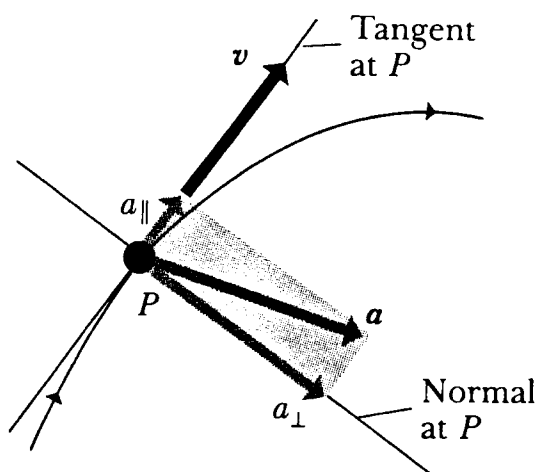
- associated with change in direction of \vec{v}

a_{\parallel} - parallel to path

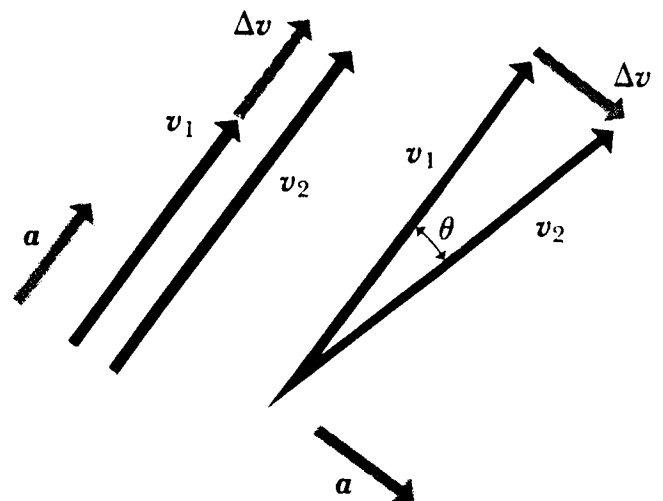
- associated with change in magnitude of \vec{v}

\vec{a} parallel to \vec{v} : Change in \vec{v}_1 during a small amount of time Δt is a vector $\Delta\vec{v}$ with same direction as \vec{a} and same direction as \vec{v}_1 . Velocity \vec{v}_2 at the end of Δt , is $\vec{v}_2 = \vec{v}_1 + \Delta\vec{v}$, same direction as \vec{v}_1 but somewhat greater in magnitude.

\vec{a} perpendicular to \vec{v} : In time interval Δt the change $\Delta\vec{v}$ is \perp to \vec{v}_1 . $\vec{v}_2 = \vec{v}_1 + \Delta\vec{v}$ but \vec{v}_1 and \vec{v}_2 differ in direction. As $\Delta t \rightarrow 0$, $\Delta\vec{v}$ is \perp to \vec{v}_1 and \vec{v}_2 . \vec{v}_1 and \vec{v}_2 have same magnitude.



(a)



(b)

(c)

$$\vec{a} = \frac{d\vec{v}}{dt}$$

$$d\vec{v} = \vec{a} dt$$

change in velocity is in same direction as acceleration vector.

using $\vec{a}_{||}$ and \vec{a}_{\perp} one easily sees that if

$$\vec{a} \cdot \vec{v} > 0 \quad \text{velocity increases}$$

$$\vec{a} \cdot \vec{v} < 0 \quad \text{velocity decreases.}$$

$$\vec{a} \cdot \vec{v} = 0 \quad \begin{array}{l} \text{velocity magnitude remains constant} \\ \text{velocity changes direction} \\ \Rightarrow \text{circular motion!!} \end{array}$$

Motion at Constant Acceleration

5-10

We had for the average acceleration

$$\vec{a} = \frac{\Delta \vec{v}}{\Delta t}$$

If initial velocity is \vec{v}_0 , then the velocity after time t is

$$\vec{v} = \vec{v}_0 + \vec{a}t \quad (1)$$

for constant acceleration.

The x , y , and z components are:

$$v_x = v_{0x} + a_x t$$

$$v_y = v_{0y} + a_y t$$

$$v_z = v_{0z} + a_z t$$

(2)

Using arguments as in the one-dimensional case the position vector becomes

$$\vec{r} = \vec{r}_0 + \vec{v}_0 t + \frac{1}{2} \vec{a} t^2 \quad (3)$$

In components:

$$x(t) = x_0 + v_{0x} t + \frac{1}{2} a_x t^2$$

$$y(t) = y_0 + v_{0y} t + \frac{1}{2} a_y t^2$$

$$z(t) = z_0 + v_{0z} t + \frac{1}{2} a_z t^2$$

(4)

In equations (2) and (4) the various components of the motion proceed independent of each other. Time is common to all motions.

x-velocity affected by x-acceleration

x-position affected by initial x-velocity and x-acceleration

