Dynamics of Uniform Circular Motion

For a particle moving with speed v in a circle of radius R, the centripetal acceleration is

$$a_c = \frac{v^2}{R}$$

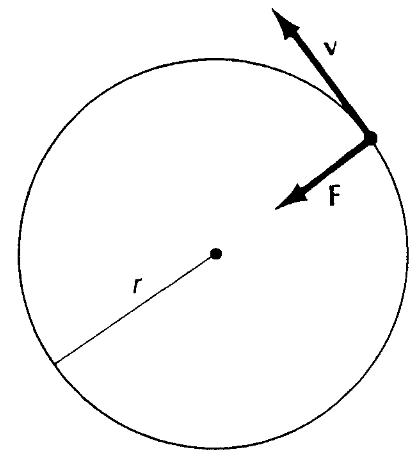


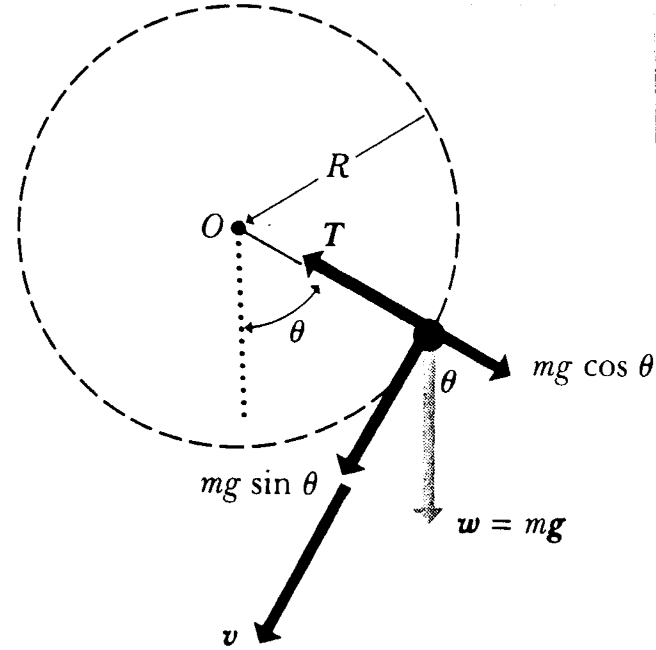
Fig. Centripetal force for a particle in uniform circular motion.

If the particle has a mass m, the required force is given by

$$F_c = m a_c = \frac{m v^2}{R}$$

Whenever a particle moves in a circle with a speed v, this force must be provided by some external agent.

Motion in a Vertical Circle.



Forces on a body whirling in a vertical circle with center at O.

- Ball whirled in a vertical circle about point O.
- Motion circular but is <u>not</u> uniform.
- Speed increases on the way down, decreases on the way up.
- v changes continuously around the path.
- ∴ We must have $a_\perp$ and $a_{\parallel}$
- Forces on ball are gravity and tension.

$$F_{\parallel} = mg \sin\theta \qquad (1)$$

$$F_{\perp} = T - mg\cos\theta \qquad (2)$$

The tangential acceleration:

$$a_{\parallel} = \frac{F_{\parallel}}{m} = g\sin\theta \qquad (3)$$

$$a_{\perp} = \frac{F_{\perp}}{m} = \frac{T - mg\cos\theta}{m} = \frac{v^2}{R} \qquad (4)$$

Solve (4) for $T = m\left(\frac{v^2}{R} + g\cos\theta\right)$

Lowest Point: $\theta = 0$

$F_{\parallel} = 0$, $a_{\parallel} = 0$, acceleration is purely radial.

$$T = m\left(\frac{v^2}{R} + g\right)$$

Highest Point: $\theta = 180^\circ$

- acceleration purely radial

$$T = m\left(\frac{v^2}{R} - g\right)$$

If speed equals a critical value v_c, tension vanishes, $T \equiv 0$.

$$0 = m\left(\frac{v_c^2}{R} - g\right)$$

$$v_c = \sqrt{Rg}.$$

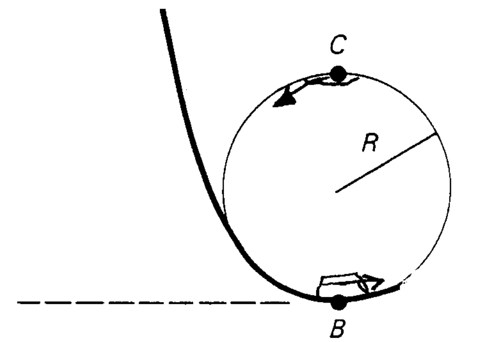
The speed at any point can be determined from energy considerations given a value at some initial point.

Example : Loop-the-Loop

$m = 75\,kg$

$R = 10\,m$

$v_0 = 15\,m/s$

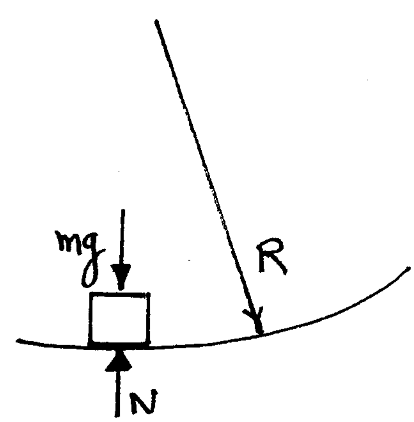


Man in a car executes motion in a vertical loop at a constant speed v_0.
Calculate the apparent force the man feels pressing against him at the bottom and top of the loop. i.e. Calculate the apparent weight.

Bottom

$$N - mg = \frac{m v_0^2}{R} \quad \text{(circle)}$$

Normal force pushing on man (weight)

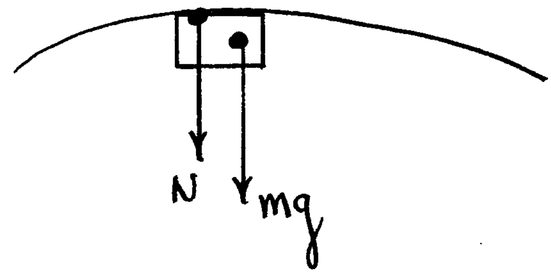


$$N = m\left(g + \frac{v_0^2}{R}\right) = mg\left[1 + \frac{1}{g}\frac{v_0^2}{R}\right]$$

$$= mg\left[1 + \frac{1}{9.81}\frac{15^2}{10}\right] = mg\left[1 + 2.29\right]$$

$$= 3.29\,(mg)$$

<u>Top</u>



$$N + mg = \frac{m v_0^2}{R}$$

$$N = mg\left[\frac{v_0^2}{Rg} - 1\right]$$

$$= mg\left[2.29 - 1\right]$$

$$= 1.29(mg)$$

What happens if v_0 is decreased?
In particular what if

$$\frac{v_0^2}{Rg} = 1 \implies v_0 = \sqrt{gR} = \sqrt{9.81 \times 10}$$

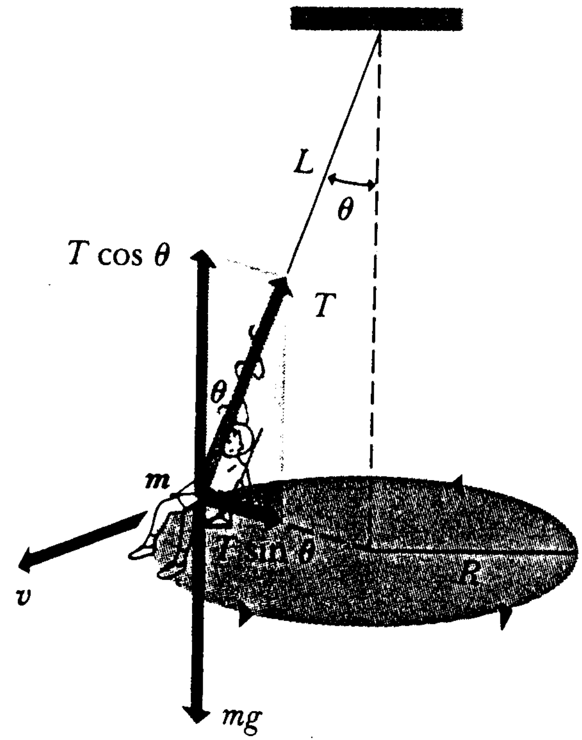
$$= 9.90 \text{ m/s}$$

$N \equiv 0$; No normal force.

What if $v_0 < \sqrt{gR}$??

Conical Pendulum

Object at end of rope of length L. Total mass is m. Object swings in a horizontal circle of radius R at constant speed v.



A child's indoor swing.

Assume period of revolution is τ.
Find tension in rope T and angle θ.

$$\sum \vec{F} = m\vec{a}$$

$$T \sin\theta = m\frac{v^2}{R} \quad ①$$

$$T\cos\theta - mg = 0 \quad ②$$

$$\tan\theta = \frac{v^2}{Rg} \quad ③$$

$$R = L\sin\theta$$

$$v = \frac{2\pi L \sin\theta}{\tau}$$

Then Eq. ③ becomes

$$\cos\theta = \frac{g\tau^2}{4\pi^2 L}$$

or $$\tau = 2\pi\sqrt{\frac{L}{g}\cos\theta}$$

Work and Energy

- Newton's Laws of Motion relating acceleration to forces acting enable us to predict future values of the position and velocity of a particle.
- In this lecture we will see how to relate force to particle motion in a second way. The scalar product of force and displacement defines work. The product of mass and the square of a particle's velocity gives twice the kinetic energy.
- Combining work and K.E. we derive the work-energy principle. This principle plays a role which is analagous to that of Newton's Second Law.

Work-Energy Principle $\longrightarrow$ Conservation of Energy

Δt(Force) = Linear Impulse $\longrightarrow$ Conservation of Momentum

Force $\otimes$ Position $\rightarrow$ Torque $\rightarrow$ Conservation of Angular Momentum

Conservation Laws $\ll\!\!\longrightarrow\!\!\gg$ Symmetry

- Conservation laws are closely connected to a symmetry associated with our mathematical description of the physics:

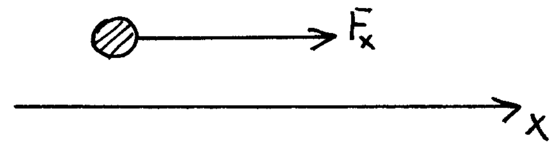
Momentum $\longleftrightarrow$ Translation in Space

Energy $\longleftrightarrow$ Translation in Time

Angular Momentum $\longleftrightarrow$ Invariance to Rotations

Electric Charge $\longleftrightarrow$ QM Shift in Phase

Work – 1 Dimension



Force F_x acting on a particle moving along x does an amount of work:

$$W = F_x \, \Delta x$$

Δx ← displacement of particle

W ← Work done by the force F_x. Work is a scalar quantity.

$[W] = N \cdot m = \underline{J}oules$

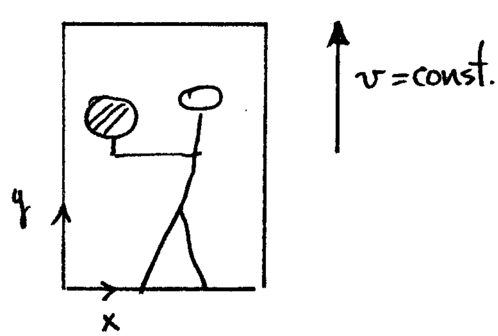
$W > 0$ • Force and displacement are in the same direction

< 0 • Force and displacement are opposed

< 0 • Represents work done <u>by</u> the particle.

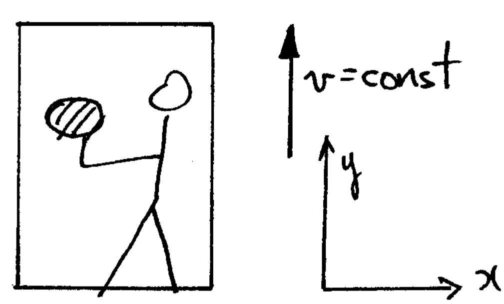
– When several forces act, add individual contributions.

Work is dependent on the frame of reference.



$W = 0$ $\Delta x = 0$

Elevator Frame



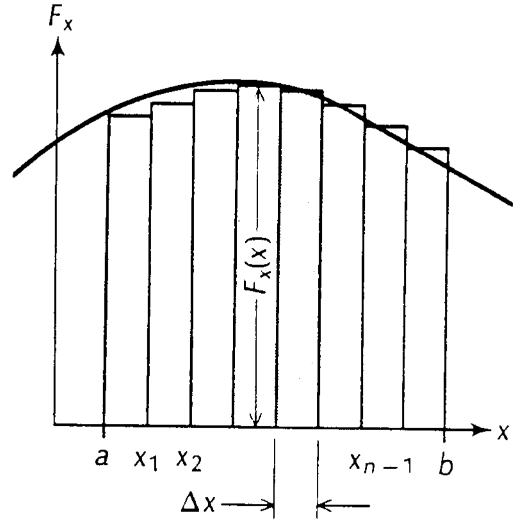
$W \neq 0$ $\Delta x \neq 0$

Laboratory Frame

Variable Force/Work.

$F_x = F_x(x)$

- Force is a function of position, spring, gravity, etc.



The total displacement from a to b has been divided into small intervals of length Δx.

What is $W(a \to b)$ in moving a particle from $x = a$ to $x = b$.

Divide the total displacement into a large number of small intervals Δx.

For each interval

$\Delta W_i = F_x(x_i)\,\Delta x \Rightarrow$ Area of a rectangle of height F_x and width Δx.

Total work from $x=a$ to $x=b$ is the sum of all such intervals.

$$W = \sum_{i=0}^{i=n-1} \Delta W_i = \sum_{i=0}^{i=n-1} F_x(x_i)\,\Delta x$$

In the limiting case $\Delta x \to 0$
$n \to \infty$

$$W = \lim_{\Delta x \to 0} \sum_i F_x(x_i)\,\Delta x$$

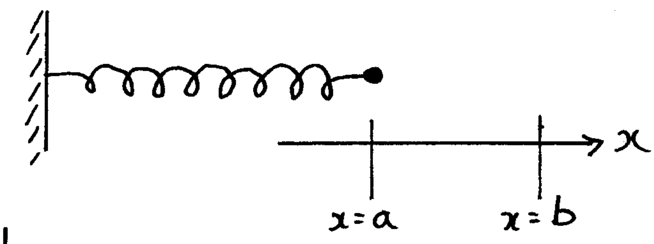
$$W = \int_a^b \underbrace{F_x(x)\,dx}_{\text{integrand}}$$

[Definite Integral]

Work $\equiv$ Area bounded by the curve $F_x(x)$ and the lines $x=a$ and $x=b$ and the x-axis

Example: Spring Force

$F(x) = -kx$



How much work is needed to move a spring (fixed at one end) from $x=a$ to $x=b$?

$$W = \int_a^b F(x)\,dx$$

$$= \int_a^b (-kx)\,dx$$

$$= -\frac{kx^2}{2}\Bigg|_a^b$$

$$= -\frac{k}{2}(b^2-a^2)$$

$W<0$, work is done on the spring!!

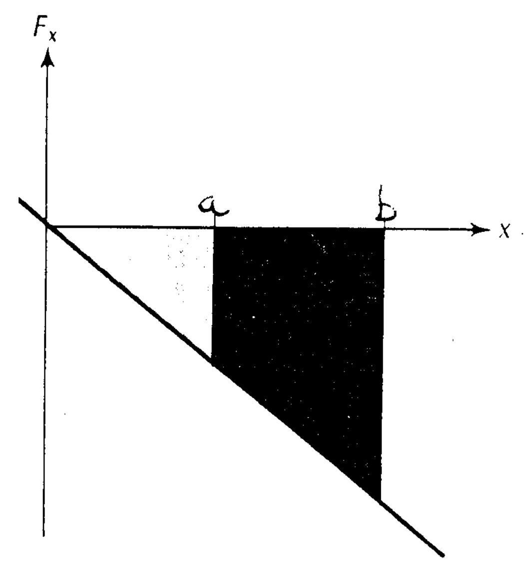


Fig. The straight line is a plot of the function $F_x(x) = -kx$. The quadrilateral area to the right of a and the left of b represents the work.

Work – 3 Dimensions

In general:

$$W = \vec{F} \cdot \vec{\Delta r}$$

↑ displacement.

constant force

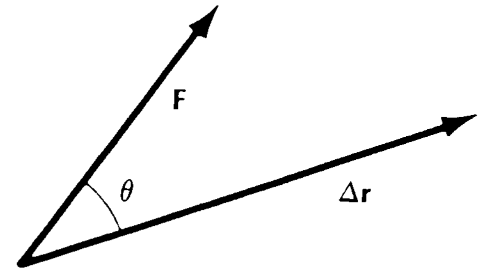


Fig. A constant force **F** acts during a displacement **Δr**.

$$= F(\Delta r)\cos\theta$$

$$W = 0 \text{ if } \vec{F} \perp \vec{\Delta r}$$

$$W = F_x(\Delta x) + F_y(\Delta y) + F_z(\Delta z)$$

[Using components]

If $F = F(x)$ is not constant,

$$\Delta W = F(\Delta r)\cos\theta$$

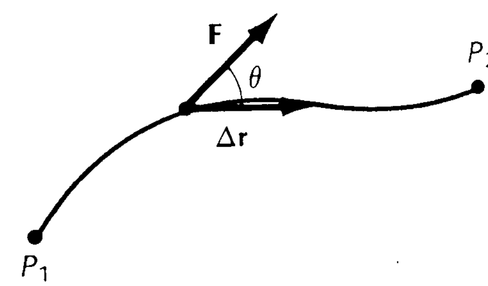


Fig. A force **F** acts during a small displacement **Δr**.

Limiting case $\Delta r \to 0$

$$W = \int_{P_1}^{P_2} \vec{F} \cdot \vec{dr}$$

[Line-Integral]

$$= \int_{P_1}^{P_2} (F_x\,dx + F_y\,dy + F_z\,dz)$$

$$W = \int_{P_1}^{P_2} F_x\,dx + \int_{P_1}^{P_2} F_y\,dy + \int_{P_1}^{P_2} F_z\,dz$$

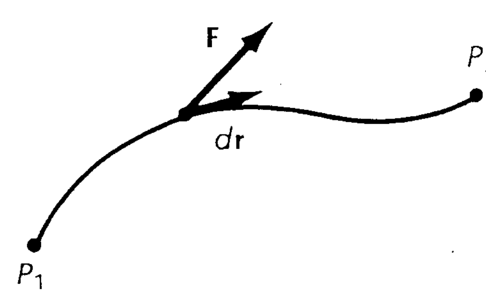


Fig. The infinitesimal displacement $d\mathbf{r}$ is tangent to the path.

Example: Gravitational Force

$$\left.\begin{aligned} F_x &= 0 \\ F_y &= 0 \\ F_z &= -mg \end{aligned}\right\} \text{Force of Gravity } \underline{\text{close}} \text{ to earth's surface.}$$

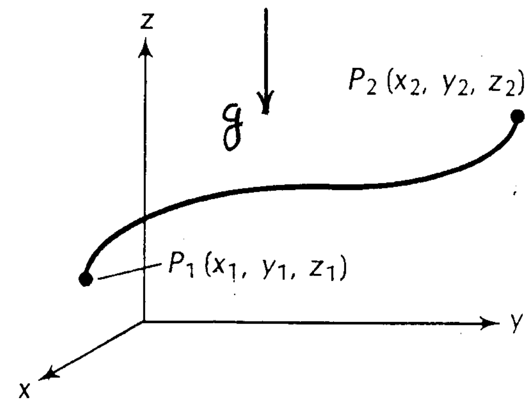


Fig. Path of a particle. The z axis is directed vertically upward.

$$W = \cancel{\int_{x_1}^{x_2} F_x\, dx} + \cancel{\int_{y_1}^{y_2} F_y\, dy} + \int_{z_1}^{z_2} F_z\, dz$$

$$W = \int_{z_1}^{z_2} -mg\, dz = -mg(z_2 - z_1)$$

$$W = -mg\,(\Delta z)$$

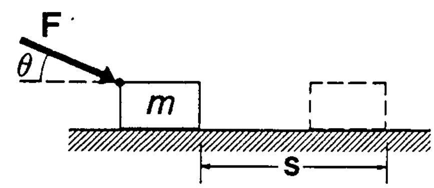
← change in height.

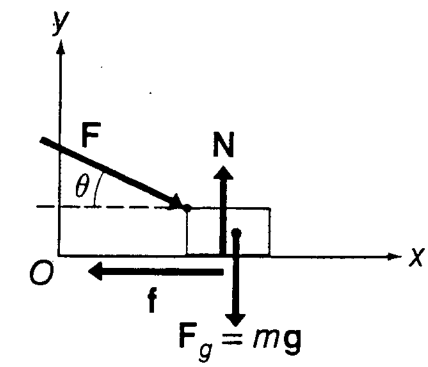
Work done by gravity depends on the vertical separation between P_1 and P_2.

NB: Complication or details of the motion do not count — only (Δz) !!

Example

A block is pushed along a horizontal plane at constant velocity. Coefficient of sliding friction is μ_k. How much work is done in a distance s by the force F?





$F\cos\theta - f = 0$ (constant v)

$-F\sin\theta + N - mg = 0$

$$f = \mu_k N = \mu_k (F\sin\theta + mg)$$

$$\therefore \quad F\cos\theta - \mu_k (F\sin\theta + mg) = 0$$

$$\therefore \quad F = \frac{\mu_k mg}{\cos\theta - \mu_k \sin\theta}$$

$$W_F = \vec{F} \cdot \vec{ds} \qquad \text{[Constant Force]}$$

$$= F s\cos\theta = \frac{\mu_k mg s}{1 - \mu_k \tan\theta}$$

Q: What happens when $\mu_k \tan\theta = 1$?

i.e. $\tan\theta = \dfrac{1}{\mu_k}$ $\qquad W \rightarrow \infty$???

$$f = F\cos\theta$$

$$= \frac{\mu_k mg\cos\theta}{\cos\theta - \mu_k \sin\theta}$$

$$W_f = \vec{f}\cdot\vec{ds}$$

$$= \frac{-\mu_k mgs\cos\theta}{\cos\theta - \mu_k\sin\theta}$$ [$\vec{f}$ and $\vec{ds}$ are opposite]

$W_N = 0$ since $\vec{N}$ and $\vec{ds}$ are $\perp$.

$W_{mg} = 0$ ✓ $m\vec{g}$ and $\vec{ds}$ are $\perp$

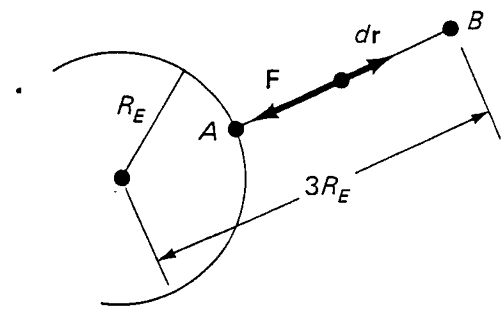
Total work on object:

$$W = W_F + W_f + W_N \overset{+W_{mg}}{} \equiv 0.$$

True since object moves at constant velocity, net force on object $\vec{F_R} \equiv 0$.

Example: Work Done on an Astronaut

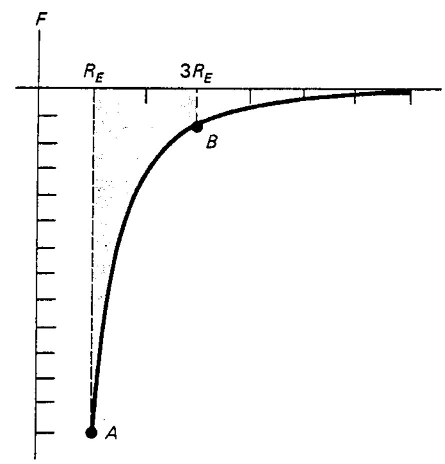
What is the work done by the force of gravity on an 80-kg astronaut in a displacement from point A at the earth's surface to point B whose altitude is $2R_E$. (earth radii).



$$F = -\frac{G M_E m}{r^2}\hat{r}$$

$$W = -\int_{R_E}^{3R_E} \frac{G M_E m}{r^2}\,dr$$

$$\int \frac{1}{r^2}\,dr = -\frac{1}{r}$$



Figure

The force acting on a object that is moving away from the earth yields the negative shaded area,

$$W = +\frac{G M_E m}{r}\Bigg|_{R_E}^{3R_E} = G M_E m\left(\frac{1}{3R_E} - \frac{1}{R_E}\right)$$

$$= -\frac{2}{3}\frac{G M_E m}{R_E} = -\frac{2}{3} m g R_E$$

Negative work: Force directed towards earth, displacement is away from earth.

$$W = -3.34 \times 10^9 \text{ J.}$$