

Kinetic Energy

14-9

Want to develop a relationship between the work done and the change in speed of a particle.

Particle moves from P_1 to P_2 under the action of a net force \vec{F} (position).

$$W = \int_{P_1}^{P_2} \vec{F} \cdot d\vec{r}$$

$$\vec{F} = F_x \hat{i} + F_y \hat{j} + F_z \hat{k}$$

$$d\vec{r} = dx \hat{i} + dy \hat{j} + dz \hat{k}$$

$$W = \int_{P_1}^{P_2} (F_x dx + F_y dy + F_z dz)$$

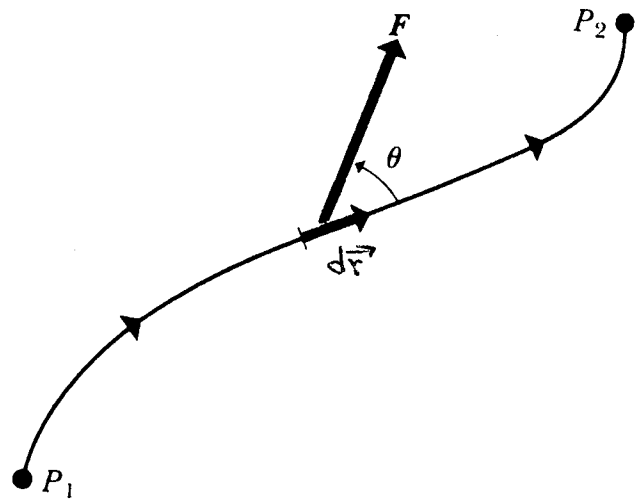
$$F_x = ma_x = m \frac{dv_x}{dt}$$

$$\int_{P_1}^{P_2} F_x dx = \int_{P_1}^{P_2} m \frac{dv_x}{dt} dx$$

\uparrow v_x is a function of time.

look at v_x as a function of position:

$$\frac{dv_x}{dt} = \frac{dv_x}{dx} \left(\frac{dx}{dt} \right) = \frac{dv_x}{dx} \cdot v_x = v_x \frac{dv_x}{dx}$$



A particle moves along a curved path from point P_1 to P_2 , acted on by a force F that varies in magnitude and direction.

$$\begin{aligned}
 \therefore \int_{P_1}^{P_2} F_x dx &= \int_{P_1}^{P_2} m \frac{dv_x}{dt} dx = \int_{P_1}^{P_2} m v_x \frac{dv_x}{dx} dx \\
 &= \int_{P_1}^{P_2} m v_x dv_x = \frac{1}{2} m v_x^2 \Big|_{v_{x1}}^{v_{x2}} \\
 &= \frac{1}{2} m (v_{x2}^2 - v_{x1}^2)
 \end{aligned}$$

v_{x1} = velocity in x-direction at P_1
 v_{x2} = velocity in x-direction at P_2 .

Do the same for terms in y and z.

$$\begin{aligned}
 W &= \frac{1}{2} m \left[v_{x2}^2 + v_{y2}^2 + v_{z2}^2 - (v_{x1}^2 + v_{y1}^2 + v_{z1}^2) \right] \\
 &= \frac{1}{2} m (v_2^2 - v_1^2)
 \end{aligned}$$

$$\boxed{W = \frac{1}{2} m v_2^2 - \frac{1}{2} m v_1^2}$$

Define: $\boxed{K = \frac{1}{2} m v^2}$ \equiv Kinetic Energy of Particle

KE: Potential for a particle to do work by virtue of its velocity.

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The work done on the particle by the net force equals the change in kinetic energy of the particle.

$$W = K_2 - K_1$$

or

$$\boxed{W = \Delta K}$$

\Rightarrow Work-Energy Theorem.

For a particle $\vec{p} = m\vec{v}$ (linear momentum)

$$\therefore K = \frac{1}{2m} p^2$$

Example:

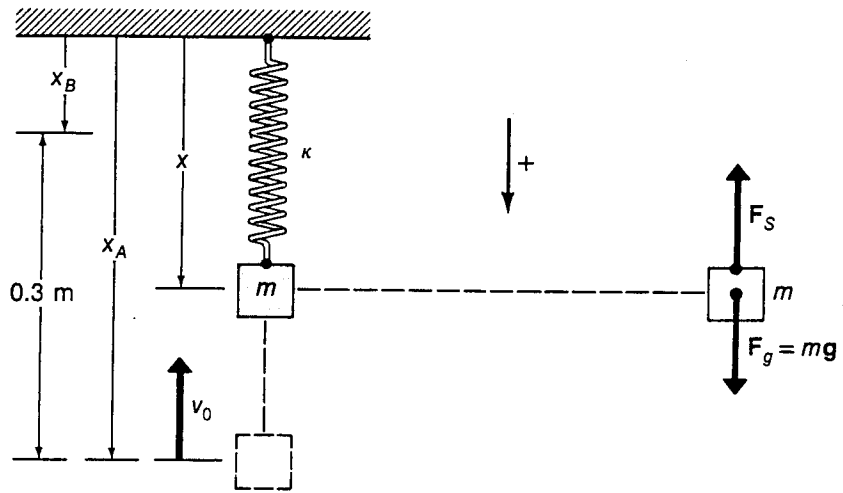
$$m = 0.50 \text{ kg.}$$

$$k = 50 \text{ N/m}$$

$$x_A = 0.50 \text{ m}$$

$$x_B = 0.20 \text{ m}$$

$$v_0 = 2.0 \text{ m/s.}$$



Mass pulled down to x_A , released by imparting an upward velocity v_0 . What is velocity of block at x_B ?

Net force on block:

$$F = F_g - F_s = mg - kx.$$

Work done, integrate $F dx$:

$$W(x_A \rightarrow x_B) = \int_{x_A}^{x_B} F dx = \int_{x_A}^{x_B} (mg - kx) dx = \left(mgx - \frac{1}{2} kx^2 \right) \Big|_{x_A}^{x_B}$$

$$W = mg(x_B - x_A) - \frac{1}{2} k (x_B^2 - x_A^2)$$

$$= 0.5 \times 9.8 (0.2 - 0.5) - \frac{1}{2} \times 50 (0.2^2 - 0.5^2)$$

$$= -1.47 + 5.25 = 3.78 \text{ J}$$

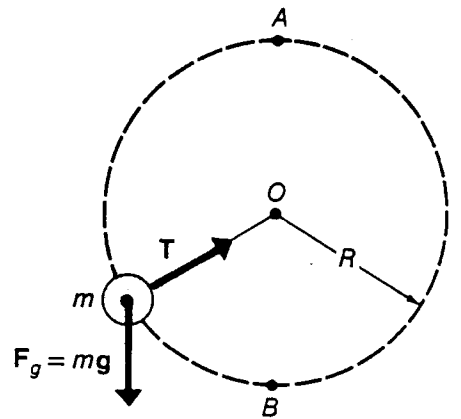
Also $W = \frac{1}{2} m v_B^2 - \frac{1}{2} m v_A^2$

$$\therefore v_B^2 = v_A^2 + \frac{2W}{m} = \left[(-2.0)^2 + \frac{2 \times 3.78}{0.5} \right] \Rightarrow v_B = \pm 4.37 \text{ m/s}$$

Example

A rock of mass m is tied to the end of a string and whirled in a circular path in the vertical plane.

What is the minimum speed v_0 that the rock has at the bottom (B) if the rock is to pass the top (A) with the string remaining taut?



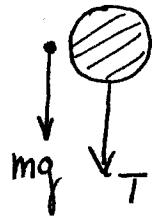
- Energy considerations alone not sufficient.
- Need to apply Newton's 2nd law.

At the top the minimum speed occurs as tension vanishes, $T \rightarrow 0$.

Net downward force is

$$F_g = mg = m \frac{v_T^2}{R}$$

$$v_T^2 = gR$$



[centripetal acceleration in circular path]

$$W(A \rightarrow B) = mgh = K_B - K_A$$

$$mg(2R) = \frac{1}{2}mv_0^2 - \frac{1}{2}mv_T^2$$

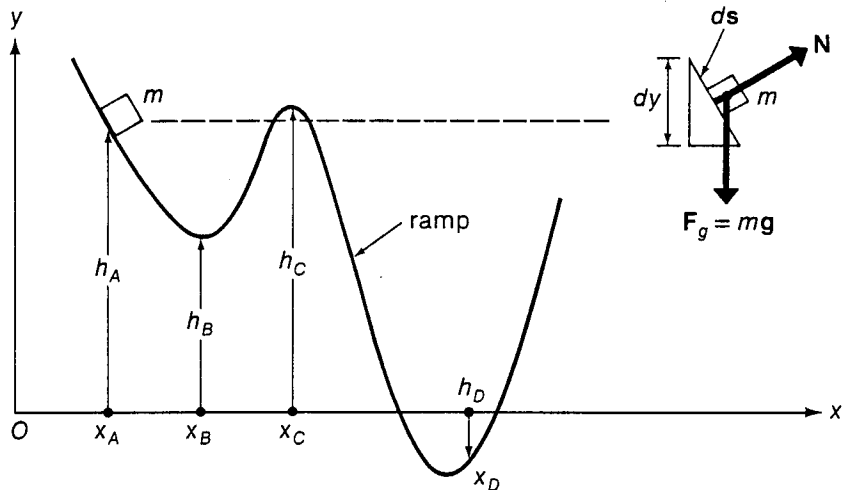
$$v_0^2 = v_T^2 + 4gR = gR + 4gR = 5gR$$

$$v_0 = \sqrt{5gR}$$

Example

$$\begin{aligned}h_A &= 7\text{m} \\h_B &= 4\text{m} \\h_C &= 7.2\text{m} \\h_D &= -1\text{m}\end{aligned}$$

$v_A = 3\text{m/s}$ downward
and tangent to ramp.



14-14

What is speed of particle at $x = x_B, x_C, x_D$?

$\vec{N} \cdot d\vec{s} \equiv 0$, so work is done only by gravitational force.

$$W(A \rightarrow B) = \int_A^B \vec{F} \cdot d\vec{s} = \int_A^B (-mg) dy$$

$$W = mg(h_A - h_B)$$

$$\text{Also } W = K_2 - K_1 = \frac{1}{2} m v_B^2 - \frac{1}{2} m v_A^2$$

$$\therefore v_B^2 = v_A^2 + \frac{2W}{m} = v_A^2 + 2g(h_A - h_B)$$

$$= 3^2 + 2 \times 9.8(7 - 4)$$

$$= 67.8 \text{ (m/s)}^2$$

$$v_B = 8.23 \text{ m/s.}$$

How far up the ramp will particle go after passing $x = x_D$?

h_m will be reached at x_m where $v_m = 0$.

$$0 = v_A^2 + 2g(h_A - h_m) \quad \Rightarrow \quad h_m = h_A + v_A^2 / 2g \quad h_m = 7.46\text{m}$$

Example

14-15

Car: $m = 1000 \text{ kg}$
 $v_1 = 20 \text{ m/s}$
 $v_2 = 30 \text{ m/s}$

$$\begin{aligned} W &= \frac{1}{2} m v_2^2 - \frac{1}{2} m v_1^2 \\ &= \frac{1}{2} m (v_2^2 - v_1^2) \\ &= \frac{1}{2} \times 1000 \times (900 - 400) \\ &= 2.5 \times 10^5 \text{ J} \end{aligned}$$

Gravitational Potential Energy

15-1

KE: Represents the capacity of a particle to do work by virtue of its velocity.

PE: Represents the capacity of a particle to do work by virtue of its position in space.

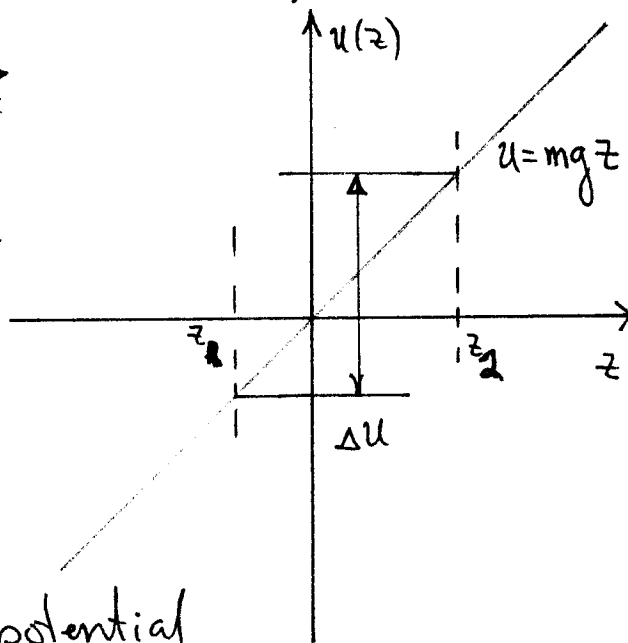
Consider a constant force of gravity

$$\vec{F}_z = -mg \hat{z},$$

acting on a particle which undergoes a displacement from (x_1, y_1, z_1) to (x_2, y_2, z_2) .

The force does an amount of work;

$$\begin{aligned} W_{gr} &= - \int_{z_1}^{z_2} \vec{F} \cdot d\vec{z} \\ &= -u(z_2) + u(z_1) = -\Delta u \end{aligned}$$



where,

$$u(z) = mgz,$$

is called the gravitational potential energy.

The change in potential energy between the points z_1 and z_2 is the negative of the work done by gravity on the particle.

Gravitational Potential Energy:

⇒ Capacity of a particle to do work by virtue of its height above the surface of an attracting mass (earth).

If the only force acting is gravity, then using the work-energy theorem,

$$W_{gr} = K_2 - K_1$$

and

$$W_{gr} = -U(z_2) + U(z_1)$$

$$\therefore K_1 + U(z_1) = K_2 + U(z_2)$$

∴ $K + U(z) \equiv$ constant of the motion.

$E = K + U(z)$: Mechanical Energy.

↳ Represents the total capacity of a particle to do work by virtue of both its velocity and its position.

If only force acting is gravity;

$$E = K + U(z) = \text{constant}$$

Law of Conservation of Mechanical Energy

$$E = \frac{1}{2}mv^2 + mgz = \text{constant}$$

z - increases \rightarrow v - decreases

z - decreases \rightarrow v - increases

Consider two different positions:

$$\frac{1}{2}mv_1^2 + mgz_1 = \frac{1}{2}mv_2^2 + mgz_2$$

$$v_1^2 + 2gz_1 = v_2^2 + 2gz_2$$

$$v_2^2 - v_1^2 = 2g(z_1 - z_2)$$

$$= -2g(z_2 - z_1)$$

Recall constant acceleration kinematics results.

Gravity + other forces

15-4

Suppose other forces act besides gravity: friction, etc.

Let W_{other} represent work by all forces ~~other~~ than gravity.

Total work by all forces equals change in KE.

$$W = W_{\text{grav}} + W_{\text{other}} = K_2 - K_1 = \Delta K$$

$$W_{\text{other}} = (mgz_2 - mgz_1) = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2$$

$$W_{\text{other}} = \left(\frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2\right) + (mgz_2 - mgz_1)$$

$$= \Delta K + \Delta U$$

↑ change in KE.
↑ change in PE

Also

$$W_{\text{other}} = \left(\frac{1}{2}mv_2^2 + mgz_2\right) - \left(\frac{1}{2}mv_1^2 + mgz_1\right)$$

$$= (K_2 + U_2) - (K_1 + U_1)$$

$$= E_2 - E_1 = \Delta E$$

The work done by all other forces acting on the body, with the exception of the gravitational force equals the change in the total mechanical energy of the body.

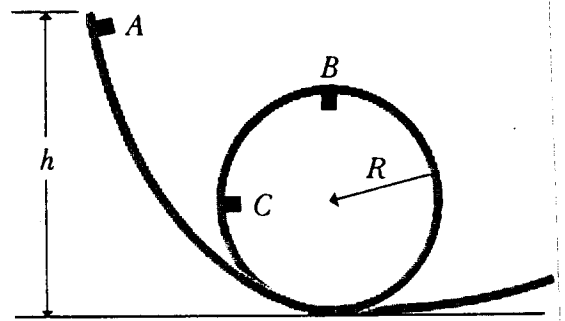
$W_{\text{other}} > 0 \Rightarrow$ Mechanical Energy increases

$W_{\text{other}} < 0 \Rightarrow$ decreases

Example: Loop-the-Loop

15-5

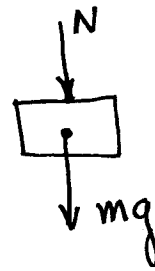
What is the minimum height H (in terms of R) such that an object moves around the loop without falling off the top (B)?



Top of Hoop:

velocity: v_T

$$F_c = mg + N = \frac{mv_T^2}{R}$$



Minimum $v_T \Rightarrow N \equiv 0$. [contact force vanishes]

$$\therefore v_T = \sqrt{gR}$$

Represents minimum velocity to make the loop.

Conservation of Energy

$$mgH + 0 = mg(2R) + \frac{1}{2}mv_T^2$$

$$v_T^2 = 2g[H - 2R]$$

$$v_T = \sqrt{2g(H - 2R)}$$

[Energy Considerations]

$$\therefore gR = 2g(H - 2R) \Rightarrow H = \frac{5}{2}R$$