Energy Curves

Assume we know the PE curve for a particle moving in one-dimension. What is the description of the particle motion as a function of time?

Given: $\mathcal{U}(x)$

Want: $x(t)$

Consider conservative forces only. Then the total mechanical energy is a constant of the motion.

$$E = K + \mathcal{U} = \text{Constant}$$

$$= \frac{1}{2} m v^2 + \mathcal{U}(x)$$

$$= \frac{1}{2} m \left(\frac{dx}{dt}\right)^2 + \mathcal{U}(x)$$

Solve for v_x:

$$v_x = \frac{dx}{dt} = \sqrt{\frac{2}{m}\left[E - \mathcal{U}(x)\right]}$$

$$\int_{x'=x_0}^{x} \frac{dx'}{\sqrt{\frac{2}{m}\left[E - \mathcal{U}(x')\right]}} = \int_{t'=0}^{t} dt' ,$$

where $x' = x_0$ at $t' = 0$.

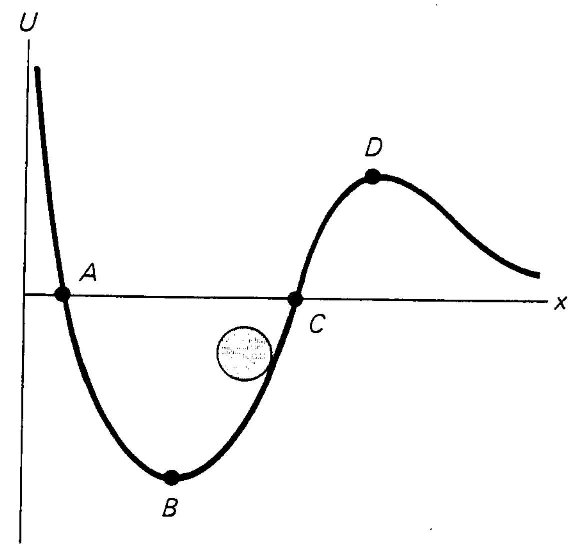
In general such an integral is not straightforward to evaluate analytically.

If we look at a PE diagram can we make some general statements about the particle motion?

Example

One dimensional potential for an alpha particle near a gold nucleus.

$r_0 \sim 10^{-13}$ cm



$$F = -\frac{\partial \mathcal{U}(x)}{\partial x}$$

i) At $x = x_B$ and $x = x_D$, $\frac{\partial \mathcal{U}}{\partial x} = 0$ and $F(x) = 0$.

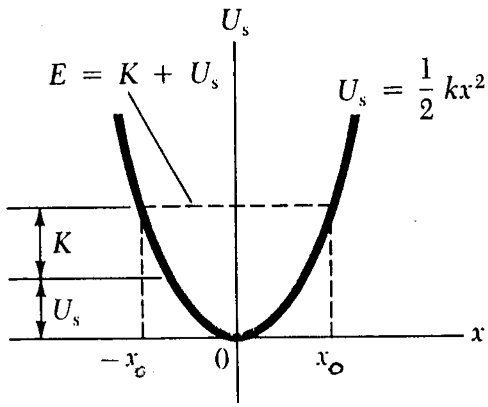
ii) If $\partial \mathcal{U}/\partial x \to 0$ as $x \to \infty$, then $F(x) \to 0$ as $x \to \infty$.

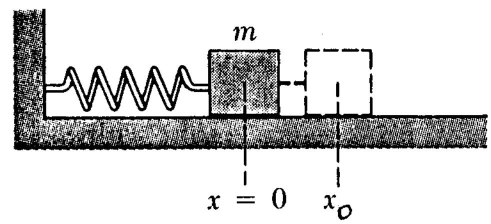
iii) $F(x) > 0$ for $0 < x < x_B$ since $\frac{\partial \mathcal{U}}{\partial x} < 0$. Also true for $x > x_D$. The force is therefore <u>repulsive</u> (away from origin)

iv) $F(x) < 0$ for $x_B < x < x_D$ since slope of $\mathcal{U}(x)$ is positive. Force is <u>attractive</u> (towards the origin).

Strategy: To remember force-potential energy relationship consider a ball rolling on a hilly surface under influence of gravity. The $\mathcal{U}$-axis is "up". The ball is pushed in the direction of F.

Example : Mass + Spring





Mass is pulled to the right $x = x_0$ and released with zero velocity.

$$E = \frac{1}{2} k x_0^2$$: Initial total energy, remains constant.

$$\therefore \frac{1}{2} m v^2 + \frac{1}{2} k x^2 = \frac{1}{2} k x_0^2$$

i) $x > 0$

$$\frac{\partial \mathcal{U}(x)}{\partial x} = kx > 0$$

$F < 0$, attractive ; accelerates mass towards equilibrium position.

ii) $x < 0$

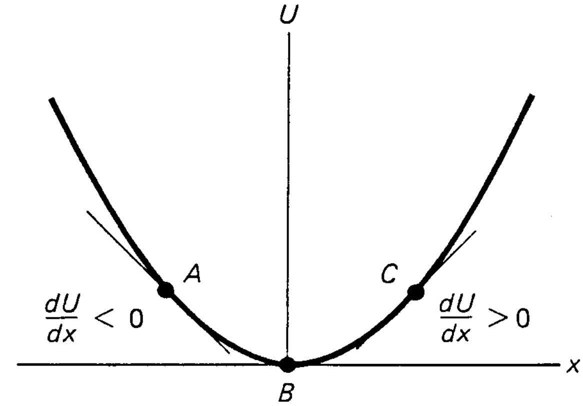
$$\frac{\partial \mathcal{U}(x)}{\partial x} = kx < 0$$

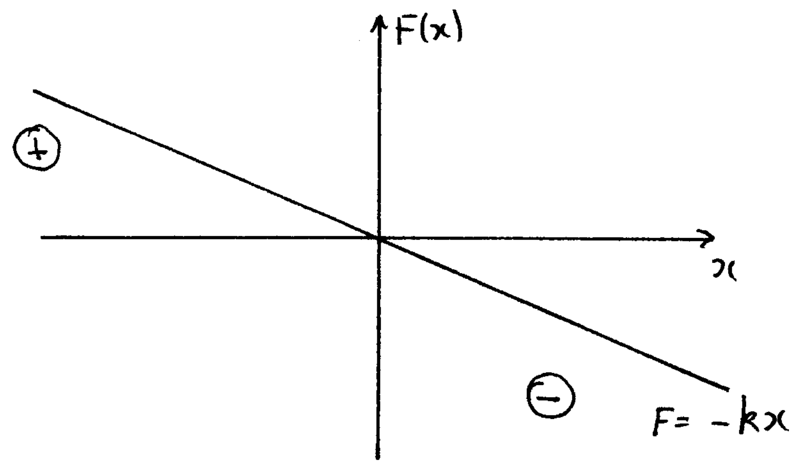
$F > 0$, attractive ; accelerates mass towards equilibrium position.

iii) $x = 0$

$$\frac{\partial U(x)}{\partial x} = 0$$

$\therefore \; F = 0$





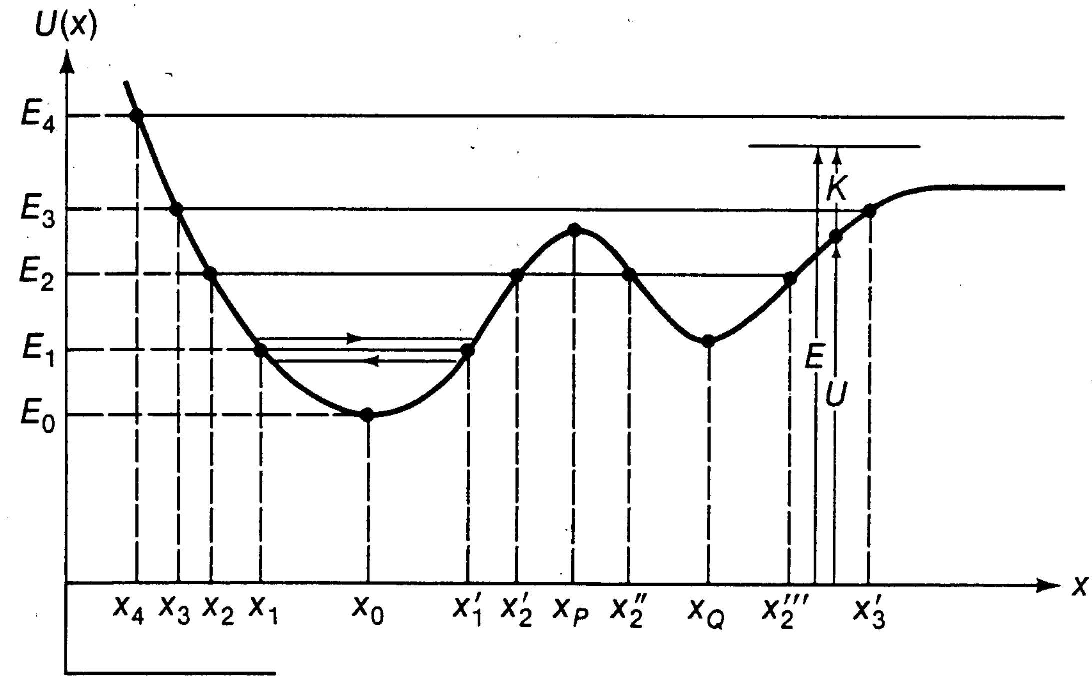
Harmonic Oscillator

Particle moves between turning points, whose limits are defined by the total energy E. As E increases, turning points move further apart. As E increases amplitude of oscillation increases.

If $E = 0$, $x = 0$ always.

Motion is always completely bounded.

Example



The graphical representation of the one-dimensional constrained potential energy function $U(x)$ for an object. Also shown are various values of the total mechanical energy.

Consider the motion for particles with different total energies $E = E_0, E_1, E_2$ and E_4. The value of E depends upon the initial conditions.

$$v_x = \sqrt{\frac{2}{m}\left(E - U(x)\right)}$$

In all cases $U(x) < E$, otherwise v_x is imaginary.

i) $E = E_0$

Particle stays fixed at $x = x_0$. It has PE but can have no KE. Point of <u>stable</u> equilibrium. It is at bottom of lowest valley in the system.

ii) $E = E_1$

Particle oscillates back and forth between the points x_1 and x_1'. Its KE at any point is given by the difference between E_1 and $U(x)$. Starting at x_1

it moves to the right [$F(x) > 0$] and increases speed up to x_0. Continues to move to right with decreasing speed to x_1', where it stops, turns around and starts back.

$\left.\begin{matrix} x_1 \\ x_1' \end{matrix}\right\}$ turning points / bounded motion [Particle is trapped in a potential well]

iii) $E = E_2$

Four turning points. Particle can move in <u>one</u> of <u>two</u> valleys, depending on where it is initially.
Quantum Mechanics: Particle can jump from one valley to the other → tunneling. Classically not allowed!!

iv) $E = E_4$

One turning point only.
$U(x) < E_4$ for all $x > x_4$.

Particle moving to left varies in speed as it passes over the valleys. It reverses direction at $x = x_4$ and moves with $x > x_4$ indefinitely → it never returns. <u>Unbounded motion.</u>

<u>Equilibrium</u>

$x = x_0$: Force on either side of x_0 acts to return particle to x_0.
"Stable Equilibrium"

$x = x_p$: Force acts to move particle away from $x = x_p$
"Unstable Equilibrium."

Equilibrium/Stability.

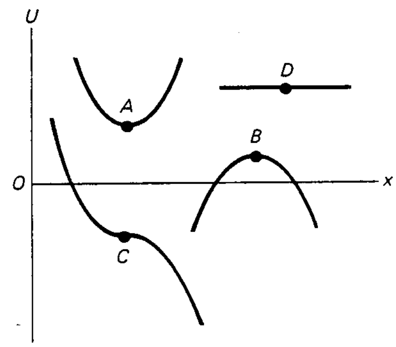
How do we tell if an equilibrium point is stable or unstable?

Mathematically: Examine sign of $\frac{d^2U}{dx^2}$!!

$\frac{d^2U}{dx^2} > 0$

Potential Minimum

Stable.



Figure

Points (A, B, C, D) on the potential-energy versus position graph with zero slope are equilibrium positions. The equilibrium is characterized as stable (A), unstable (B), or neutral (D), according to the graph in the neighborhood of the equilibrium point. For a curve like that near point C the equilibrium cannot be characterized simply as stable, unstable, or neutral.

$\frac{d^2U}{dx^2} < 0$

Potential Maximum

Unstable.

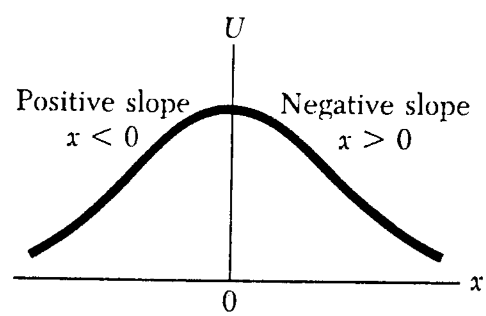


Figure A plot of U versus x for a system that has a position of unstable equilibrium, located at $x = 0$. In this case, the force on the system for finite displacements is directed away from $x = 0$.

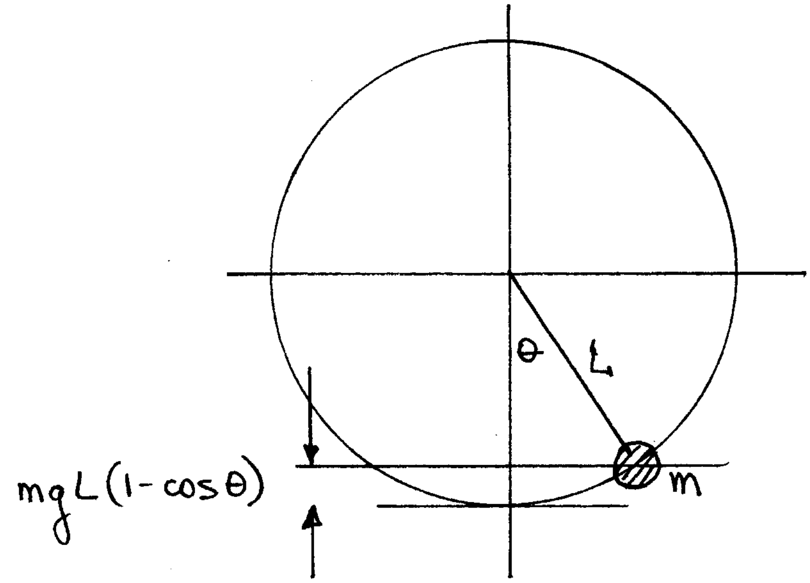
$\frac{d^2U}{dx^2} = 0$

U is constant over a region.

$F = 0$

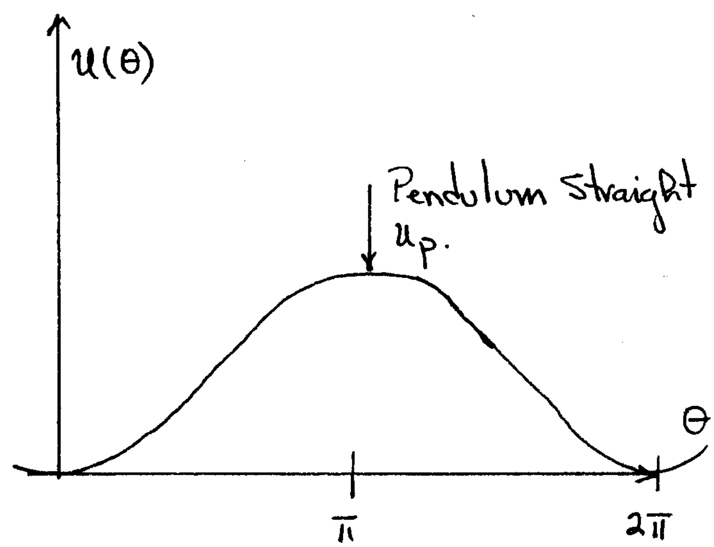
Neutral equilibrium.

Pendulum



$\dfrac{d^2U}{d\theta^2} > 0$ at $\theta = 0$ (stable)

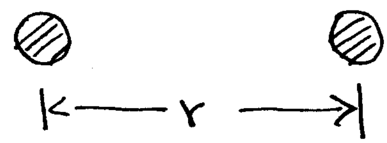
$\dfrac{d^2U}{d\theta^2} < 0$ at $\theta = \pi$ (unstable)



Example: Diatomic Molecule

Force between atoms in a diatomic molecule has its origin in the interactions between the electrons and nuclei in each atom. For simple atoms the potential energy is, to a good approx., represented by the Lennard-Jones potential (6, 12).

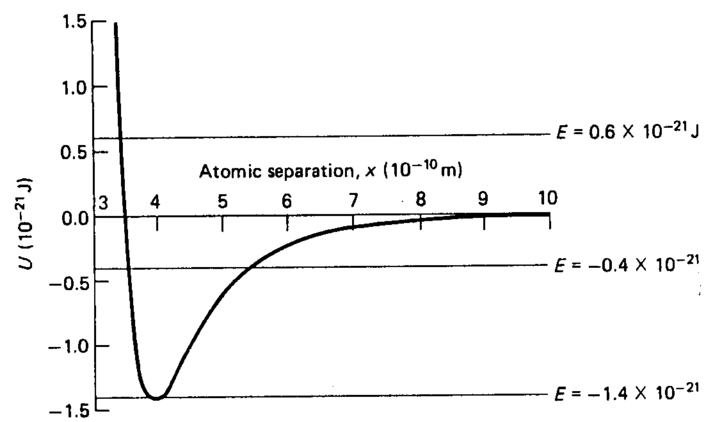
$$U(r) = U_0\left[\left(\frac{a}{r}\right)^{12} - \left(\frac{a}{r}\right)^{6}\right]$$



$r \equiv$ distance between atoms

$U_0, a \equiv$ constants.

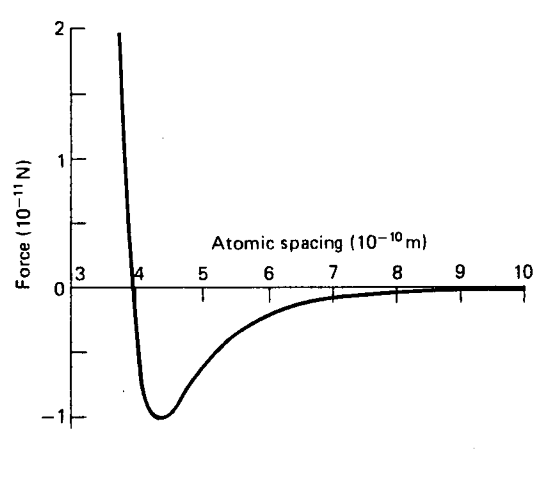
$$\left.\begin{array}{l} U_0 = 5.6 \times 10^{-21}\ \mathrm{J} \\ a = 3.5 \times 10^{-10}\ \mathrm{m} \end{array}\right\} O_2$$



$$F = -\frac{dU}{dr}$$

$$= -U_0\left[-12\frac{a^{12}}{r^{13}} + 6\frac{a^{6}}{r^{7}}\right]$$

$$= \frac{6U_0}{a}\left[2\left(\frac{a}{r}\right)^{13} - \left(\frac{a}{r}\right)^{7}\right]$$



At equilibrium $F = 0$

$$\therefore\ 2\left(\frac{a}{r_0}\right)^{13} - \left(\frac{a}{r_0}\right)^{7} = 0$$

$$\left(\frac{a}{r_0}\right)^{6} = \frac{1}{2}$$

$$r_0 = 2^{1/6}a = 2^{1/6}(3.5 \times 10^{-10}\ \mathrm{m})$$

$$= 3.9 \times 10^{-10}\ \mathrm{m} \quad (O_2 \text{ molecule})$$

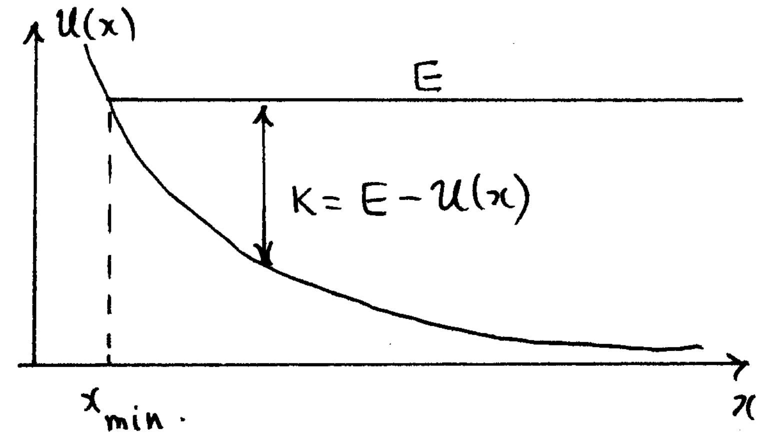
$$\boxed{2^{1/6} = 1.122}$$

Example

Repulsive square law force. [eg. like point charges]

$$F(x) = \frac{A}{x^2} \qquad A > 0$$

$$U = \frac{A}{x}$$



Assume particles have total energy E.
Particle approaching the origin comes as close as x_{min}, reverses direction and then moves out forever.

$$x_{min} = \frac{A}{E} \qquad \left[\text{i.e. } U(x_{min}) = E \right]$$

Brachistochrone Problem

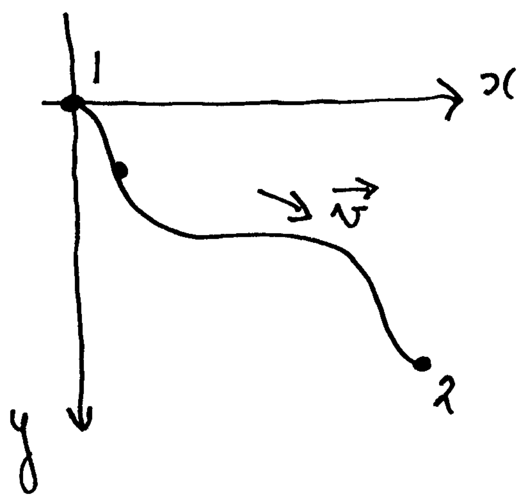
- Find shape of curve joining two points so that a particle starting from rest falls under gravity in the least time.

Let v be speed along the curve
Time to fall an arc length ds is

$$dt = \frac{ds}{v}$$

Total time

$$T = \int_1^2 \frac{ds}{v}$$



Measure y down from point of release:

$$\frac{1}{2}mv^2 = mgy$$

$$v = \sqrt{2gy}$$

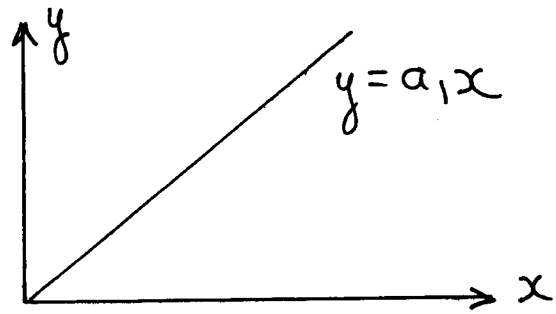
$$T = \int_1^2 \sqrt{\frac{1+\left(\frac{dy}{dx}\right)^2}{2gy}}\; dx$$

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\; dx$$

Least Time Trajectories

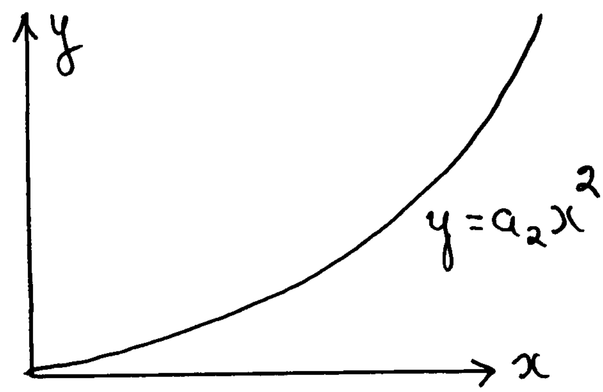
1. Straight line:

$$y = a_1 x$$



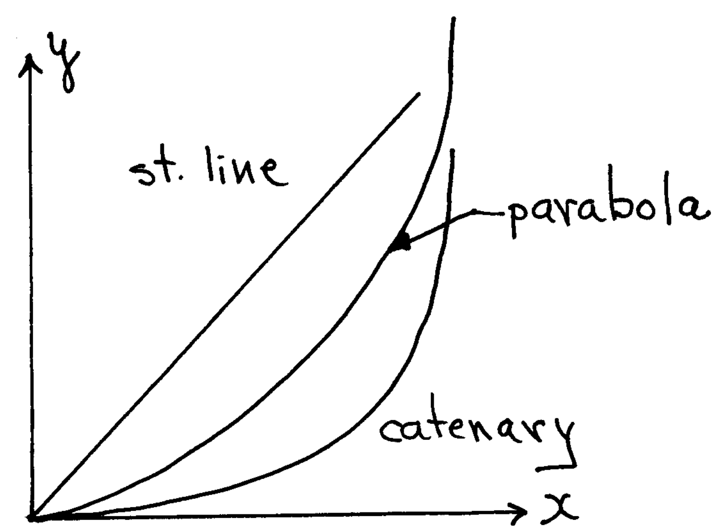
2. Parabola:

$$y = a_2 x^2$$



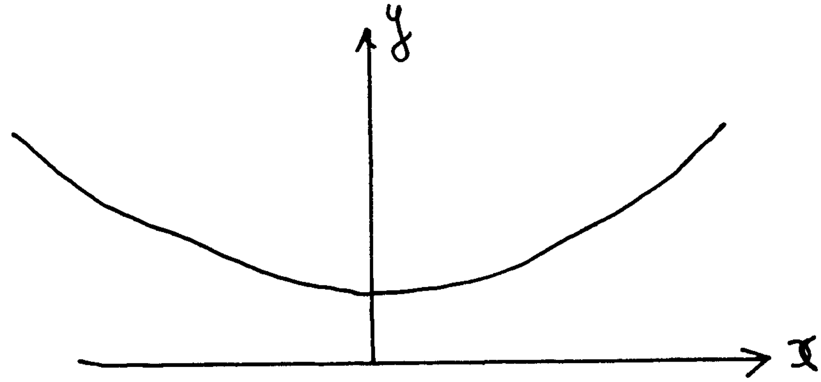
3. Catenary:

$$y = \frac{1}{a} \cosh ax$$



Catenary

- Suspend a cable between two posts and let it hang under own weight.
- The shape the cable takes is called a catenary

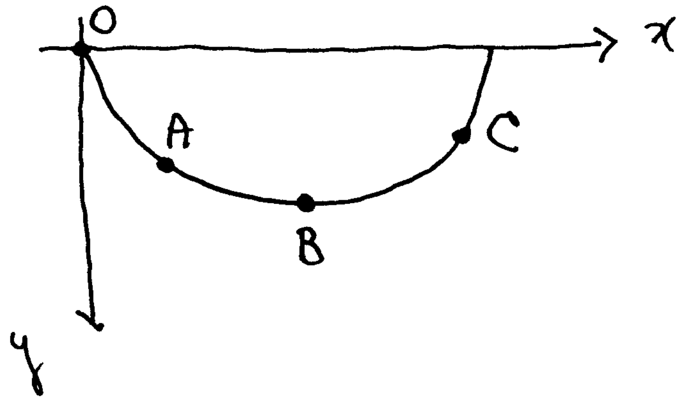


$$y = \frac{1}{2a}\left(e^{ax} + e^{-ax}\right)$$

$$= \frac{1}{a}\cosh ax$$

$$y(x=0) = 1/a$$

- catenary has lowest average center-of-gravity
- catenary has smallest potential energy
- nature shapes curve to minimize PE



- Beads released on the cycloidal wire at O, A, C will reach B in the same amount of time.

Forms of Energy

- electrical
- chemical
- heat
- nuclear

Atomic/Nuclear:

$$1\,\text{eV} = \text{electron-volt} = 1.602 \times 10^{-19}\ \text{J}$$

Power Plants:

$$1\text{-kilowatt hour} = 3.6 \times 10^{6}\ \text{J}$$

Fuels:

$$1\ \text{kilocalorie} = 4.187 \times 10^{3}\ \text{J}$$

$$1\ \text{Btu} = 1.055 \times 10^{3}\ \text{J}$$

Relativity/Einstein

$$\text{Mass} \longleftrightarrow \text{Energy}$$

$$E = mc^2$$

$c = 3 \times 10^{8}\ \text{m/s}$

mass has energy
energy has mass
} nuclear power
nuclear weapons
particle reactions

Power

- Power is defined as the time rate of doing work.
- If an external force applied to an object does an amount of work ΔW in the time interval Δt, the average power is

$$\overline{P} = \frac{\Delta W}{\Delta t}.$$

- The instantaneous power, P, is the limiting value of the average power as Δt approaches zero.

$$P \equiv \lim_{\Delta t \to 0} \frac{\Delta W}{\Delta t} = \frac{dW}{dt}$$

$$[P] = J/s$$

$$= \text{Watts}$$

British: $1\,hp = 550\ ft \cdot lbs/s$

$$= 745.7\,W$$

Power ⟷ Force

We can express the power in terms of the force acting and the velocity of the object

For a small displacement $d\vec{r}$ the work done is

$$dW = \vec{F} \cdot d\vec{r}$$

$$P = \frac{dW}{dt} = \vec{F} \cdot \frac{d\vec{r}}{dt} = \vec{F} \cdot \vec{v}$$

Note: Velocity depends on the frame of reference.
Power also depends upon reference frame.

Energy ⟷ Power

$$E = \int_{t_1}^{t_2} P \, dt$$

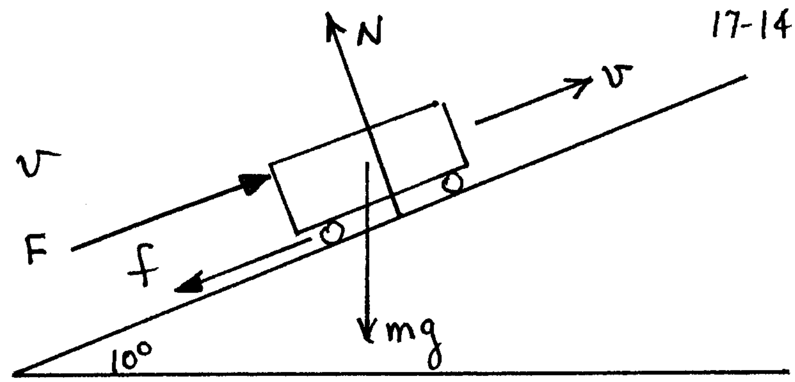
$$= Pt \qquad [\text{Constant } P]$$

$$1\,\text{kWh} = 3.6 \times 10^6\ \text{J} \qquad [1000\,\text{W rate for } 1\,\text{h}]$$

$$\left.\begin{array}{ll} 1\,\text{kilocalorie} & = 4.187 \times 10^3\ \text{J} \\ 1\,\text{Btu} & = 1.055 \times 10^3\ \text{J} \end{array}\right\} \text{Fuels}$$

Example

Car moves at constant v up a plane



$m = 1400 kg$
$v = 80 km/h = 22 m/s$
$f = 700 N$

Net force exerted by engine to move car up at constant v:

$$F = f + mg \sin\theta$$

$$= 700 + 1400 \times 9.81 \times \sin 10^\circ$$

$$= 3100 N$$

$P = \vec{F} \cdot \vec{v}$ [$\vec{F}$ is parallel to $\vec{v}$]

$$= 3100 \times 22 = 6.8 \times 10^4 W$$

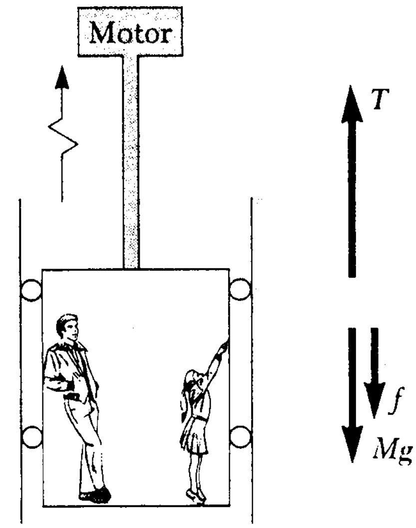
$$= 91 hp.$$

$1 hp = 746 W$

Example

m (elevator+load) = 1800 kg.

f = 4000 N



a) What is horsepower required to move elevator up at constant speed of 3 m/s?

Motor supplies required tension.

$$T - f - Mg = 0$$

$$T = f + Mg$$
$$= 4\times10^3 + 1.8\times10^3 \times 9.81$$
$$= 2.16\times10^4\ \text{N}$$

$$P = \vec{T}\cdot\vec{v} = Tv$$
$$= 2.16\times10^4 \times 3$$
$$= 6.48\times10^4\ \text{W}$$
$$= 64.8\ \text{kW}$$
$$= 86.9\ \text{hp}.$$

b) If elevator accelerates upward at $1.0\ \text{m/s}^2$?

$$T - f - Mg = Ma$$
$$T = M(a+g) + f$$
$$= 2.34\times10^4\ \text{N}$$

$v \equiv$ instantaneous speed.

$P = Tv = (2.34\times10^4)v$ Watts Power increases with inc. speed