MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Physics

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QUIZ 12
FORMULA SHEET
Quiz Date: Friday, May 6, 2005

For motion in one dimension:

$$v_{\mathrm{av}} \equiv \frac{\Delta x}{\Delta t} \qquad \text{Average velocity;}$$

$$v \equiv \frac{\mathrm{d}x}{\mathrm{d}t} \qquad \text{Instantaneous velocity;}$$

For motion in three dimensions:

$$\vec{\mathbf{v}} \equiv \frac{\mathrm{d}\vec{\mathbf{r}}}{\mathrm{d}t}\,; \qquad \vec{\mathbf{a}} \equiv \frac{\mathrm{d}\vec{\mathbf{v}}}{\mathrm{d}t} = \frac{\mathrm{d}^2\vec{\mathbf{r}}}{\mathrm{d}t^2}\,; \qquad \vec{\mathbf{r}}(t_1) = \vec{\mathbf{r}}_0 + \int_0^{t_1} \vec{\mathbf{v}}\,\mathrm{d}t\,; \qquad \vec{\mathbf{v}}(t_1) = \vec{\mathbf{v}}_0 + \int_0^{t_1} \vec{\mathbf{a}}\,\mathrm{d}t\ .$$

For *constant* acceleration $\vec{\mathbf{a}}$, if $\vec{\mathbf{r}} = \vec{\mathbf{r}}_0$ and $\vec{\mathbf{v}} = \vec{\mathbf{v}}_0$ at time $t = 0$, then

$$\vec{\mathbf{v}}(t) = \vec{\mathbf{v}}_0 + \vec{\mathbf{a}}t$$

$$\vec{\mathbf{r}}(t) = \vec{\mathbf{r}}_0 + \vec{\mathbf{v}}_0 t + \frac{1}{2}\vec{\mathbf{a}}t^2\ .$$

For one-dimensional motion with constant acceleration a:

$$v^2 = v_0^2 + 2a(x - x_0)\ .$$

For circular motion at constant speed v:

$$a = \frac{v^2}{r}\ ,$$

where r is the radius of the circle, and the acceleration is directed towards the center of the circle.

If an object has position $\vec{\mathbf{r}}$ and velocity $\vec{\mathbf{v}}$, its position and velocity relative to an observer with position $\vec{\mathbf{r}}_0$ and velocity $\vec{\mathbf{v}}_0$ are given respectively by

$$\vec{\mathbf{r}}' = \vec{\mathbf{r}} - \vec{\mathbf{r}}_0\ , \qquad \vec{\mathbf{v}}' = \vec{\mathbf{v}} - \vec{\mathbf{v}}_0\ .$$

Average velocity and acceleration are given by

$$\vec{\mathbf{v}}_{\mathrm{average}} \equiv \frac{\Delta\vec{\mathbf{r}}}{\Delta t}\ , \qquad \vec{\mathbf{a}}_{\mathrm{average}} \equiv \frac{\Delta\vec{\mathbf{v}}}{\Delta t}\ .$$

Mass, Acceleration, and Force:

$$\vec{\mathbf{F}} = m\vec{\mathbf{a}} \qquad \text{(Newton's second law);}$$

$$\vec{\mathbf{F}} = -\frac{GMm}{r^2}\hat{\boldsymbol{r}} \qquad \text{(the gravitational force between two particles);}$$

$$\vec{\mathbf{F}} = \frac{1}{4\pi\epsilon_0}\frac{Qq}{r^2}\hat{\boldsymbol{r}} \qquad \text{(the electrostatic force between two particles);}$$

$$F_x = -kx \qquad \text{(Hooke's law);}$$

where $\hat{\boldsymbol{r}}$ is a unit vector pointing from the particle which is the source of the force, toward the particle on which the force is acting.

Friction:

$$\left|\vec{\mathbf{F}}_k\right| = \mu_k \left|\vec{\mathbf{N}}\right| \qquad \text{(kinetic friction);}$$

$$\left|\vec{\mathbf{F}}_s\right| \le \mu_s \left|\vec{\mathbf{N}}\right| \qquad \text{(static friction).}$$

Kinetic Energy, Work, and Potential Energy:

Description	**1 Dimension**	**3 Dimensions**
Work done by a constant force $\vec{\mathbf{F}}$	$W \equiv F\Delta x$	$W \equiv \vec{\mathbf{F}} \cdot \overrightarrow{\mathbf{\Delta r}}$
Work done by a varying force $\vec{\mathbf{F}}$	$W \equiv \int F(x)\,\mathrm{d}x$	$W \equiv \int_{\vec{\mathbf{r}}_1}^{\vec{\mathbf{r}}_2} \vec{\mathbf{F}} \cdot \overrightarrow{\mathbf{dr}}$
Potential energy derived from force $\vec{\mathbf{F}}$	$U(x_p) \equiv U_0 - \int_{x_0}^{x_p} F\,\mathrm{d}x$	$U(\vec{\mathbf{r}}_p) \equiv U_0 - \int_{\vec{\mathbf{r}}_0}^{\vec{\mathbf{r}}_p} \vec{\mathbf{F}} \cdot \overrightarrow{\mathbf{dr}}$
Force derived from potential energy	$F = -\frac{\mathrm{d}U}{\mathrm{d}x}$	$\vec{\mathbf{F}} = \left[-\frac{\partial U}{\partial x}, -\frac{\partial U}{\partial y}, -\frac{\partial U}{\partial z}\right]$

$\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} \equiv \lvert\vec{\mathbf{a}}\rvert\lvert\vec{\mathbf{b}}\rvert \cos\theta$ $= a_x b_x + a_y b_y + a_z b_z$	(scalar (or dot) product of two vectors);
$E_k \equiv \frac{1}{2}mv^2$	(kinetic energy of a particle);
$W_{\text{tot}} = E_{k,f} - E_{k,i}$	(work-energy theorem: always true if W_{tot} includes work due to all forces; a non-rigid object can do work on itself!);
$E_{k,i} + U_i + W_{\text{other}}$ $= E_{k,f} + U_f$	(generalized work-energy theorem: always true if W_{other} includes work due to all forces not included in U);
$\frac{1}{2}mv^2 + U(x) = \text{constant}$	(conservation of mechanical energy: true in the absence of dissipative forces);
$\frac{1}{2}mv^2 + mgh = \frac{1}{2}mv_0^2$	(conservation of mechanical energy for a projectile: true in the absence of dissipative forces);
$W = \frac{1}{2}kx^2$	(work to compress a spring);
$U = \frac{1}{2}kx^2$	(potential energy for spring force);
$W = mgh$	(work to lift a body near the surface of the Earth);
$U = mgh$	(gravitational potential energy, near the surface of the Earth);
$U = -\frac{GMm}{r}$	(gravitational potential energy, spherical bodies);
$U = \frac{1}{4\pi\epsilon_0}\frac{Qq}{r}$	(electrostatic potential energy, spherical charges).

Momentum, Center of Mass, and Systems of Particles:

$\vec{\mathbf{F}}_{\text{AB}} = -\vec{\mathbf{F}}_{\text{BA}}$	(Newton's third law);
$\vec{\mathbf{p}} \equiv m\vec{\mathbf{v}}$	(momentum);
$\frac{\mathrm{d}\vec{\mathbf{P}}_{\text{tot}}}{\mathrm{d}t} = 0$	(conservation of momentum in absence of external force)
$\vec{\mathbf{F}} = \frac{\mathrm{d}\vec{\mathbf{p}}}{\mathrm{d}t}$	(Newton's second law in terms of momentum);
$\vec{\mathbf{r}}_{\text{cm}} \equiv \frac{1}{M_{\text{tot}}}\sum_i m_i \vec{\mathbf{r}}_i$	(position of center of mass);

$$\vec{\mathbf{v}}_{\text{cm}} \equiv \frac{\mathrm{d}\vec{\mathbf{r}}_{\text{cm}}}{\mathrm{d}t} = \frac{1}{M_{\text{tot}}} \sum_i m_i \vec{\mathbf{v}}_i \qquad \text{(velocity of center of mass);}$$

$$\vec{\mathbf{F}}^{\text{ext}}_{\text{tot}} = M_{\text{tot}} \vec{\mathbf{a}}_{\text{cm}} = \frac{\mathrm{d}\vec{\mathbf{P}}_{\text{tot}}}{\mathrm{d}t} \qquad \text{(acceleration of a system of particles);}$$

$$\vec{\mathbf{P}}_{\text{tot}} = \sum_i m_i \vec{\mathbf{v}}_i = M_{\text{tot}} \vec{\mathbf{v}}_{\text{cm}} \qquad \text{(momentum of a system of particles);}$$

$$K_{\text{tot}} = \frac{1}{2} M_{\text{tot}} v_{\text{cm}}^2 + \sum_i \frac{1}{2} m_i \left(\vec{\mathbf{v}}_i - \vec{\mathbf{v}}_{\text{cm}}\right)^2 \qquad \text{(K.E. of a system of particles);}$$

$$\vec{\mathbf{J}} = \int_{t_1}^{t_2} \vec{\mathbf{F}}\, \mathrm{d}t = \int_{t_1}^{t_2} \frac{\mathrm{d}\vec{\mathbf{p}}}{\mathrm{d}t}\, \mathrm{d}t = \vec{\mathbf{p}}_2 - \vec{\mathbf{p}}_1 \qquad \text{(impulse-momentum theorem).}$$

Rotation in Two Dimensions:

Most of the equations for this topic are most easily remembered in the context of the analogous equations for linear motion in one dimension:

TRANSLATION (one dimension)		ROTATION (about fixed axis)	
Name	Symbol	Name	Symbol
Position	x	Orientation	θ
Velocity	$v = \frac{\mathrm{d}x}{\mathrm{d}t}$	Angular velocity	$\omega = \frac{\mathrm{d}\theta}{\mathrm{d}t}$
Acceleration	$a = \frac{\mathrm{d}v}{\mathrm{d}t}$	Angular acceleration	$\alpha = \frac{\mathrm{d}\omega}{\mathrm{d}t}$
Mass	$M = \sum_i m_i$	Moment of inertia	$I = \sum_i m_i R_i^2$
Force	F	Torque	$\boldsymbol{\tau} = F_\perp R$ $= \pm\lvert\vec{\mathbf{F}}\rvert R_\perp$
Force equation	$\sum_i \vec{\mathbf{F}}^{\text{ext}} = M\vec{\mathbf{a}}_{\text{cm}}$	Torque equation	$\sum_i \boldsymbol{\tau}^{\text{ext}} = I\alpha$
Momentum	$p = Mv$	Angular momentum	$L = I\omega$
Kinetic energy	$\frac{1}{2}Mv^2$	Kinetic energy	$\frac{1}{2}I\omega^2$
Work done	$\vec{\mathbf{F}} \cdot \overrightarrow{\boldsymbol{\Delta r}}$	Work done	$\boldsymbol{\tau}\, \Delta\theta$

Other equations about rotation in two dimensions:

$v_r = 0\ ;\quad v_\perp = R\omega$	(velocity of point on rotating body);
$a_r = -\dfrac{v^2}{R} = -R\omega^2\ ;\quad a_\perp = R\alpha$	(acceleration of point on rotating body);
$v = \pm R\lvert\omega\rvert$	(rolling without slipping);
$\left.\begin{aligned} \sum \vec{\mathbf{F}}^{\text{ext}} &= M\vec{\mathbf{a}}_{\text{cm}} = \frac{\mathrm{d}\vec{\mathbf{p}}}{\mathrm{d}t} \\ \sum \boldsymbol{\tau}^{\text{ext}} &= I_{\text{cm}}\alpha = \frac{\mathrm{d}L}{\mathrm{d}t} \end{aligned}\right\}$	(combined translational and rotational motion);
$K_{\text{tot}} = \frac{1}{2}Mv_{\text{cm}}^2 + \frac{1}{2}I_{\text{cm}}\omega^2$	(kinetic energy for combined translational and rotational motion);
$I_\parallel = I_{\text{cm}} + Md^2$	(parallel-axis theorem);
$I_z = I_x + I_y$	(perpendicular-axis theorem).

TABLE OF STANDARD MOMENTS OF INERTIA:

Slender uniform rod of length ℓ, axis through center and perpendicular to axis of rod	$\frac{1}{12}m\ell^2$
Rectangular plate with dimensions $a \times b$, axis along one of the b edges	$\frac{1}{3}ma^2$
Thin-walled hollow cylinder of radius R, axis along axis of cylinder	mR^2
Uniform solid cylinder of radius R, axis along axis of cylinder	$\frac{1}{2}mR^2$
Thin-walled hollow sphere of radius R, axis through center	$\frac{2}{3}mR^2$
Solid uniform sphere of radius R, axis through center	$\frac{2}{5}mR^2$

Rotations in Vector Notation:

$$\begin{aligned} c_x &= a_y b_z - a_z b_y \ ; \\ c_y &= a_z b_x - a_x b_z \ ; \\ c_z &= a_x b_y - a_y b_x \ . \end{aligned}$$ (vector cross product, component form);

$$|\vec{\mathbf{c}}| = |\vec{\mathbf{a}}||\vec{\mathbf{b}}| \sin\theta$$ (magnitude of vector cross product);

$$\vec{\mathbf{v}} = \vec{\boldsymbol{\omega}} \times \vec{\mathbf{r}}$$ (velocity of atom in rotating body with a fixed point);

$$\vec{\mathbf{v}} = \vec{\mathbf{v}}_P + \vec{\boldsymbol{\omega}} \times \left(\vec{\mathbf{r}} - \vec{\mathbf{r}}_P\right)$$ (velocity of atom in rotating body, general case);

$$\vec{\mathbf{L}} = \sum_i \vec{\mathbf{r}}_i \times \vec{\mathbf{p}}_i$$ (angular momentum, as vector product);

$$\vec{\boldsymbol{\tau}} = \sum_i \vec{\mathbf{r}}_i \times \vec{\mathbf{F}}_i$$ (vector torque, as vector product);

$$\vec{\boldsymbol{\tau}} = \frac{\mathrm{d}\vec{\mathbf{L}}}{\mathrm{d}t}$$ (torque equation);

$$\left.\begin{aligned} \sum \vec{\mathbf{F}}^{\mathrm{ext}} &= M\vec{\mathbf{a}}_{\mathrm{cm}} = \frac{\mathrm{d}\vec{\mathbf{p}}}{\mathrm{d}t} \\ \sum \vec{\boldsymbol{\tau}}^{\mathrm{ext}} &= \frac{\mathrm{d}\vec{\mathbf{L}}_{\mathrm{cm}}}{\mathrm{d}t} \end{aligned}\right\}$$ (combined translational and rotational motion);

$$\begin{aligned} \vec{\mathbf{L}} &= \vec{\mathbf{r}}_{\mathrm{cm}} \times \vec{\mathbf{p}}_{\mathrm{tot}} \\ &\quad + \sum_i \vec{\mathbf{r}}_{\mathrm{rel},i} \times m_i \vec{\mathbf{v}}_{\mathrm{rel},i} \end{aligned}$$ (angular momentum decomposition);

$$\begin{aligned} \vec{\boldsymbol{\tau}} &= \vec{\mathbf{r}}_{\mathrm{cm}} \times \vec{\mathbf{F}}_{\mathrm{tot}} \\ &\quad + \sum_i \vec{\mathbf{r}}_{\mathrm{rel},i} \times \vec{\mathbf{F}}_i \end{aligned}$$ (torque decomposition).

For Static Bodies:

$$\sum \vec{\mathbf{F}}^{\mathrm{ext}} = 0$$ (total external force vanishes);

$$\sum \vec{\boldsymbol{\tau}}^{\mathrm{ext}} = 0$$ (total external torque about ANY point vanishes).

Gravitation:

$F_g = \dfrac{Gm_1 m_2}{r^2}$	(Newton's law of gravity);
$g = \dfrac{Gm_E}{R_E^2}$	(Acceleration due to gravity at Earth's surface);
$U = -\dfrac{Gm_E m}{r}$	(Gravitational potential energy);
$v = \sqrt{\dfrac{Gm_E}{r}}$	(Speed in cicular orbit);
$T = \dfrac{2\pi r}{v} = \dfrac{2\pi r^{3/2}}{\sqrt{Gm_E}}$	(Period of a circular orbit);
$R_S = \dfrac{2GM}{c^2}$	(Schwarzschild radius).

Periodic Motion:

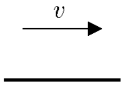
$f = \dfrac{1}{T}$	(Frequency and period);
$\omega = 2\pi f = \dfrac{2\pi}{T}$	(angular frequency);
$F_x = -kx$	(simple harmonic oscillator);
$\omega = \sqrt{\dfrac{k}{m}}$	(angular frequency of simple harmonic oscillator);
$x = A\cos(\omega t + \phi)$	(motion of simple harmonic oscillator);
$\omega = \sqrt{\dfrac{\kappa}{I}}$	(angular frequency ω in terms of torsion constant κ and moment of inertia I)

Special Relativity:

(1) TIME DILATION: Any clock which is moving at speed v relative to a given reference frame will appear (to an observer using that reference frame) to run slower than normal by a factor denoted by the Greek letter γ (gamma), and given by

$$\gamma \equiv \frac{1}{\sqrt{1-\beta^2}} \ , \qquad \beta \equiv v/c \ .$$

(2) LORENTZ-FITZGERALD CONTRACTION: Any rod which is moving at a speed v along its length relative to a given reference frame will appear (to an observer using that reference frame) to be shorter than its normal length by the same factor γ. A rod which is moving perpendicular to its length does not undergo a change in apparent length.



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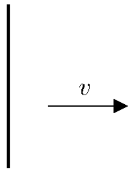
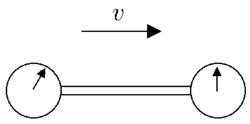


Uncontracted

(3) RELATIVITY OF SIMULTANEITY: Suppose a rod which has rest length ℓ_0 is equipped with a clock at each end. The clocks can be synchronized in the rest frame of the system by using light pulses. If the system moves at speed v along its length, then the trailing clock will appear to read a time which is later than the leading clock by an amount $\beta\ell_0/c$. If, on the other hand, the system moves perpendicular to its length, then the synchronization of the clocks is not disturbed.



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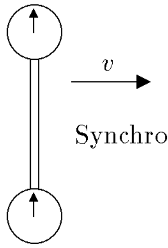

Synchronized

There are a few minor qualifications that must be appended to the above statements. First, they hold only for inertial reference frames— they do not hold for rotating or accelerating reference frames. Any reference frame which moves at a uniform velocity relative to an inertial reference frame is also an inertial reference frame. Second, one must define the word "appear" which occurs in each of the three statements. In plain English, the word "appear" normally refers to the perception of the human eyes. However, in these situations the perception of the human eyes would be very complicated. The complication is that one sees with light, and light travels with a less-than-infinite speed. Thus, when an object is moving toward you, the light which you see coming from the front of the object has left the object later than the light which you see coming from the back of the object. Effects of this kind lead to complicated distortions, which are not taken into account in the statements above. For purposes of interpreting these statements, one can imagine that each reference frame is covered by an infinite number of local observers, each of which observes only events so close that the time delay for light travel is negligible. Each observer carries a clock which has been synchronized with the others by light pulses. The "appearance" is then the description which is assembled after the fact by combining the reports of these local observers.

The Lorentz Transformation:

If an (x', y', z', t') inertial coordinate system is moving to the right (positive x direction) with speed v relative to an (x, y, z, t) inertial coordinate system, then the coordinates are related by

$$\begin{aligned} x' &= \gamma(x - vt) \\ t' &= \gamma\left(t - \frac{vx}{c^2}\right) \\ y' &= y \\ z' &= z \ . \end{aligned}$$

and

$$\begin{aligned} x &= \gamma(x' + vt') \\ t &= \gamma\left(t' + \frac{vx'}{c^2}\right) \\ y &= y' \\ z &= z' \ . \end{aligned}$$

Relativistic velocity addition and subtraction:

$$\begin{aligned} v'_x &= \frac{v_x - v}{1 - \frac{vv_x}{c^2}} \\ v_x &= \frac{v'_x + v}{1 + \frac{vv'_x}{c^2}} \ . \end{aligned}$$