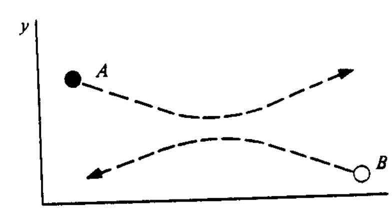
Relativistic Momentum

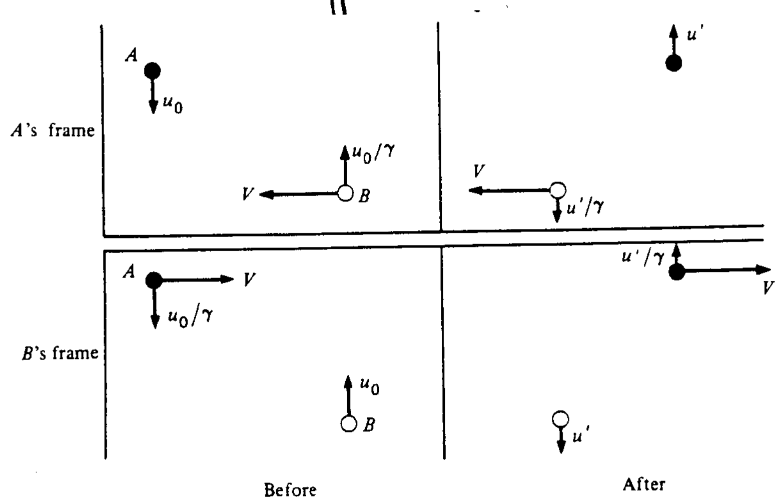
- What does Special Relativity do to Dynamics?
- What has to be changed to preserve 'Conservation of Momentum' – a pillar principle of Dynamics.

- Elastic collision of two identical particles A and B:
  Frame-A : moves along x-axis fixed to A.
  Frame-B : moves along x-axis fixed to B.

- Collision completely symmetrical:
- Each particle has the same speed u_0 in its own frame along the y-axis



- Collision alters y-velocities leaving x-motion constant.
- Relative x-velocity of the frames is V.



- The law of transformation of velocities gives the y-velocity of the opposite particle as

$$\frac{u_0}{\gamma} = u_0\sqrt{1-\frac{v^2}{c^2}}$$

- After the collision the y-velocities are both reversed.

Assume :

$$\vec{p} = m(w)\,\vec{w}$$

$m(w)$: Something that may depend on particle ~~s~~speed $\vec{w}$.

Frame-A :

x-momentum entirely due to particle-B

Before collision B's speed

$$w^2 = V^2 + \frac{u_0^2}{\gamma^2}$$

After collision

$$w'^2 = V^2 + \frac{u'^2}{\gamma^2}$$

writing conservation of momentum along-x

$$m(w)\,V = m(w')\,V$$

$$\therefore w = w'$$

$$\therefore\ u' = u_0$$

Write conservation of momentum in Frame-A along y:

$$-m(u_0)\,u_0 + m(w)\,\frac{u_0}{\gamma} = m(u_0)\,u_0 - m(w)\,\frac{u_0}{\gamma}$$

$$\therefore\ m(w) = \gamma\, m(u_0)$$

Let $m(u_0) = m_0$ as $u_0 \to 0$.

m_0 : rest mass. Mass measured in a frame where it is stationary.

In this limit $w = V$.

$$\therefore\ m(V) = \gamma\, m_0 = \frac{m_0}{\sqrt{1 - v^2/c^2}}$$

- Can be interpreted as the dependance of m on speed.

In general:

$$\vec{p} = \frac{m_0 \vec{u}}{\sqrt{1 - u^2/c^2}} = m\vec{u} = \gamma m_0 \vec{u}$$

$$m = \frac{m_0}{\sqrt{1 - u^2/c^2}} = \gamma\, m_0$$

Relativistic Energy

- Generalize classical concept
- Preserve "Conservation of Energy"
- Classical Approach

We defined:

$$K_b - K_a = \int_a^b \frac{d\vec{p}}{dt} \cdot d\vec{r}$$

Consider a particle moving with velocity $\vec{u}$

$$\vec{p} = m\vec{u} \qquad m = \text{constant here}$$

$$K_b - K_a = \int_a^b \frac{d}{dt}(m\vec{u}) \cdot d\vec{r}$$

$$\equiv \int_a^b m\frac{d\vec{u}}{dt} \cdot \vec{u}\, dt$$

$$= \int_a^b m\,\vec{u} \cdot d\vec{u}$$

Use $\vec{u} \cdot d\vec{u} = \frac{1}{2} d(\vec{u} \cdot \vec{u}) = \frac{1}{2} d(u^2) = u\,du$

$$\therefore K_b - K_a = \frac{1}{2} m u_b^2 - \frac{1}{2} m u_a^2$$

Relativity:

Try a similar approach

$$\vec{p} = \gamma m_o \vec{u}$$

$$K_b - K_a = \int_a^b \frac{d\vec{p}}{dt} \cdot d\vec{r}$$

$$= \int_a^b \frac{d}{dt}\left[\frac{m_o \vec{u}}{\sqrt{1-u^2/c^2}}\right] \cdot \vec{u}\, dt$$

$$= \int_a^b \vec{u} \cdot d\left[\frac{m_o \vec{u}}{\sqrt{1-u^2/c^2}}\right]$$

The integrand is $\vec{u} \cdot d\vec{p} = d(\vec{u}\cdot\vec{p}) - \vec{p}\cdot d\vec{u}$

$$K_b - K_a = \vec{u}\cdot\vec{p}\Big|_a^b - \int_a^b \vec{p}\cdot d\vec{u}$$

$$= \frac{m_o u^2}{\sqrt{1-u^2/c^2}}\Bigg|_a^b - \int_a^b \frac{m_o u\, du}{\sqrt{1-u^2/c^2}}$$

we used $\vec{u}\cdot d\vec{u} = u\,du$ [see earlier]

Integral is elementary,

$$K_b - K_a = \frac{m_o u^2}{\sqrt{1-u^2/c^2}}\Bigg|_a^b + m_o c^2 \sqrt{1-\frac{u^2}{c^2}}\,\Bigg|_a^b$$

b is an arbitrary point
Assume at point a, $u_a = 0$

$$K = \gamma m_o u^2 + \frac{m_o c^2}{\gamma} - m_o c^2$$

$$= \gamma m_o \left[u^2 + c^2 (1 - u^2/c^2)\right] - m_o c^2$$

$$= \gamma m_o c^2 - m_o c^2$$

$$K = mc^2 - m_o c^2 \qquad \text{where } m = \gamma m_o$$

What happens when $v \ll c$

$$\gamma \sim 1 + \frac{1}{2} u^2 c^2 + \cdots$$

$$K = \frac{m_o c^2}{\sqrt{1 - u^2/c^2}} - m_o c^2$$

$$\simeq m_o c^2 \left[1 + \frac{1}{2}\frac{u^2}{c^2} - 1\right]$$

$$K = \frac{1}{2} m_o u^2 \qquad \text{[Classical Result]}$$

KE results from work done on particle to bring it from rest to speed u.

Rewrite:

$$mc^2 = K + m_0c^2$$

$$= \text{Work Done} + m_0c^2$$

Einstein:

$$mc^2 \equiv \text{Total Energy, } E\text{, of particle}$$

$$= \text{External Work} + \text{"Rest" Energy}$$

$$\therefore E = mc^2 = \gamma m_0 c^2$$

If an energy ΔE is added to an object, its mass changes by

$$\Delta m = \frac{\Delta E}{c^2}$$

Energy / Momentum

Classically, $E = \frac{1}{2} m v^2 = \frac{p^2}{2m}$ $\quad u = 0$

$$\vec{p} = m\vec{u} = \frac{m_0 \vec{u}}{\sqrt{1 - u^2/c^2}} = \gamma m_0 \vec{u}$$

$$E = mc^2 = \gamma m_0 c^2$$

$$p^2 = \gamma^2 m_0^2 u^2 = \frac{1}{1 - u^2/c^2} m_0^2 u^2$$

$$\frac{u^2}{c^2} = \frac{p^2}{p^2 + m_0^2 c^2}$$

$$\gamma = \frac{1}{\sqrt{1 - u^2/c^2}} = \sqrt{1 + \frac{p^2}{m_0^2 c^2}}$$

$$\therefore E = m_0 c^2 \sqrt{1 + \frac{p^2}{m_0^2 c^2}}$$

$$\therefore E^2 = (pc)^2 + (m_0 c^2)^2$$

$$E^2 - (pc)^2 = (m_0 c^2)^2$$

4-Vector Invariant Quantity

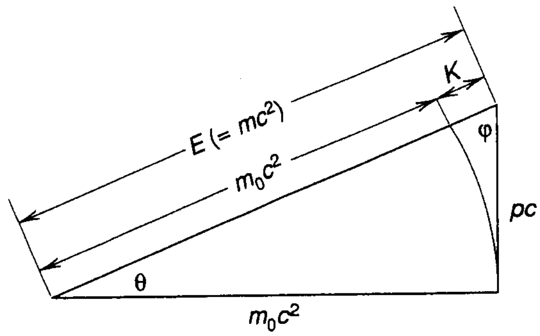


FIGURE 3-3. A mnemonic device, using a right triangle and the Pythagorean relation, to help in remembering the relations between total energy E, rest energy m_0c^2, and momentum p; see Eq. 3-13b. Shown also is the relation $E = m_0c^2 + K$ between total energy, rest energy, and kinetic energy. You can show that $\sin\theta = \beta$ and $\sin\phi = 1/\gamma$.

$$\sin\theta = \beta = \frac{u}{c}$$

$$\sin\varphi = \frac{1}{\gamma}$$

Massless Particles

$$E^2 = (pc)^2 + (m_0 c^2)^2$$

If $m_0 = 0$

$$E = pc$$

$$p = \gamma m_0 \vec{u} = \frac{m_0}{\sqrt{1 - u^2/c^2}} \vec{u}$$

As $m_0 \to 0$ $\vec{p}$ must remain finite

Only possible if $u \to c$ as $m_0 \to 0$

$\therefore$ massless particles must travel with c!

Photons have $m_0 = 0$
Neutrinos have $m_\nu \sim 0$ [close/maybe?]

Forces and Relativity

$$\vec{F} = \frac{d\vec{p}}{dt} = \frac{d}{dt}(m\vec{v})$$

$$= m\frac{d\vec{v}}{dt} + \vec{v}\frac{dm}{dt} \qquad ①$$

$$m = E/c^2$$

$$\therefore \frac{dm}{dt} = \frac{1}{c^2}\frac{dE}{dt} = \frac{1}{c^2}\frac{d}{dt}(K + m_0c^2) = \frac{1}{c^2}\frac{dK}{dt}$$

but

$$\frac{dK}{dt} = \frac{\vec{F}\cdot d\vec{l}}{dt} = \vec{F}\cdot\frac{d\vec{l}}{dt} = \vec{F}\cdot\vec{v}$$

so

$$\frac{dm}{dt} = \frac{1}{c^2}\vec{F}\cdot\vec{v}$$

Sub. into ①

$$\vec{F} = m\frac{d\vec{v}}{dt} + \frac{\vec{v}(\vec{F}\cdot\vec{v})}{c^2}$$

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{1}{m}\left[\vec{F} - \frac{\vec{v}}{c^2}(\vec{F}\cdot\vec{v})\right]$$

$\vec{a}$ is no longer // to $\vec{F}$,
has a component // to $\vec{v}$ itself.

Relativistic Force / Acceleration