

**QUIZ 10**  
**FORMULA SHEET**  
**Quiz Date: Friday, April 15, 2005**

For motion in one dimension:

$$v_{\text{av}} \equiv \frac{\Delta x}{\Delta t} \quad \text{Average velocity;}$$

$$v \equiv \frac{dx}{dt} \quad \text{Instantaneous velocity;}$$

For motion in three dimensions:

$$\vec{v} \equiv \frac{d\vec{r}}{dt}; \quad \vec{a} \equiv \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2}; \quad \vec{r}(t_1) = \vec{r}_0 + \int_0^{t_1} \vec{v} dt; \quad \vec{v}(t_1) = \vec{v}_0 + \int_0^{t_1} \vec{a} dt .$$

For *constant* acceleration  $\vec{a}$ , if  $\vec{r} = \vec{r}_0$  and  $\vec{v} = \vec{v}_0$  at time  $t = 0$ , then

$$\vec{v}(t) = \vec{v}_0 + \vec{a}t$$

$$\vec{r}(t) = \vec{r}_0 + \vec{v}_0t + \frac{1}{2}\vec{a}t^2 .$$

For one-dimensional motion with constant acceleration  $a$ :

$$v^2 = v_0^2 + 2a(x - x_0) .$$

For circular motion at constant speed  $v$ :

$$a = \frac{v^2}{r} ,$$

where  $r$  is the radius of the circle, and the acceleration is directed towards the center of the circle.

If an object has position  $\vec{r}$  and velocity  $\vec{v}$ , its position and velocity relative to an observer with position  $\vec{r}_0$  and velocity  $\vec{v}_0$  are given respectively by

$$\vec{r}' = \vec{r} - \vec{r}_0 , \quad \vec{v}' = \vec{v} - \vec{v}_0 .$$

Average velocity and acceleration are given by

$$\vec{v}_{\text{average}} \equiv \frac{\Delta\vec{r}}{\Delta t} , \quad \vec{a}_{\text{average}} \equiv \frac{\Delta\vec{v}}{\Delta t} .$$

### Mass, Acceleration, and Force:

$$\vec{\mathbf{F}} = m\vec{\mathbf{a}} \quad (\text{Newton's second law});$$

$$\vec{\mathbf{F}} = -\frac{GMm}{r^2}\hat{\mathbf{r}} \quad (\text{the gravitational force between two particles});$$

$$\vec{\mathbf{F}} = \frac{1}{4\pi\epsilon_0} \frac{Qq}{r^2}\hat{\mathbf{r}} \quad (\text{the electrostatic force between two particles});$$

$$F_x = -kx \quad (\text{Hooke's law});$$

where  $\hat{\mathbf{r}}$  is a unit vector pointing from the particle which is the source of the force, toward the particle on which the force is acting.

### Friction:

$$|\vec{\mathbf{F}}_k| = \mu_k |\vec{\mathbf{N}}| \quad (\text{kinetic friction});$$

$$|\vec{\mathbf{F}}_s| \leq \mu_s |\vec{\mathbf{N}}| \quad (\text{static friction}).$$

### Kinetic Energy, Work, and Potential Energy:

Description	1 Dimension	3 Dimensions
Work done by a constant force $\vec{\mathbf{F}}$	$W \equiv F\Delta x$	$W \equiv \vec{\mathbf{F}} \cdot \vec{\Delta\mathbf{r}}$
Work done by a varying force $\vec{\mathbf{F}}$	$W \equiv \int F(x) dx$	$W \equiv \int_{\vec{\mathbf{r}}_1}^{\vec{\mathbf{r}}_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$
Potential energy derived from force $\vec{\mathbf{F}}$	$U(x_p) \equiv U_0 - \int_{x_0}^{x_p} F dx$	$U(\vec{\mathbf{r}}_p) \equiv U_0 - \int_{\vec{\mathbf{r}}_0}^{\vec{\mathbf{r}}_p} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$
Force derived from potential energy	$F = -\frac{dU}{dx}$	$\vec{\mathbf{F}} = \left[ -\frac{\partial U}{\partial x}, -\frac{\partial U}{\partial y}, -\frac{\partial U}{\partial z} \right]$

$\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} \equiv  \vec{\mathbf{a}}  \vec{\mathbf{b}}  \cos \theta$	(scalar (or dot) product of two vectors);
$= a_x b_x + a_y b_y + a_z b_z$	
$E_k \equiv \frac{1}{2} m v^2$	(kinetic energy of a particle);
$W_{\text{tot}} = E_{k,f} - E_{k,i}$	(work-energy theorem: always true if $W_{\text{tot}}$ includes work due to all forces; a non-rigid object can do work on itself!);
$E_{k,i} + U_i + W_{\text{other}}$	(generalized work-energy theorem: always true if $W_{\text{other}}$ includes work due to all forces not included in $U$ );
$= E_{k,f} + U_f$	
$\frac{1}{2} m v^2 + U(x) = \text{constant}$	(conservation of mechanical energy: true in the absence of dissipative forces);
$\frac{1}{2} m v^2 + mgh = \frac{1}{2} m v_0^2$	(conservation of mechanical energy for a projectile: true in the absence of dissipative forces);
$W = \frac{1}{2} k x^2$	(work to compress a spring);
$U = \frac{1}{2} k x^2$	(potential energy for spring force);
$W = mgh$	(work to lift a body near the surface of the Earth);
$U = mgh$	(gravitational potential energy, near the surface of the Earth);
$U = -\frac{GMm}{r}$	(gravitational potential energy, spherical bodies);
$U = \frac{1}{4\pi\epsilon_0} \frac{Qq}{r}$	(electrostatic potential energy, spherical charges).

### Momentum, Center of Mass, and Systems of Particles:

$\vec{\mathbf{F}}_{\text{AB}} = -\vec{\mathbf{F}}_{\text{BA}}$	(Newton's third law);
$\vec{\mathbf{p}} \equiv m\vec{\mathbf{v}}$	(momentum);
$\frac{d\vec{\mathbf{P}}_{\text{tot}}}{dt} = 0$	(conservation of momentum in absence of external force)
$\vec{\mathbf{F}} = \frac{d\vec{\mathbf{p}}}{dt}$	(Newton's second law in terms of momentum);
$\vec{\mathbf{r}}_{\text{cm}} \equiv \frac{1}{M_{\text{tot}}} \sum_i m_i \vec{\mathbf{r}}_i$	(position of center of mass);

$$\vec{v}_{\text{cm}} \equiv \frac{d\vec{r}_{\text{cm}}}{dt} = \frac{1}{M_{\text{tot}}} \sum_i m_i \vec{v}_i \quad (\text{velocity of center of mass});$$

$$\vec{\mathbf{F}}_{\text{tot}}^{\text{ext}} = M_{\text{tot}} \vec{\mathbf{a}}_{\text{cm}} = \frac{d\vec{\mathbf{P}}_{\text{tot}}}{dt} \quad (\text{acceleration of a system of particles});$$

$$\vec{\mathbf{P}}_{\text{tot}} = \sum_i m_i \vec{v}_i = M_{\text{tot}} \vec{v}_{\text{cm}} \quad (\text{momentum of a system of particles});$$

$$K_{\text{tot}} = \frac{1}{2} M_{\text{tot}} v_{\text{cm}}^2 + \sum_i \frac{1}{2} m_i (\vec{v}_i - \vec{v}_{\text{cm}})^2 \quad (\text{K.E. of a system of particles});$$

$$\vec{\mathbf{J}} = \int_{t_1}^{t_2} \vec{\mathbf{F}} dt = \int_{t_1}^{t_2} \frac{d\vec{\mathbf{p}}}{dt} dt = \vec{\mathbf{p}}_2 - \vec{\mathbf{p}}_1 \quad (\text{impulse-momentum theorem}).$$

### Rotation in Two Dimensions:

Most of the equations for this topic are most easily remembered in the context of the analogous equations for linear motion in one dimension:

TRANSLATION (one dimension)		ROTATION (about fixed axis)	
Name	Symbol	Name	Symbol
Position	$x$	Orientation	$\theta$
Velocity	$v = \frac{dx}{dt}$	Angular velocity	$\omega = \frac{d\theta}{dt}$
Acceleration	$a = \frac{dv}{dt}$	Angular acceleration	$\alpha = \frac{d\omega}{dt}$
Mass	$M = \sum_i m_i$	Moment of inertia	$I = \sum_i m_i R_i^2$
Force	$F$	Torque	$\tau = F_{\perp} R$ $= \pm  \vec{\mathbf{F}}  R_{\perp}$
Force equation	$\sum_i \vec{\mathbf{F}}^{\text{ext}} = M \vec{\mathbf{a}}_{\text{cm}}$	Torque equation	$\sum_i \tau^{\text{ext}} = I \alpha$
Momentum	$p = Mv$	Angular momentum	$L = I\omega$
Kinetic energy	$\frac{1}{2} Mv^2$	Kinetic energy	$\frac{1}{2} I\omega^2$
Work done	$\vec{\mathbf{F}} \cdot \Delta \vec{\mathbf{r}}$	Work done	$\tau \Delta \theta$

Other equations about rotation in two dimensions:

$$v_r = 0 ; \quad v_{\perp} = R\omega \quad (\text{velocity of point on rotating body});$$

$$a_r = -\frac{v^2}{R} = -R\omega^2 ; \quad a_{\perp} = R\alpha \quad (\text{acceleration of point on rotating body});$$

$$v = \pm R|\omega| \quad (\text{rolling without slipping});$$

$$\left. \begin{aligned} \sum \vec{\mathbf{F}}^{\text{ext}} &= M\vec{\mathbf{a}}_{\text{cm}} = \frac{d\vec{\mathbf{p}}}{dt} \\ \sum \vec{\boldsymbol{\tau}}^{\text{ext}} &= I_{\text{cm}}\alpha = \frac{dL}{dt} \end{aligned} \right\} \quad (\text{combined translational and rotational motion});$$

$$K_{\text{tot}} = \frac{1}{2}Mv_{\text{cm}}^2 + \frac{1}{2}I_{\text{cm}}\omega^2 \quad (\text{kinetic energy for combined translational and rotational motion});$$

$$I_{\parallel} = I_{\text{cm}} + Md^2 \quad (\text{parallel-axis theorem});$$

$$I_z = I_x + I_y \quad (\text{perpendicular-axis theorem}).$$

**TABLE OF STANDARD MOMENTS OF INERTIA:**

Slender uniform rod of length $\ell$ , axis through center and perpendicular to axis of rod	$\frac{1}{12}m\ell^2$
Rectangular plate with dimensions $a \times b$ , axis along one of the $b$ edges	$\frac{1}{3}ma^2$
Thin-walled hollow cylinder of radius $R$ , axis along axis of cylinder	$mR^2$
Uniform solid cylinder of radius $R$ , axis along axis of cylinder	$\frac{1}{2}mR^2$
Thin-walled hollow sphere of radius $R$ , axis through center	$\frac{2}{3}mR^2$
Solid uniform sphere of radius $R$ , axis through center	$\frac{2}{5}mR^2$

## Rotations in Vector Notation:

$$\begin{aligned} c_x &= a_y b_z - a_z b_y ; \\ c_y &= a_z b_x - a_x b_z ; \\ c_z &= a_x b_y - a_y b_x . \end{aligned} \quad (\text{vector cross product, component form});$$

$$|\vec{c}| = |\vec{a}||\vec{b}| \sin \theta \quad (\text{magnitude of vector cross product});$$

$$\vec{v} = \vec{\omega} \times \vec{r} \quad (\text{velocity of atom in rotating body with a fixed point});$$

$$\vec{v} = \vec{v}_P + \vec{\omega} \times (\vec{r} - \vec{r}_P) \quad (\text{velocity of atom in rotating body, general case});$$

$$\vec{L} = \sum_i \vec{r}_i \times \vec{p}_i \quad (\text{angular momentum, as vector product});$$

$$\vec{\tau} = \sum_i \vec{r}_i \times \vec{F}_i \quad (\text{vector torque, as vector product});$$

$$\vec{\tau} = \frac{d\vec{L}}{dt} \quad (\text{torque equation});$$

$$\left. \begin{aligned} \sum \vec{F}^{\text{ext}} &= M\vec{a}_{\text{cm}} = \frac{d\vec{p}}{dt} \\ \sum \vec{\tau}^{\text{ext}} &= \frac{d\vec{L}_{\text{cm}}}{dt} \end{aligned} \right\} (\text{combined translational and rotational motion});$$

$$\begin{aligned} \vec{L} &= \vec{r}_{\text{cm}} \times \vec{p}_{\text{tot}} \\ &+ \sum_i \vec{r}_{\text{rel},i} \times m_i \vec{v}_{\text{rel},i} \end{aligned} \quad (\text{angular momentum decomposition});$$

$$\begin{aligned} \vec{\tau} &= \vec{r}_{\text{cm}} \times \vec{F}_{\text{tot}} \\ &+ \sum_i \vec{r}_{\text{rel},i} \times \vec{F}_i \end{aligned} \quad (\text{torque decomposition}).$$

## For Static Bodies:

$$\sum \vec{F}^{\text{ext}} = 0 \quad (\text{total external force vanishes});$$

$$\sum \vec{\tau}^{\text{ext}} = 0 \quad (\text{total external torque about ANY point vanishes}).$$