

Simple Pendulum

32-2

Point mass m suspended on a massless cord of length L . Motion occurs in a vertical plane driven by the force of gravity.

Forces acting are the tension, \vec{T} , and the weight $m\vec{g}$. The tangential component of $m\vec{g}$ is a restoring force always acting toward $\theta=0$.

$$F_t = -mg \sin \theta = m \frac{d^2 s}{dt^2}$$

$$s = L \theta$$

(arc length)

$$\frac{d^2 \theta}{dt^2} + \frac{g}{L} \sin \theta = 0$$

For small θ : $\sin \theta = \theta - \frac{\theta^3}{3!} + \dots$

$$\frac{d^2 \theta}{dt^2} + \frac{g}{L} \theta = 0$$

$\theta \ll 1$

Solving: $\theta = \theta_0 \cos(\omega t + \delta)$

$$\omega = \sqrt{\frac{g}{L}}$$

$$T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{L}{g}}$$

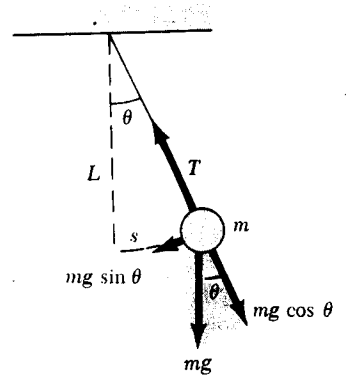


Figure When θ is small, the simple pendulum oscillates with simple harmonic motion about the equilibrium position ($\theta=0$). The restoring force is $mg \sin \theta$, the component of weight tangent to the circle.

The period and frequency of a simple pendulum depend only on the length of the string and the acceleration of gravity.

More exact solutions give \bar{T} as a power series in terms of $\sin(\theta_0/2)$:

$$\bar{T} = 2\pi \sqrt{\frac{L}{g}} \left[1 + \frac{1}{4} \sin^2 \frac{\theta_0}{2} + \frac{9}{64} \sin^4 \frac{\theta_0}{2} + \dots \right]$$

$$\theta = 15^\circ$$

$$\frac{\theta - \sin \theta}{\theta} = 1.15\%$$

Alternate: Torques

Torques relative to origin at top of string.

$$\tau = -mgL \sin \theta \quad (\text{Restoring})$$

Moment of inertia of pendulum

$$I = mL^2$$

$$I\alpha = \tau \quad \text{Rigid-Body Dynamics}$$

$$mL^2 \frac{d^2\theta}{dt^2} = -mgL \sin \theta$$

$$\frac{d^2\theta}{dt^2} + \frac{g}{L} \sin \theta = 0$$

Physical Pendulum

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Any rigid body suspended from a fixed axis not passing through the c.m. It will oscillate if displaced from its equilibrium position.

Body pivoting a distance d from the c.m. Torque provided by the force of gravity.

$$\tau = I\alpha$$

$$-mgd \sin \theta = I \frac{d^2 \theta}{dt^2}$$

$$\frac{d^2 \theta}{dt^2} + \frac{mgd}{I} \theta = 0 \quad (\theta \ll 1)$$

$$\theta = \theta_0 \cos(\omega t + \delta) \quad (\theta \ll 1)$$

$$\omega = \sqrt{\frac{mgd}{I}} \quad \text{and} \quad T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{I}{mgd}}$$

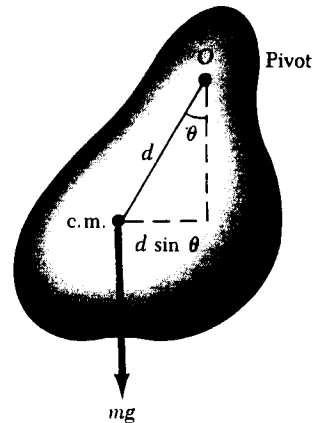


Figure The physical pendulum consists of a rigid body pivoted at the point O, and not through the center of mass. At equilibrium, the weight vector passes through O, corresponding to $\theta = 0$. The restoring torque about O when the system is displaced through an angle θ is $mgd \sin \theta$.

An ideal pendulum of length

$$L = \frac{I}{md}$$

would have the same oscillation frequency.

Physical pendulum suspended from O oscillates as if it were an ideal pendulum with all its mass concentrated at O' a distance L from O on the extension through C .

Point O' is called the center of oscillation of the pendulum.

O and O' are conjugate points.

Consider oscillations about O' .

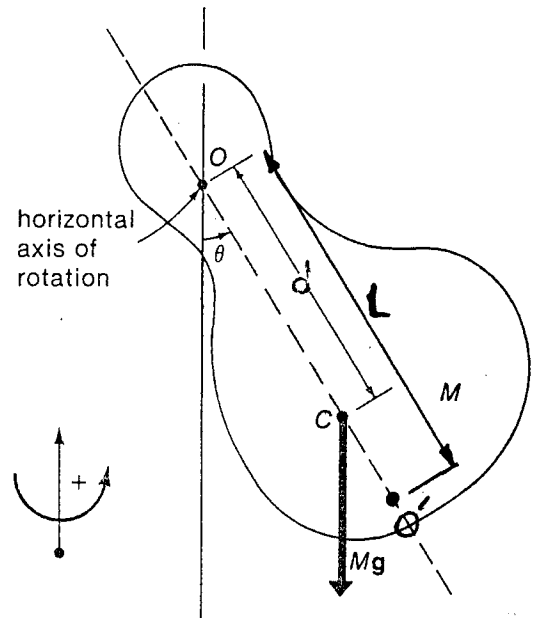
$$T' = 2\pi \sqrt{\frac{I'}{mg(L-d)}}$$

$$L-d = \frac{I}{md} - d = \frac{I - md^2}{md} = \frac{I_c}{md}$$

$$L \rightarrow = L$$

$$I = I_c + md^2$$

$$I' = I_c + m(L-d)^2$$



$$[I = I_c + md^2]$$

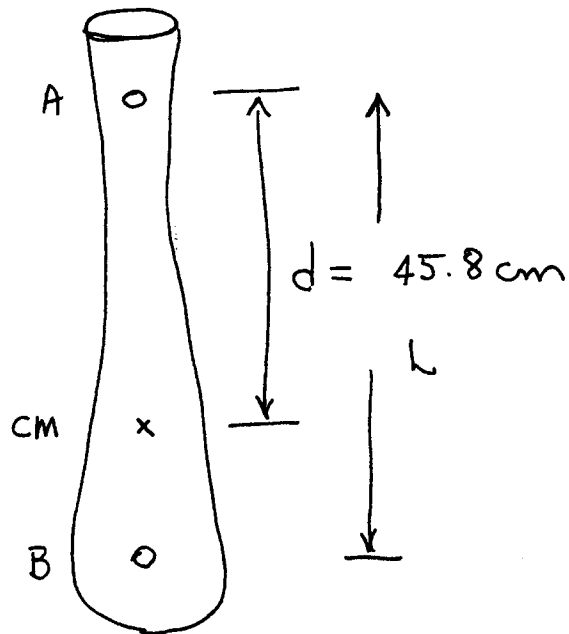
$$\begin{aligned} T' &= 2\pi \sqrt{\frac{I'}{mg(L-d)}} = 2\pi \sqrt{\frac{I_c + m(L-d)^2}{mg(L-d)}} \\ &= 2\pi \sqrt{\frac{I_c + m\left(\frac{I_c}{md}\right)^2}{mg\left(\frac{I_c}{md}\right)}} \end{aligned}$$

$$T' = 2\pi \sqrt{\frac{md^2 + I_c}{mgd}} = 2\pi \sqrt{\frac{I}{mgd}}$$

$$\therefore T' = T$$

[Periods are equal]

Example [Demo]



$$M = 0.910 \text{ kg.}$$

$$T_A = 1.5 \text{ s}$$

$$L = \frac{I}{md}$$

$$T_A = 2\pi \sqrt{\frac{I}{mgd}} = 2\pi \sqrt{\frac{L}{g}}$$

$$L = \left(\frac{T_A}{2\pi}\right)^2 g = \left(\frac{1.5}{2\pi}\right)^2 \times 9.81 = 55.9 \text{ cm.}$$

$T_B = T_A$ Conjugate Points

Torsional Pendulum

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A rigid body is suspended by a wire attached at the top to a fixed support.

When the body is twisted through a small angle θ , the twisted wire exerts a restoring torque proportional to the angular displacement.

$$\tau = -K\theta$$

K : torsion constant of the wire.

$$\tau = -K\theta = I \frac{d^2\theta}{dt^2}$$

$$\frac{d^2\theta}{dt^2} + \frac{K}{I} \theta = 0$$

$$\omega = \sqrt{\frac{K}{I}} \quad \text{and} \quad T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{I}{K}}$$

- balance wheel of watch.
- galvanometers
- Cavendish torsional balance.

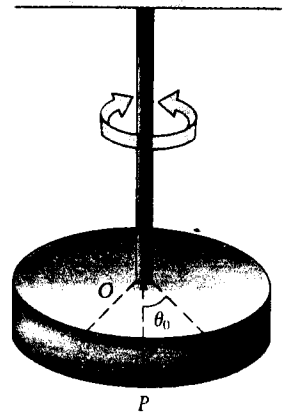
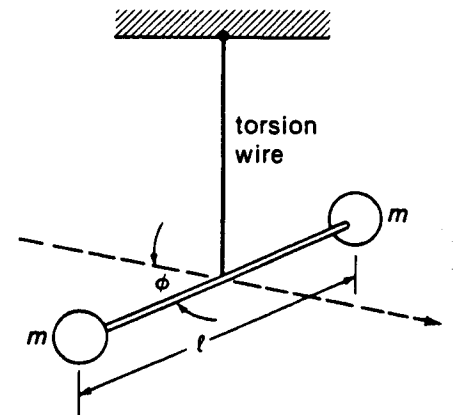


Figure ... A torsional pendulum consists of a rigid body suspended by a wire attached to a rigid support. The body oscillates about the line OP with an amplitude θ_0 .



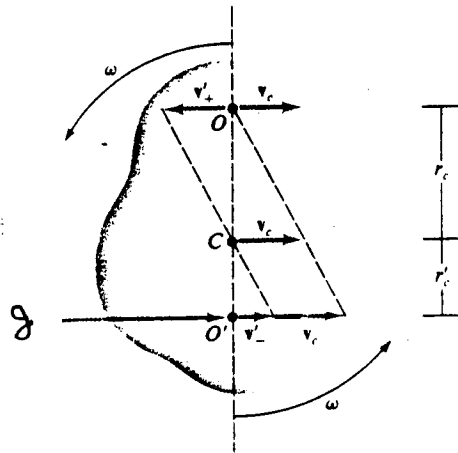


figure Motion of a physical pendulum immediately after receiving an impulse \vec{J} at O' , the center of oscillation when suspended at O . At O , the speed $v = v_c - \omega r_c = 0$; and at O' , speed $v = v_c + \omega r_c' = (\vec{J}/m)(1 + r_c'/r_c)$.

Body suspended from point O . Receives an impulse at O' .

$$\vec{J} = \Delta(mv_c) = \int \vec{F} dt$$

The resulting velocity of the CM is

$$v_c = \frac{\vec{J}}{m}$$

The impulse imparts an angular velocity to the body with respect to the CM of

$$\omega = \frac{J r_c'}{I_c} \quad [\Delta L = \Delta(I_c \omega) = \vec{J} dt = J r_c']$$

The body translates and rotates about CM.

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The linear velocity \vec{v} at any point in the body is the vector sum of \vec{v}_c and the tangential velocity \vec{v}' about the cm due to its rotation.

At O' the velocities \vec{v}' and \vec{v}_c add. At O they oppose each other.

$$v_o = v_c - v' = v_c - \omega r_c = \frac{d}{dt} - \omega r_c = \frac{d}{dt} \left(1 - \frac{m r_c r_c'}{I_c} \right)$$

But $r_c r_c' = \frac{I_c}{m}$ so that $v_c = \omega r_c$

$$\therefore v_o = 0$$

Object appears to be rotating about axis through O .

Center of oscillation is also called the center of percussion.

Example: Potential Well

- Particle of mass m rolling around the potential minimum of a cored surface of radius R . Sphere radius = r .

Assume: $\theta \ll 1$

$$x \sim R\theta$$

$$U(\theta) = mgR(1 - \cos \theta)$$

$$\approx mgR \frac{\theta^2}{2}$$

$$U(0) = 0$$

Let $\theta = \theta_0$ represent maximum amplitude of the displacement.

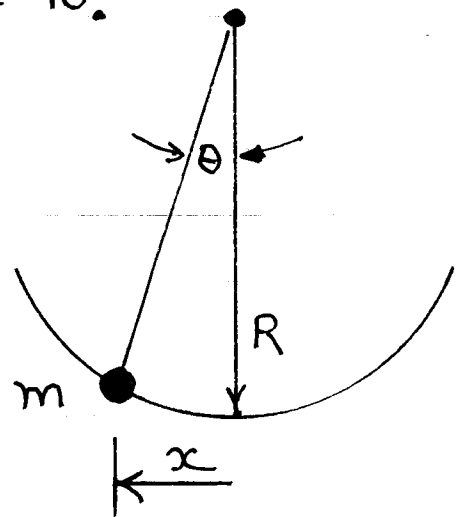
$$x_0 = R\theta_0 \quad (\text{maximum } x)$$

$$\frac{1}{2} I \omega^2 + \frac{1}{2} m v^2 + mgR \frac{\theta^2}{2} = mgR \frac{\theta_0^2}{2} \quad [\text{Cons of } E]$$

$$\frac{1}{2} I \omega^2 + \frac{1}{2} m v^2 + \frac{mg}{2R} x^2 = \frac{mg}{2R} x_0^2$$

Compare with:

$$\frac{1}{2} m v^2 + \frac{1}{2} k x^2 = \frac{1}{2} k x_0^2$$



Rewrite:

$$\frac{1}{2} \beta \frac{m \kappa^2}{\pi^2} (\pi \omega)^2 + \frac{1}{2} m v^2 + \frac{m g}{2R} x^2 = \frac{1}{2} \frac{m g}{R} x_0^2$$

$$\frac{1}{2} m v^2 [\beta + 1] + \frac{m g}{2R} x^2 = \frac{1}{2} \frac{m g}{R} x_0^2$$

$$\omega^2 = \frac{k}{m} = \frac{g}{R} \frac{1}{(\beta + 1)}$$

$$T = 2\pi \sqrt{\frac{R}{g} (\beta + 1)}$$

[Demo]

$$R = 90.6 \text{ cm}$$

$$\kappa =$$

$$T_{\text{thy}} =$$

Damped Oscillations

- No friction
 - amplitude constant with time
- Real oscillators stop
 - pendula
 - mass/spring
- Dissipative / Frictional Forces
 - ↳ damping
 - damped oscillation
 - assume damping force is proportional to velocity of body

$$F = -bv$$

↑ strength of damping

$$-kx = bv = m \frac{d^2 x}{dt^2}$$

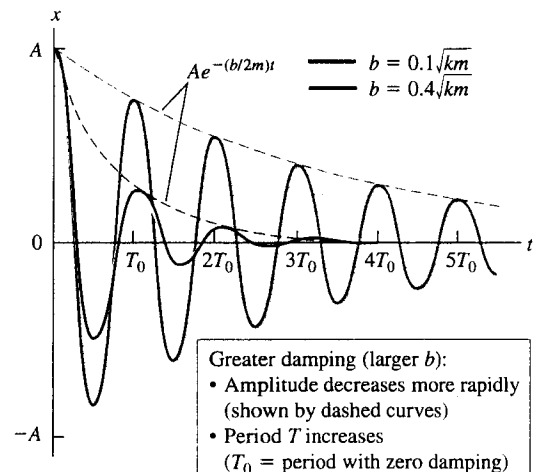
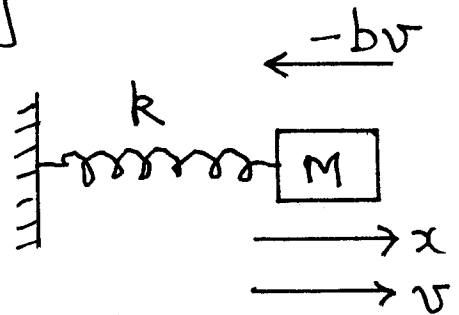
Solve de:

$$x = A_0 e^{-\frac{b}{2m}t} \cos(\omega' t + \varphi)$$

$$= A(t) \cos(\omega' t + \varphi)$$

$$\omega'^2 = \frac{k}{m} - \frac{b^2}{4m^2}$$

$$\omega'^2 < \omega_0^2 = \frac{k}{m}$$



- Amplitude decreases with time

$$E = \frac{1}{2} k A^2 \quad \text{SHM, Total Energy}$$

$$E(t) = \frac{1}{2} k A_0^2 e^{-\frac{b}{m}t}$$

$$E(t) = E_0 e^{-\frac{b}{m}t}$$

$$\gamma = \frac{b}{m} : \text{measures}$$

how long oscillator takes for energy to reach $1/e$ of its initial value.

Critical Damping:

$$\text{let } \frac{k}{m} - \frac{b^2}{4m^2} = 0$$

$$\Rightarrow b_c = 2\sqrt{km}$$

$$\text{Recall: } \omega'^2 = \omega_0^2 - \frac{\gamma^2}{4}$$

$$\text{If } b > b_c, \omega'^2 < 0$$

\therefore No oscillation!!

$$x(t) = A_1 e^{-(\frac{\gamma}{2} + \beta)t} + A_2 e^{-(\frac{\gamma}{2} - \beta)t}$$

$$\text{where, } \beta^2 = \frac{\gamma^2}{4} - \omega_0^2$$

$$b = b_c = 2\sqrt{km}$$

$$x(t) = (A + Bt) e^{-\gamma t/2}$$

Critical damping important for mechanical systems. A constant force suddenly applied to a system which was at rest results in a new position of equilibrium with no overshoot or oscillation.

Note:

$$E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$$

$$\frac{dE}{dt} = mv \frac{dv}{dt} + kx \frac{dx}{dt}$$

$$= v(ma + kx)$$

$$= -bv^2$$

If $b=0$, no damping, $\frac{dE}{dt} = 0$; $E = \text{constant}$

If $b \neq 0$, E decreases with time.