ii) Elastic Collisions - Two Dimensions

- Assume special case where target particle is initially at rest, $\vec{v}_2 = 0$.

- Particle collide by interacting: For many forces, gravity, electromag., etc. forces act along line joining the particles.
- The initial particle and scattered particles define a common plane - 2-dimensional problem.

Given initial velocities, we have four unknowns following collision:

$$\left.\begin{matrix} v'_{1x}, & v'_{1y} \\ v'_{2x}, & v'_{2y} \end{matrix}\right\} \text{ x and y velocities of particles -1 and -2.}$$

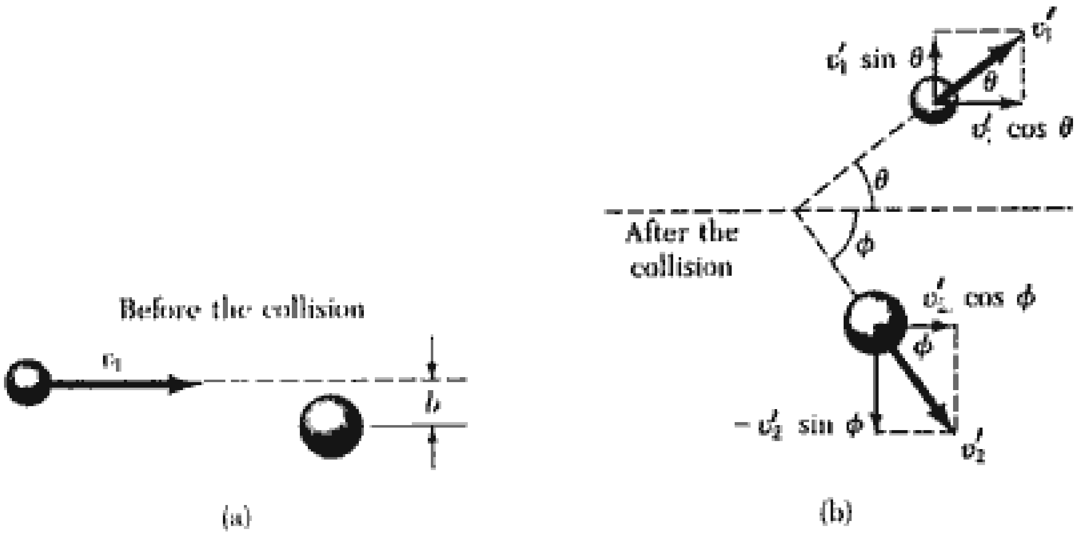


Figure 9.11 Schematic representation of an elastic glancing collision between two particles: (a) before the collision and (b) after the collision. Note that the impact parameter, b, must be greater than zero for a glancing collision.

Cons. of Momentum: 2 equations (x, y)
Cons. of Energy: 1 equation

$\therefore$ Can only get restrictions on the final motion or at least one other quantity must be known.

Cons of Momentum:

$$m_1 \vec{v}_1 = m_1 \vec{v}_1' + m_2 \vec{v}_2'$$

$$m_1 v_1 = m_1 v_1' \cos\theta + m_2 v_2' \cos\phi$$

$$0 = m_1 v_1' \sin\theta - m_2 v_2' \sin\phi$$

Cons. of Energy:

$$\tfrac{1}{2} m_1 v_1^2 = \tfrac{1}{2} m_1 v_1'^2 + \tfrac{1}{2} m_2 v_2'^2$$

<u>Special Case:</u> $m_1 = m_2$

$$v_1^2 = v_1'^2 + v_2'^2 \qquad (1)$$

$$v_2' \cos\phi = v_1 - v_1' \cos\theta \qquad (2)$$

$$v_2' \sin\phi = v_1' \sin\theta \qquad (3)$$

$②^2 + ③^2 \quad v_2'^2 = v_1^2 - 2v_1 v_1' \cos\theta + v_1'^2$

Use Eq. ① to eliminate v_2'

$$v_1' = v_1 \cos\theta$$

$$\frac{1}{2} m v_1'^2 = \frac{1}{2} m v_1^2 \cos^2\theta$$

KE of deflected incident projectile after scattering.

We had cons. of momentum:

$$\vec{v_1} = \vec{v}_1' + \vec{v}_2'$$

$$v_1^2 = v_1'^2 + v_2'^2 + 2v_1' v_2' \cos(\theta + \phi) \qquad \text{[square]}$$

Comparing with Eq. ①, we must have that

$$\theta + \varphi = \pi/2$$

When particles of equal mass collide, the sum of their scattering angles is $90°$.

If $\quad m_1 > m_2 \quad \theta \leq \pi/2$

$\quad m_1 < m_2 \quad 0 \leq \theta \leq \pi$

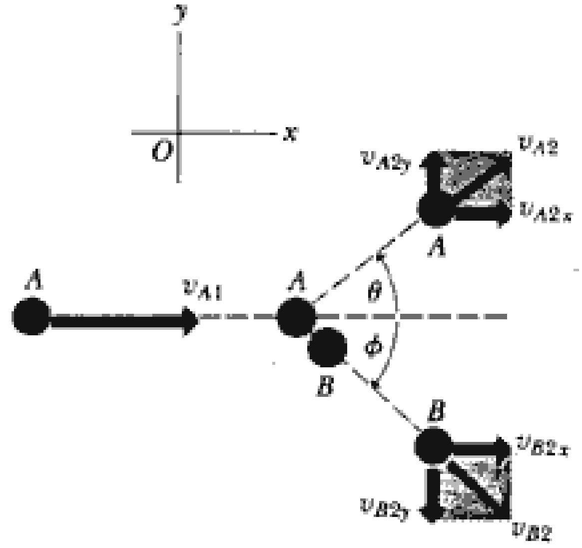
Example: 2-D Elastic Collision

$m_A = 5\,kg \quad v_{A1} = 4\,m/s$

$m_B = 3\,kg \quad v_{B1} = 0$ [stationary]

Assume after collision, $v_{A2} = 2\,m/s$

Find v_{B2}, θ and ϕ



Since collision is elastic

$$KE_f = KE_i$$

$$\frac{1}{2}(5kg)(4m/s)^2 = \frac{1}{2}(5kg)(2m/s)^2 + \frac{1}{2}(3kg)v_{B2}^2$$

Solving,

$$v_{B2} = 4.47\,m/s$$

Conservation of x- and y-components of momentum gives,

① $(5kg)(4m/s) = (5kg)(2m/s)\cos\theta + (3kg)(4.47m/s)\cos\phi$

② $0 = (5kg)(2m/s)\sin\theta - (3kg)(4.47m/s)\sin\phi$

Solve Eq.① for $\cos\phi$ } square and add
Solve Eq.② for $\sin\phi$ } $\sin^2\phi + \cos^2\phi = 1$

Then solve for θ and finally ϕ:

$\theta = 36.9^\circ$

$\phi = 26.6^\circ$

Ballistic Pendulum

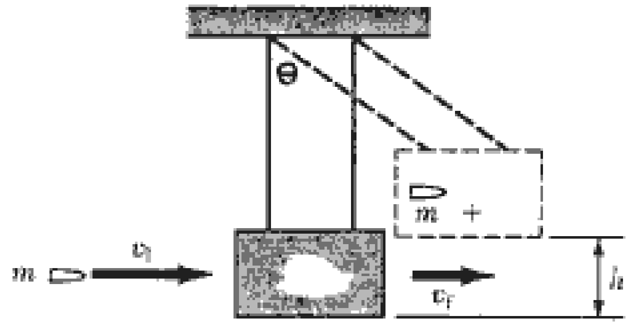


Figure Diagram of a ballistic pendulum. Note that v_f is the velocity of the system right after the perfectly inelastic collision.

A method used to measure the speed of a projectile such as a bullet.

Bullet: mass = m
speed = v_1 (initially)

Block: mass = $M \gg m$
speed = 0 (initially)

After collision, mass $(m+M)$ moves up a height h.

Collision in two parts:
i) Collision and bullet stopped in block.
ii) Motion of block and bullet to maximum height h.
– Recoil –

i) Collision:
- collision time is short
- block does not move during collision
- no net external force during collision which is perfectly inelastic and momentum is conserved.
- The velocity right after the collision is given by:

$$m v_1 = (m + M) v_f \qquad ①$$

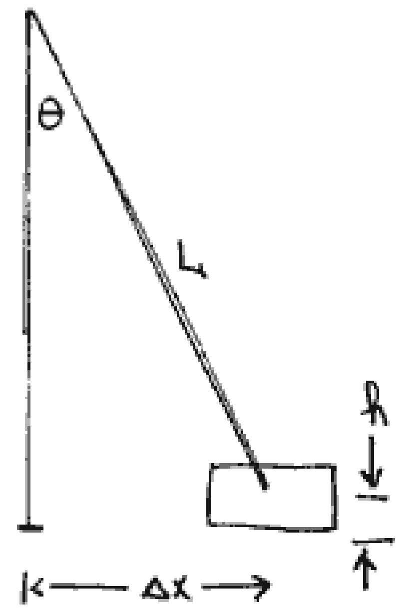
ii) Recoil:

- after collision the system $(m+M)$ has a Kinetic Energy.
- Energy is conserved.
- KE at the bottom is transformed to PE in the block and the bullet at the height h.

$$\frac{1}{2}(m+M) v_f^2 = (m+M) g h$$

$$\therefore v_f = \sqrt{2gh} \qquad ②$$

From ① $\quad v_1 = \frac{(m+M)}{m}\sqrt{2gh}$



For $\theta \ll 1$

$$\Delta x = L\sin\theta \simeq L\theta \Rightarrow \theta \sim \frac{\Delta x}{L}$$

$$h = L(1-\cos\theta) \sim \frac{L\theta^2}{2} \sim \frac{(\Delta x)^2}{2L}$$

$$v_1 = \left(\frac{m+M}{m}\right)\sqrt{2g\left(\frac{\Delta x^2}{2L}\right)}$$

$$= \left(\frac{m+M}{m}\right)\sqrt{\frac{g}{L}}\,(\Delta x)$$

$m = 2.7\,\text{gm} \qquad L = 1.14\,\text{m}$

$M = 3840\,\text{gm} \qquad \Delta x = 6.5\,\text{cm}$

$v_1 = 293\,\text{m/s}$

Ballistic Pendulum / Kinetic Energies

$$K_i = \frac{1}{2} m v_1^2$$

$$K_f = \frac{1}{2}(m+M)\, v_f^2 = \frac{1}{2}(m+M)\left(\frac{m}{m+M}\right)^2 v_1^2 = \frac{m^2}{2(m+M)} v_1^2$$

$$\frac{K_f}{K_i} = \frac{m}{m+M} \lll 1$$

Most of the initial KE is lost.

Collision Time

Assume bullet decelerates uniformly in a distance of 0.10 m.

$$v_f^2 - v_0^2 = 2aS \qquad v_f \cong 0.0$$

$$a = -\frac{v_0^2}{2S} \simeq -\frac{(300)^2}{2 \times 0.10} \quad m/s^2$$

Also $v_f = v_0 + at$

$$t \sim \frac{-v_0}{a} = \frac{-300}{\frac{-(300)^2}{2 \times 0.10}} = .00067\, s$$

Period of pendulum: $T = 2\pi\sqrt{\frac{L}{g}} = 2\pi\sqrt{\frac{1.14}{9.81}} = 2.14\, s$

$t \lll T$ [$\therefore$ good approx.]

COLLISIONS

1. Conditions:

An event is a collision if $\Delta t \ll \Delta T$

- time can be separated into before, during + after
- Δt collision time
- ΔT observation time

An event is a collision if $|J_{ext}| \ll |I_{coll}|$ – the impulse of external forces can be neglected and momentum is conserved.

2. Collision Classifications:

Elastic – KE is conserved

Inelastic – KE is not conserved

Completely inelastic – particles stick together after

3. Notation

m_1, m_2 – masses of the two particles

v_{1i}, v_{2i} – initial (before collision) velocities of parts 1, 2.

v_{1f}, v_{2f} – final (after collision) velocities of particles 1, 2.

4. Equations

Conservation of momentum – valid for all collisions:

$$m_1 \vec{v}_{1i} + m_2 \vec{v}_{2i} = m_1 \vec{v}_{1f} + m_2 \vec{v}_{2f}$$

Conservation of kinetic energy – valid only for elastic collisions.

$$\frac{1}{2} m_1 v_{1i}^2 + \frac{1}{2} m_2 v_{2i}^2 = \frac{1}{2} m_1 v_{1f}^2 + \frac{1}{2} m_2 v_{2f}^2 .$$

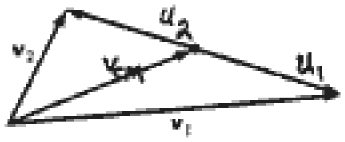
Center-of-Mass Frame / Collisions / KE

- 2 Particle System

$\vec{u}_1, \vec{u}_2$: CM velocities

$\vec{v}_1, \vec{v}_2$: Lab velocities

v_{cm} : Velocity of CM.



We had for the Kinetic Energy

$$K = \frac{1}{2} M v_{cm}^2 + K_{INT}$$

$$= \frac{1}{2} M v_{cm}^2 + \underbrace{\frac{1}{2} m_1 u_1^2 + \frac{1}{2} m_2 u_2^2}_{\text{Internal Energy}}$$

u_1, u_2 : velocities of particles relative to CM frame

$\frac{1}{2} M \bar{v}_{cm}^2 \rightarrow$ Translational motion of CM. When no external forces act must be conserved.

$$\vec{u}_1 = \vec{v}_1 - \vec{v}_{cm} = v_1 - \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2} = \frac{m_2 (\vec{v}_1 - \vec{v}_2)}{m_1 + m_2}$$

$$\vec{u}_2 = \vec{v}_2 - \vec{v}_{cm} = v_2 - \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2} = -\frac{m_1 (\vec{v}_1 - \vec{v}_2)}{m_1 + m_2}$$

$$\therefore K = \frac{1}{2} M v_{cm}^2 + \frac{1}{2} \frac{m_1 m_2}{m_1 + m_2} (\vec{v}_1 - \vec{v}_2)^2$$

Relative velocity.

2ND Term: Internal Energy. Represents maximum energy available for a totally inelastic collision.

let $\vec{v}_{rel} = \vec{u}_1 - \vec{u}_2 = \vec{v}_1 - \vec{v}_2$ } relative velocity of two particles

$$K = \frac{1}{2} M v_{cm}^2 + \frac{1}{2} \mu v_{rel}^2$$

$\mu = \dfrac{m_1 m_2}{m_1 + m_2}$ $\Longrightarrow$ reduced mass for a 2-particle system.

Collisions

In any collision, whether elastic or inelastic, when external forces can be neglected, total momentum is conserved.

$$\vec{P} = m_1 \vec{v}_1 + m_2 \vec{v}_2 \qquad [\text{constant}]$$

Value of $\vec{P}$ depends on coordinate system, but conservation is true in all frames.

2-Particles

$$\vec{r}_{cm} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2}$$

$$\vec{V}_c = \frac{m_1 \vec{v}_1 + m_2 \vec{v}_2}{m_1 + m_2}$$ (on line joining, $\vec{v}_1$, $\vec{v}_2$)

Momenta in C-System:

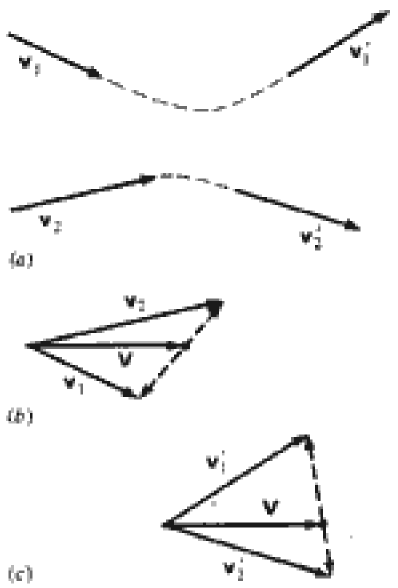
$$\vec{P}_{1c} = m_1 \vec{u}_1 = \frac{m_1 m_2}{m_1+m_2}(\vec{v}_1 - \vec{v}_2) = \mu v_{rel}$$

$$\vec{P}_{2c} = m_2 \vec{u}_2 = \frac{-m_1 m_2}{m_1+m_2}(\vec{v}_1 - \vec{v}_2) = -\mu v_{rel}$$

Total Momentum:

$$\vec{P}_C = \vec{P}_{1c} + \vec{P}_{2c} \equiv 0. \qquad [\text{CM-Frame}]$$

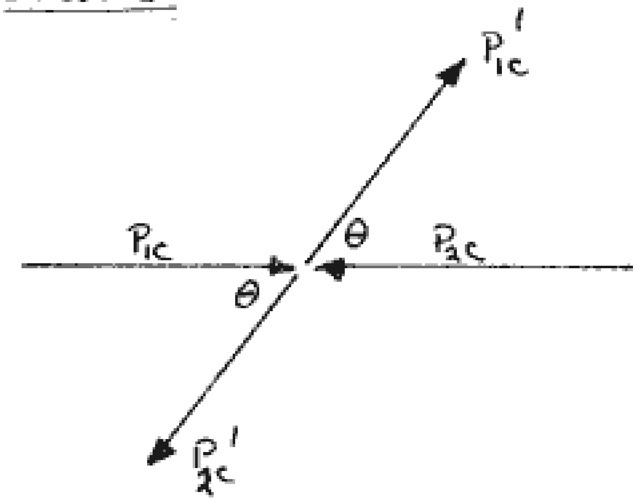
$$\vec{P}_L = \vec{P}_1 + \vec{P}_2 = (m_1 + m_2) V_{cm}$$



What does collision look like in CM Frame?

- Initial and final velocities determine the scattering plane.
- Each particle is scattered through the same angle θ.

Elastic:



$\longrightarrow V_{cm}$

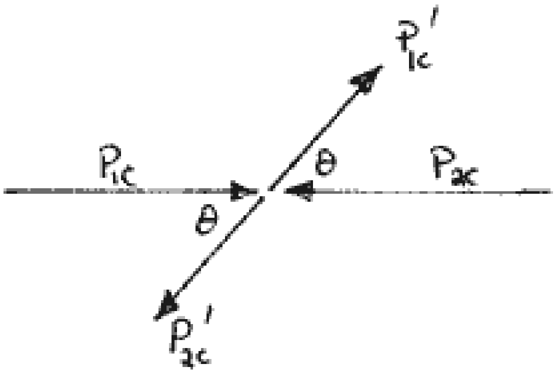
[cm is moving at constant speed before, during and after collision]

$|P_{1c}| = |P_{1c}'|$

$|P_{2c}| = |P_{2c}'|$

lengths of momentum vectors before and after collision are equal. Energy conserved. Collisions are back-to-back. $\vec{P} = 0$.

Inelastic



Back-to-Back
Momentum magnitudes are reduced following collision.
Loss in KE
Inelastic Collision.