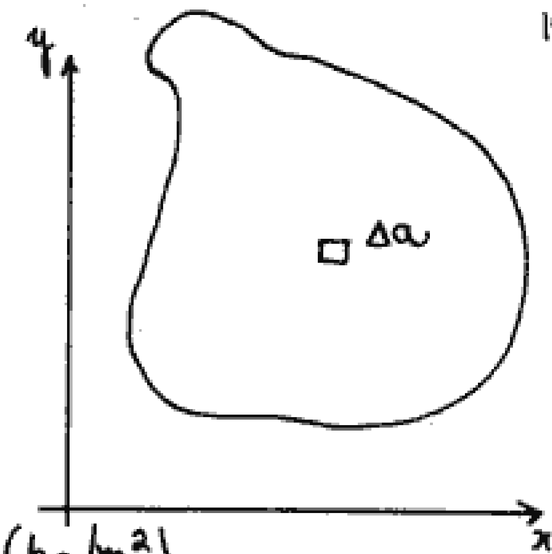
## Areas




If object is in the form of a plane sheet of constant thickness t

$$\Delta m_i = \sigma \, \Delta a_i$$

$\sigma$ — areal mass density (kg/m²)

$$x_{cm} = \frac{1}{M}\int \sigma x \, da$$

$$y_{cm} = \frac{1}{M}\int \sigma y \, da$$

$\Rightarrow$ First moments of the area

$$da = dx\,dy$$

If $\sigma(x,y) = \sigma_0$, a constant

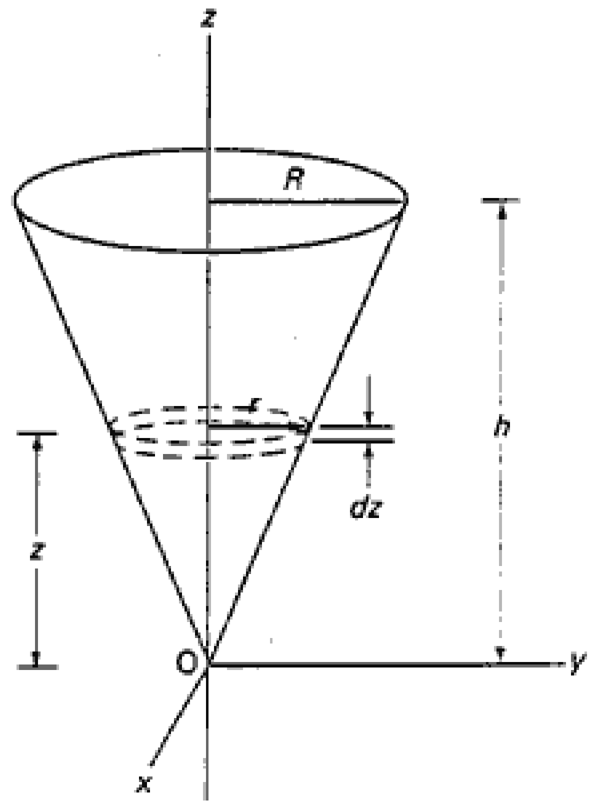
$$x_{cm} = \frac{1}{A}\int x \, da$$

$$y_{cm} = \frac{1}{A}\int y \, da$$

## Example: Right Circular Cone

By symmetry the cm must lie on the axis of the cone.

$x_{cm} = 0 \qquad y_{cm} = 0$




$$z_{cm} = \frac{1}{M}\int z\,dm = \frac{1}{M}\int_0^h \rho z \pi r^2 dz$$

Cone is divided into many cylinders of radius r and thickness dz. Assume $\rho$ = constant

Mass of each cylinder

$dm = \rho\, dV = \rho \pi r^2 dz$

The radius of each cylinder is related to its location z by

$$\frac{r}{z} = \frac{R}{h} \quad \text{or} \quad r = Rz/h$$

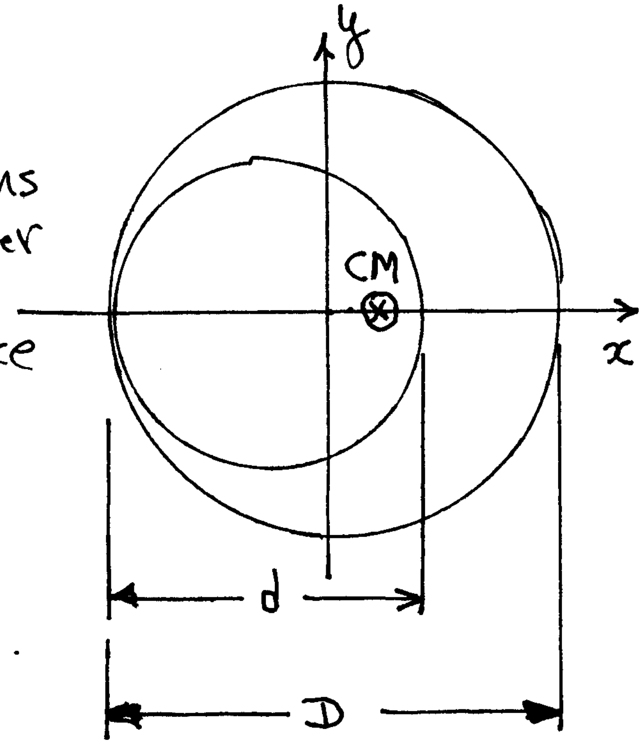
$$\therefore \quad z_{cm} = \frac{1}{M}\int_0^h \frac{\rho\pi R^2}{h^2} z^3 dz = \frac{\rho\pi R^2}{Mh^2}\frac{z^4}{4}\Bigg|_0^h = \frac{\rho\pi R^2 h^2}{4M}$$

The total mass of the cone is equal to its density times the volume.

$M = \rho\pi R^2 h/3 \qquad\qquad \therefore\ z_{cm} = \frac{3}{4}h$ or $\frac{1}{4}h$ from base

## Example : Disk + Hole

A homogeneous disk of diameter $D = 16$ cm, contains a circular hole of diameter $d = 12$ cm. The hole is tangent to the circumference as shown.
Assume the aerial mass density of the disk is $\sigma$ g/cm$^2$.
Locate the center-of-mass.




If object had no hole its cm would be at

$$x_{cm} = 0$$
$$y_{cm} = 0$$

The body of diameter d (negative mass) has a cm at $x_d = -2$cm.

Treat problem as two particles. Find common cm.
By symmetry $y_{cm} = 0$

$$x_{cm} = \frac{\left(\frac{\pi D^2}{4}\sigma\right) \times 0 - \left(\frac{\pi d^2}{4}\sigma\right)(-2\,\text{cm})}{\frac{\pi D^2}{4}\sigma - \frac{\pi d^2}{4}\sigma}$$

$$x_{cm} = 2.6 \text{ cm}.$$

# Motion of the Center-of-Mass

Suppose we take the time derivative of the position vector of the cm. Assuming $M$ is constant (no particles enter or leave system) then we get for the velocity of the cm:

$$\vec{v}_{cm} = \frac{d\vec{r}_{cm}}{dt} = \frac{1}{M}\left[ m_1 \frac{d\vec{r}_1}{dt} + m_2 \frac{d\vec{r}_2}{dt} + \cdots \cdots \frac{d\vec{r}_n}{dt} \right]$$

$$= \frac{1}{M}\left( m_1 \vec{v}_1 + m_2 \vec{v}_2 + \cdots m_n \vec{v}_n \right)$$

$$= \frac{\vec{P}}{M}$$

or $\quad \vec{P} = M \vec{v}_{cm}$

Total momentum of the system is its total mass multiplied by the velocity of the cm. i.e. total $\vec{P}$ is that of a <u>single particle</u> of <u>mass $M$</u> moving with a <u>velocity $\vec{v}_{cm}$</u>.

Differentiate again to get the acceleration of the cm:

$$\vec{a}_{cm} = \frac{d\vec{v}_{cm}}{dt} = \frac{1}{M} \Sigma m_i \frac{d\vec{v}_i}{dt} = \frac{1}{M} \Sigma m_i \vec{a}_i$$

$$M \vec{a}_{cm} = \Sigma \vec{F}_i$$

$\vec{F}_i \equiv$ Force on particle $i$.

Forces on any particle have two sources:

- External (from outside the system)
- Internal (from within the system)

From Newton's 3'rd Law the sum over 'Internal' forces cancells in pairs and the overall sum vanishes.

The <u>net</u> force on the system is due only to <u>external</u> forces.

$$\therefore \sum \vec{F}_{ext} = M\vec{a}_c = \frac{d\vec{P}}{dt}.$$

"The cm <u>moves</u> like an <u>imaginary particle of mass M</u> under the influence of the <u>resultant external force</u> on the system".

If $\sum \vec{F}_{ext} = 0$

$$\frac{d\vec{P}}{dt} = M\vec{a}_c = 0$$

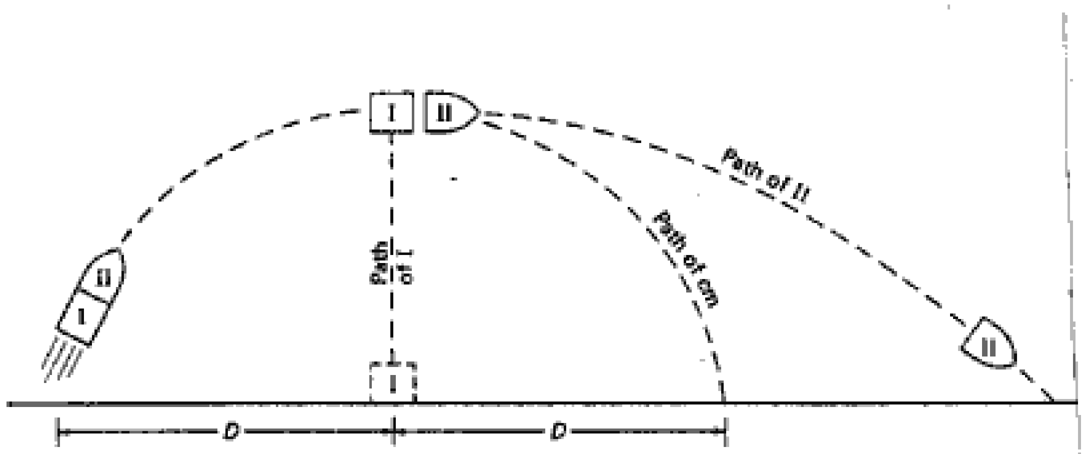
and

$$\boxed{\vec{P} = M\vec{v}_c = \text{constant.}} \quad (\text{when } \sum \vec{F}_{ext} = 0)$$

Total linear momentum of a system is conserved if there are no external forces acting on it.
For an isolated system of particles, both the total momentum and velocity of the cm are constant in time.

# Example: Exploding Projectile

Rocket fired into the air. At its highest point a distance D from the origin it separates into two parts of equal mass.
Part I falls vertically to the earth.
Where does Part II land? Assume $g$ = constant.




$m_I = m_{II}$

cm follows ballistic trajectory of particle with mass $M = m_I + m_{II}$ and intercepts ground a distance $2D$ from the origin.

Since masses $m_I$ and $m_{II}$ are equal, Part II hits ground a distance D beyond cm or a total distance $3D$ from the origin.

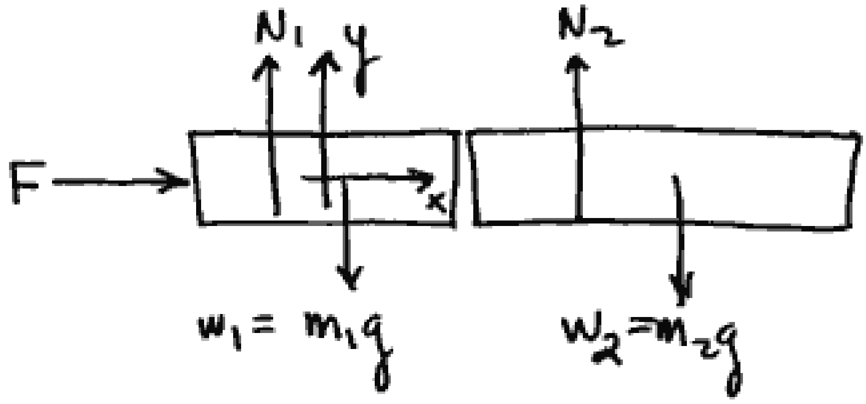
Note: External forces vanish for horizontal motion only. External forces are present for vertical motion.

Example : cm motion

-Uniform blocks

$m_1 = 2\,kg \quad L_1 = 20\,cm$

$m_2 = 4\,kg \quad L_2 = 40\,cm.$




a) $F = 12\,\hat{i}$ (N)

$M = m_1 + m_2 = 6$

$$\sum F_{ext} = Ma_{cm}$$

$$\vec{F} + \vec{N_1} + \vec{N_2} + \vec{w_1} + \vec{w_2} = M\vec{a}_{cm}$$

$$12\hat{i} = 6\,\vec{a}_{cm}$$

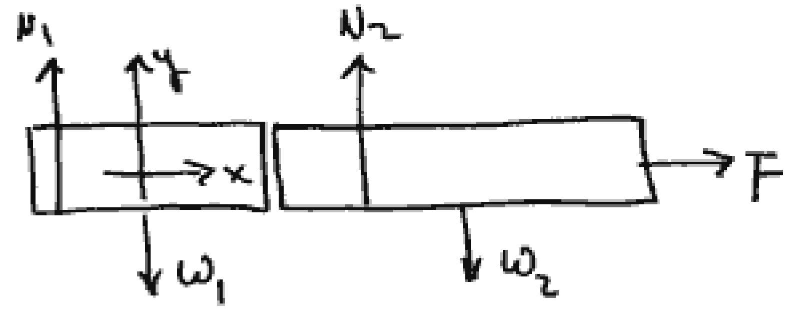
$$\vec{a}_{cm} = 2\hat{i} \quad (m/s^2)$$

$$x_{cm} = \frac{2 \times 0 + 4 \times 30}{2+4} = 20\,cm.$$

$a_{m_1} = 2\hat{i}$

$a_{m_2} = 2\hat{i}$

(b)




$\vec{a}_1 = 0$

$\vec{a}_2 = \frac{12}{4}\hat{i} = 3\hat{i}$

$\vec{a}_{cm} = ?$ [same as before??]

$\vec{a}_1 = 0 \qquad \vec{a}_2 = 3\hat{i}$

$\vec{v}_1 = 0 \qquad \vec{v}_2 = 3t\,\hat{i}$

$\vec{r}_1 = 0 \qquad \vec{r}_2 = (0.3\text{m} + 1.5t^2)\,\hat{i}$

$M\vec{r}_{cm} = \sum m_i \vec{r}_i$

$M\vec{v}_{cm} = \sum m_i \vec{v}_i$

$M\vec{a}_{cm} = \sum m_i \vec{a}_i$

$\therefore \quad \vec{a}_{cm} = \frac{1}{6}\left[0 + 4 \times 3\hat{i}\right] = 2\hat{i} \quad (\text{m/s}^2)$

$\vec{v}_{cm} = \frac{1}{6}\left[0 + 4 \times 3t\hat{i}\right] = 2t\,\hat{i} \quad (\text{m/s})$

$\vec{r}_{cm} = \frac{1}{6}\left[0 + 4\left[0.3 + 1.5t^2\right]\hat{i}\right]$

$= (0.2 + 1.0t^2)\,\hat{i}$

cm in this case does not remain fixed relative to centers of the two blocks.

## Energy of a System of Particles

Momentum : $\vec{P} = M\vec{v}_{cm}$

KE : $K = \cancel{\frac{1}{2} M v_{cm}^2}$ No!!!

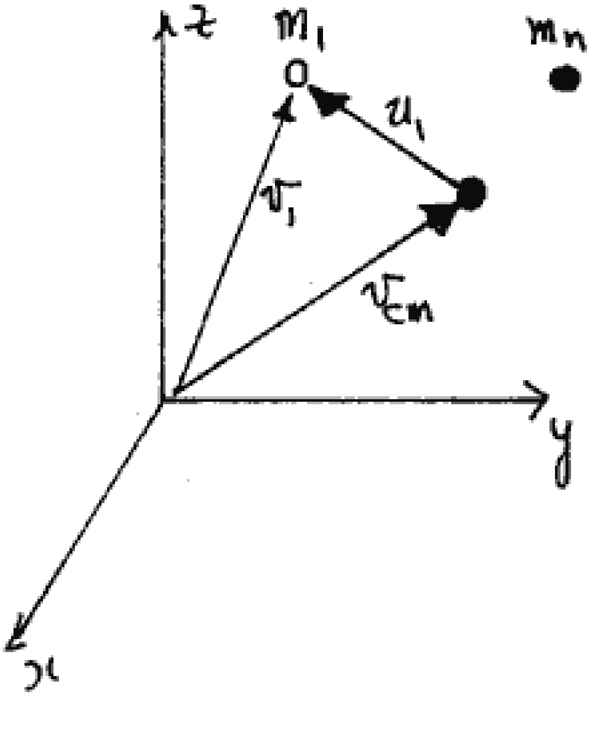
The total KE is the sum of the individual particle KE's.

$$K = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 + \cdots \frac{1}{2} m_n v_n^2$$

$$= \sum_i \frac{1}{2} m_i v_i^2$$

We'd like to write this in terms of $\vec{v}_{cm}$. Let us look at the motion in an inertial reference frame moving with the cm: "cm Frame".

Particle velocities in cm frame are given by the Galilean transformations.




$\vec{u}_1 = \vec{v}_1 - \vec{v}_{cm}$

$\vdots$

$\vec{u}_n = \vec{v}_n - \vec{v}_{cm}$

$\vec{v}_1 = \vec{u}_1 + \vec{v}_{cm}$

$\vdots$

etc.

$$K = \frac{1}{2}m_1(\vec{u}_1 + \vec{v}_{cm})^2 + \cdots\cdots + \frac{1}{2}m_n(\vec{u}_n + \vec{v}_{cm})^2$$

$$= \frac{1}{2}m_1(u_1^2 + 2\vec{u}_1 \cdot \vec{v}_{cm} + v_{cm}^2) + \cdots\cdots$$

$$= \left[\frac{1}{2}m_1 u_1^2 + \frac{1}{2}m_2 u_2^2 + \cdots\right] + \left[m_1\vec{u}_1 + m_2\vec{u}_2 + \cdots\right] \cdot \vec{v}_{cm}$$

$$+ \frac{1}{2}\left[m_1 + m_2 \cdots\right] v_{cm}^2 + \cdots \text{All other terms.}$$

First Term:

$$K_{Int} = \frac{1}{2}m_1 u_1^2 + \frac{1}{2}m_2 u_2^2 + \cdots \frac{1}{2}m_n u_n^2$$

$\Rightarrow$ Internal Kinetic Energy.

Second Term:

$$[\qquad] \cdot \vec{v}_{cm} = \left[m_1(\vec{v}_1 - \vec{v}_{cm}) + m_2(\vec{v}_2 - \vec{v}_{cm}) + \cdots\right] \cdot \vec{v}_{cm}$$

$$= \left[(m_1\vec{v}_1 + m_2\vec{v}_2 + \cdots m_n\vec{v}_n) - (m_1 + m_2 + \cdots m_n)\vec{v}_{cm}\right] \cdot \vec{v}_{cm}$$

$$= \left[\, m\vec{v}_{cm} - m\vec{v}_{cm}\right] \cdot \vec{v}_{cm}$$

$$\equiv 0$$

Third Term:

$$\frac{1}{2} M v_{cm}^2$$

$$\therefore \quad K = K_{Int} + \frac{1}{2} M v_{cm}^2$$

Translation of CM

Internal Kinetic Energy.

$K_{Int}$:

- Rotations
- Heat

## Potential Energy.

If internal and external conservative forces act on a body, the system will also have potential energy.

$\mathcal{U} \equiv$ Function of position of all the particles.

$$\boxed{\text{Total } E = \text{Total } K + \text{Total } \mathcal{U}.}$$

## Gravitational PE - Extended Body.

For a system of particles:

$$\mathcal{U} = (m_1 z_1 + m_2 z_2 + \cdots m_n z_n) g$$

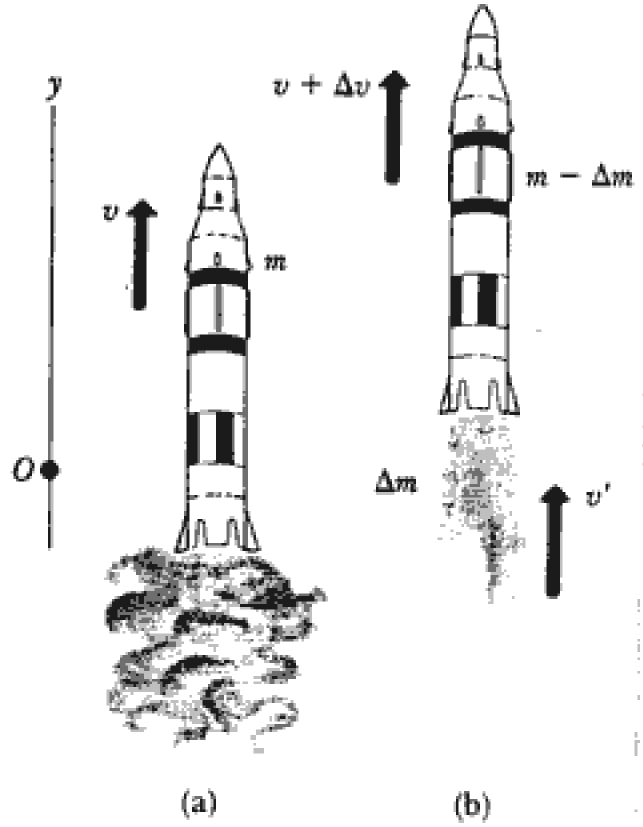
$$= M z_{cm} g.$$

Behaves as if entire mass is located at the CM.

# Variable Mass / Rocket Equation

$$F_{ext} = \frac{d\vec{p}}{dt} = \frac{d(m\vec{v})}{dt}$$

What happens in cases where the mass of the system varies?




8–11 (a) Rocket at time $t$ after takeoff, with mass $m$ and upward velocity $v$. Its momentum is $mv$. (b) At time $t + \Delta t$, the mass of the rocket (and unburned fuel) is $m - \Delta m$, its velocity is $v + \Delta v$, and its momentum is $(m - \Delta m)(v + \Delta v)$. The ejected gas has momentum $\Delta m(v - v_r)$.

<u>Rockets</u>: Propelled forward by the ejection of gases. Force exerted by the gases on the rocket accelerates it.

Mass of the rocket decreases,

$$\frac{dm}{dt} < 0.$$

Consider a rocket of mass $(M+\Delta m)$ which ejects a mass $\Delta m$ in a time $\Delta t$

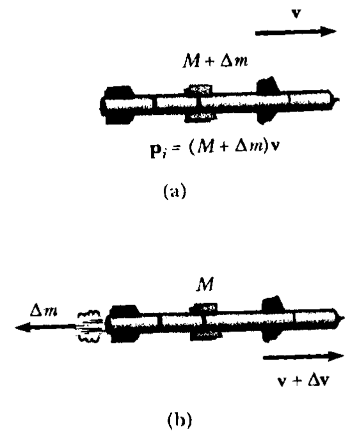



**FIGURE 9.27** Rocket propulsion. (a) The initial mass of the rocket is $M + \Delta m$ at a time $t$, and its speed is $v$. (b) At a time $t + \Delta t$, the rocket's mass has reduced to $M$, and an amount of fuel $\Delta m$ has been ejected. The rocket's speed increases by an amount $\Delta v$.

| Time | Mass | Momentum |
|---|---|---|
| $t$ | $M+\Delta m$ | $(M+\Delta m)v$ |
| $t+\Delta t$ | $M$ | $(v+\Delta v)M$ |
| $t+\Delta t$ | $\Delta m$ | $\Delta m(v-v_e)$ |

$M(t)$ : mass of rocket at any time $t$.

$v$ : velocity of rocket in inertial frame

$v_e$ : exhaust speed $\sim 3000$ m/s
velocity of gases relative to rocket.

$v-v_e$ : velocity of gases relative to inertial frame

$$d\vec{P} = \vec{P}_f - \vec{P}_i \qquad \text{[Change in Momentum]}$$

$$= [M(v+\Delta v) + \Delta m(v-v_e)] - (M+\Delta m)v$$

$$= M\Delta v - \Delta m\, v_e$$

$$\vec{F}_{ext} = \frac{d\vec{P}}{dt} \approx M\frac{\Delta v}{\Delta t} - \frac{\Delta m}{\Delta t} v_e$$

$$= M\frac{dv}{dt} - v_e\frac{dm}{dt}$$

Note: $\frac{dm}{dt} = -\frac{dM}{dt}$

$$\therefore F_{ext} = M\frac{dv}{dt} + v_e \frac{dM}{dt}$$

$$M\frac{dv}{dt} = F_{ext} - v_e \frac{dM}{dt}$$

Assume $F_{ext} = -Mg$ $[g = \text{constant}]$

$$M\frac{dv}{dt} = -Mg - v_e \frac{dM}{dt}$$

$$\int_{v_i}^{v_f} dv = -g\int_0^t dt - v_e \int_{M_i}^{M_f} \frac{dM}{M}$$

$$v_f = v_i - gt + v_e \ln\left(\frac{M_i}{M_f}\right)$$

- Change in velocity of rocket prop. to $v_e$
- Increase in speed prop. to $\ln\left(\frac{M_i}{M_f}\right)$
- Empty rocket should weigh very little.
- Thrust $= M\frac{dv}{dt} = v_e \frac{dM}{dt}$

## Example: Rocket

$m_0 = 21{,}000$ kg

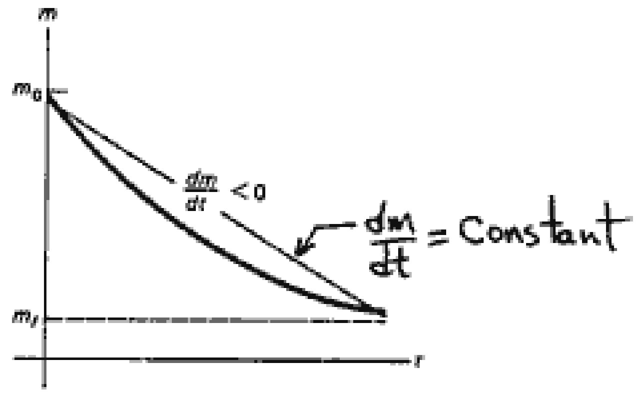
$m_f = 6{,}000$ kg (After burnout)

$\frac{dm}{dt} = -190$ kg/s (Rate of fuel exh.)

$v_r = 2800$ m/s

$g = 9.81$ m/s$^2$

A rocket whose initial total mass is $m_0$ loses mass as the engine burns fuel. The detailed shape of the curve depends on the burning rate versus time.




a) Thrust: $v_r \frac{dm}{dt} = (2800)(190) = 5.3 \times 10^5$ N

b) $F_{ext} = Mg = 2.1 \times 10^4$ kg $(9.81) = 2.1 \times 10^5$ N (initially)

$= (0.6 \times 10^4)(9.81) = 5.9 \times 10^4$ N (at burn-out)

Net Force on M:

start: $(5.3 \times 10^5 - 2.1 \times 10^5) = 3.2 \times 10^5$ N  $a = 1.52$ gees.

Just Prior to Burn-Out: $(5.3 \times 10^5 - 5.9 \times 10^4) = 4.7 \times 10^5$ N  $a = 8.0$ gees

Just After: $= -gM$  $a = -1$ gee

Burn-Out Time: $t = \frac{1.5 \times 10^4 \text{ kg}}{190 \text{ kg/s}} = 79$ s

If $v_0 = 0$

$v = -9.81(79) + (-2800 \text{ m/s}) \ln \frac{6000}{21000}$

$= 2830$ m/s

opposite to $v$.