MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Physics

Physics 8.01 Spring 2005

FINAL EXAM SOLUTIONS

Exam Date: Tuesday, May 17, 2005

FAMILY (LAST) NAME

GIVEN (FIRST) NAME

STUDENT ID NUMBER

Please check (✔) your class ☞

	L01	MTW 10:00	Walter Lewin
	L02	MTW 11:00	Walter Lewin
	L03	MTW 2:00	Min Chen
	L04	MTW 3:00	Min Chen

INSTRUCTIONS:

1. The FORMULA SHEET is in the back of this exam. You may tear it off. There is also an extra BLANK PAGE in case you need it.
2. This is a closed book exam. CALCULATORS, BOOKS, and NOTES are NOT ALLOWED.
3. Unless otherwise stated, to earn full credit you must show a valid DERIVATION and/or EXPLANATION of your answer, and you must express it in terms of the GIVEN VARIABLES.

Part I: Quiz 13

Problem	**Maximum**	**Score**	**Grader**
1	20		
2	30		
3	35		
4	15		
TOTAL	100		

Part II: Make-Up Exam

Problem	**Maximum**	**Score**	**Grader**
5	20		
6	35		
7	30		
8	35		
9	30		
10	35		
11	15		
TOTAL	200		

Name ______________________________

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FINAL EXAM CORRECTIONS

Tuesday, May 17, 2005

Problem 8a:

Express your answer in terms of P_E, M_E, and R_E.

Problem 9:

Assume that the mass of the spring is negligible.

Problem 10c:

The skater is twirling about a **fixed** axis, and L denotes the angular momentum about that axis.

Note on Procedure:

If you decide that you do not want us to grade the make-up part of the exam, put an X over the grade box for the Make-Up Exam on the first sheet of the exam. Remember, however, that if you want us to grade what you have written, there is no penalty. The Make-Up part of the final will count only if it brings your average up.

Name ______________________

PART I: QUIZ 13

Part I is the first of two parts. It replaces Quiz 13, and will count toward your final grade the same as any other weekly quiz. The only difference is that lowest of the first 12 quizzes will be dropped, while Quiz 13 will not be dropped. Part I is intended to be one third of the exam, but you are free to divide your time as you wish. Part II, which is optional, starts on p. 12.

Problem 1: Basic concepts about special relativity (*20 points*)

(a) (*5 points*) Which of the following statements best describes the relativistic definition of momentum and its relationship to the Newtonian definition, $\vec{\mathbf{p}} \equiv m\vec{\mathbf{v}}$?

(i) The relativistic definition of momentum is the same as the Newtonian definition of the momentum.

(ii) The Newtonian definition is consistent with all the principles of relativity, but experimentally it has been found to fail at velocities near the speed of light. It therefore must be replaced by the relativistic definition.

[(iii)] The Newtonian definition of momentum is inconsistent with the principles of relativity, because the relativistic transformation equations (i.e., the Lorentz transformations) imply that if the Newtonian momentum were conserved in one inertial frame of reference, it would not be conserved in other inertial frames of reference.

(iv) The relativistic definition of momentum is different from the Newtonian definition because in Newtonian mechanics the total momentum of a system is always conserved when no external forces are acting, while in relativity it is not.

(v) The relativistic definition of momentum differs from the Newtonian definition very significantly for low velocities, but the two definitions are approximately equal for velocities approaching that of light.

Explanation: The relativistic definition is $\vec{\mathbf{p}} = \gamma m\vec{\mathbf{v}}$, where $\gamma = 1/\sqrt{1-(v^2/c^2)}$ (as listed in the Formula Sheet), so it is not the same as the Newtonian definition, eliminating (i). For low velocities $v \ll c$, one sees that $\gamma \approx 1$, so the two definitions are approximately equal, while for high velocities they can differ by a lot. Thus (v) is backwards. Momentum is conserved in the absence of external forces in both Newtonian and relativistic physics, so (iv) is false. That leaves (ii) and (iii), which plainly contradict each other. (iii) is correct.

— Problem 1 Continues —

Name ______________________

(b) (*5 points*) Which of the following statements best describes the relativistic definition of velocity and its relationship to the Newtonian definition, $\vec{\mathbf{v}} \equiv \mathrm{d}\vec{\mathbf{x}}/\mathrm{d}t$?

[(i)] The relativistic definition of velocity is the same as the Newtonian definition of the velocity.

(ii) The Newtonian definition is consistent with all the principles of relativity, but experimentally it has been found to fail at velocities near the speed of light. It therefore must be replaced by the relativistic definition.

(iii) The relativistic definition of velocity differs from the Newtonian definition very significantly for low velocities, but the two definitions are approximately equal for velocities approaching that of light.

(iv) The relativistic definition of velocity differs from the Newtonian definition very significantly for high velocities, comparable to that of light, but the two definitions are approximately equal for velocities very small compared to that of light.

Explanation: When special relativity was discussed in the textbook and in class, did anybody give you a new definition of velocity? No. The same definition is used both in Newtonian mechanics and in special relativity.

(c) (*5 points*) Which of the following statements best describes the behavior of the velocity and energy of a relativistic particle (of nonzero rest mass) as it is accelerated to very high velocities?

(i) As in Newtonian physics, both the velocity and energy can in principle increase without limit. The velocity and energy are limited only by the capacity of the device that is causing the acceleration.

[(ii)] The velocity of the particle is limited by the speed of light, which can never be reached, while the energy can grow arbitrarily large, limited only by the capacity of the accelerating device.

(iii) Both the velocity and the energy of the particle have finite limits, which are the speed of light and the corresponding energy. No matter what device is used to accelerate the particle, the speed can never quite reach the (finite) speed of light, and the energy can never quite reach the (finite) energy corresponding to travel at light speed.

(iv) The energy of the particle is limited by a certain maximal value, which can never be reached, while the speed can grow arbitrarily large, limited only by the capacity of the accelerating device.

Explanation: The velocity v of a particle is limited by the speed of light, but the Formula Sheet shows that

$$E = \frac{mc^2}{\sqrt{1 - v^2/c^2}} \ .$$

It can be seen that $E \to \infty$ as $v \to c$, so the energy can grow arbitrarily large.

— *Problem 1 Continues* —

Name ______________________

(d) (*5 points*) A muon is a particle like an electron, except that its mass is about 207 times larger. A high energy electron (e^-) can collide with a high energy positron (e^+), resulting in the annihilation (disappearance) of the two particles, producing a muon (μ^-) and an antimuon (μ^+). If the process happens in empty space, with no external forces acting, which of the following statements correctly described the conservation laws that apply? Circle as many answers as you believe to be valid. *Hint: For a relativistic particle there are three kinds of energy that can be defined: the rest energy, the kinetic energy, and the total energy, where the total energy is the sum of the rest energy and the kinetic energy.*

(i) The total momentum and the total kinetic energy are conserved.

[(ii)] The total momentum and the total energy are conserved.

(iii) The total momentum and the total rest energy are conserved.

(iv) The total momentum is not conserved, but the total kinetic energy is.

(v) The total momentum is not conserved, but the total energy is.

(vi) The total momentum is not conserved, but the total rest energy is.

Explanation: Total momentum is always conserved in the absence of external forces, so (iv)-(vi) are eliminated. In the process described rest energy is clearly not conserved, since the two muons in the final state have 207 times the rest energy of the two electrons in the initial state. The increase in rest energy is compensated by a decrease in kinetic energy, so the total energy remains constant.

Name

Problem 2: Relativistic Momentum vs. Newtonian Momentum (*30 points*)

For the following questions, you are not expected to numerically simplify your answers. For example, an expression such as

$$c\sqrt{1+\left(\frac{5}{183}\right)^{7/2}+\left(\frac{17}{365}\right)^{4/11}+\frac{\pi}{2}}$$

would be a perfectly acceptable, albeit implausible, answer.

(a) (*10 points*) At what speed is the momentum of a particle three times as great as the result obtained from the nonrelativistic expression mv? Express your answer in terms of the speed of light.

(b) (*10 points*) A force is applied to a particle along its direction of motion. At what speed is the magnitude of force required to produce a given acceleration three times as great as the force required to produce the same acceleration when the particle is at rest? Express your answer in terms of the speed of light.

(c) (*10 points*) A force is applied to a particle perpendicular to its direction of motion. At what speed is the magnitude of force required to produce a given acceleration three times as great as the force required to produce the same acceleration when the particle is at rest? Express your answer in terms of the speed of light.

(a) The relativistic expression for the momentum is

$$\vec{\mathbf{p}} = \gamma m \vec{\mathbf{v}} \ ,$$

where

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \ .$$

So the momentum is three times the Newtonian expression when $\gamma = 3$:

$$\frac{1}{\sqrt{1 - v^2/c^2}} = 3 \quad \Longrightarrow \quad \sqrt{1 - \frac{v^2}{c^2}} = \frac{1}{3} \quad \Longrightarrow \quad 1 - \frac{v^2}{c^2} = \frac{1}{9} \quad \Longrightarrow$$

$$\frac{v^2}{c^2} = \frac{8}{9} \quad \Longrightarrow \quad v = c\sqrt{\frac{8}{9}} \quad \Longrightarrow \quad \boxed{v = \frac{2\sqrt{2}\,c}{3} \ .}$$

(b) When a force is applied in the same direction as the velocity, one sees from the formula sheet that

$$F = \gamma^3 ma \ .$$

Thus, if the force is three times as large as the force required to produce the same acceleration when the particle is at rest, then $\gamma^3 = 3$:

$$\left(\frac{1}{\sqrt{1 - v^2/c^2}}\right)^3 = 3 \quad \Longrightarrow \quad \frac{1}{\sqrt{1 - v^2/c^2}} = 3^{1/3} \quad \Longrightarrow \quad \sqrt{1 - \frac{v^2}{c^2}} = 3^{-1/3} \quad \Longrightarrow$$

$$1 - \frac{v^2}{c^2} = 3^{-2/3} \quad \Longrightarrow \quad \frac{v^2}{c^2} = 1 - 3^{-2/3} \quad \Longrightarrow \quad \boxed{v = c\sqrt{1 - 3^{-2/3}} \ .}$$

(c) When a force is applied perpendicular to the velocity, one sees from the formula sheet that

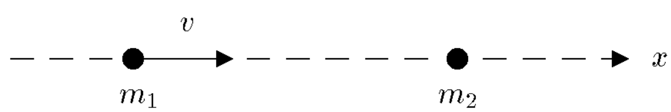
$$F = \gamma ma \ .$$

So, if the force is three times as large as the force ma which is required to produce an acceleration a when the particle is at rest, then $\gamma = 3$. This is the same condition we found in part (a), so again

$$\boxed{v = \frac{2\sqrt{2}\,c}{3} \ .}$$

Problem 3: A Totally Inelastic Particle Collision (*35 points*)

A particle of rest mass m_1 moves with relativistic speed v along the x-axis, in the positive direction. It collides with a particle of rest mass m_2, which is at rest. The two stick together, and continue to move as one particle.



In the following sequence of questions, you may express the answers to each part in terms of the original given variables and/or the answers to any previous part, whether or not you correctly answered the previous part. You need not carry out the algebra to express each answer in terms of the original given variables.

(a) (*5 points*) What is $\vec{\mathbf{P}}_i$, the total momentum of the two-particle system, before the collision? Express your answer in terms of some or all of the variables m_1, m_2, and v. Here "i" stands for "initial".

(b) (*5 points*) What is the total energy E_i of the two-particle system before the collision? Again express your answer in terms of some or all of the variables m_1, m_2, and v.

(c) (*5 points*) After the collision, what is the energy E_f and the momentum $\vec{\mathbf{P}}_f$ of the single particle. Here "f" stands for "final". You may express your answer in terms of some or all of the variables m_1, m_2, v, $\vec{\mathbf{P}}_i$, and E_i.

(d) (*5 points*) What is the x-velocity u ($u \equiv dx/dt$) of the final particle? You may express your answer in terms of some or all of the variables m_1, m_2, v, $\vec{\mathbf{P}}_i$, E_i, $\vec{\mathbf{P}}_f$, E_f.

(e) (*5 points*) What is the rest mass M of the final particle?

(f) (*5 points*) Is the rest mass M of the final particle larger than, smaller than, or equal to $m_1 + m_2$. Explain briefly how you know this.

(g) (*5 points*) Suppose the collision is observed from a frame of reference that moves to the right at speed u (the same u found in part (d)). Let $\vec{\mathbf{k}}_1$ and $\vec{\mathbf{k}}_2$ denote the momenta of the two initial particles, as seen in this frame. State whether $\left|\vec{\mathbf{k}}_1\right| > \left|\vec{\mathbf{k}}_2\right|$, $\left|\vec{\mathbf{k}}_1\right| < \left|\vec{\mathbf{k}}_2\right|$, or $\left|\vec{\mathbf{k}}_1\right| = \left|\vec{\mathbf{k}}_2\right|$, and explain briefly how you know this.

Name ______________________

(a) The relativistic expression for the momentum of a particle is

$$\vec{\mathbf{p}} = \gamma m v = \frac{m\vec{\mathbf{v}}}{\sqrt{1 - v^2/c^2}} \ .$$

Before the collision only particle 1 has a nonzero velocity and hence a nonzero momentum, so the total momentum is

$$\boxed{\vec{\mathbf{P}}_i = \frac{m_1 v \hat{\boldsymbol{\imath}}}{\sqrt{1 - v^2/c^2}} \ .}$$

(b) The total energy of the two-particle system is

$$E_i = E_1 + E_2 .$$

The relativistic expression for the energy is given as

$$E = \gamma m c^2 .$$

For particle 1, moving with a speed v, we have

$$E_1 = \gamma m_1 c^2 = \frac{m_1 c^2}{\sqrt{1 - v^2/c^2}} .$$

Initially particle 2 is stationary, so $\gamma_2 = 1$, and

$$E_2 = m_2 c^2 .$$

The total energy of the two-particle system is then

$$\boxed{E_i = \frac{m_1 c^2}{\sqrt{1 - v^2/c^2}} + m_2 c^2 \ .}$$

(c) Conservation of energy demands that

$$\boxed{E_f = E_i \ ,}$$

while conservation of momentum demands that

$$\boxed{\vec{\mathbf{P}}_f = \vec{\mathbf{P}}_i \ .}$$

(d) Since the momentum and energy of a particle are given by

$$\vec{\mathbf{p}} = \gamma m \vec{\mathbf{v}}$$

and

$$E = \gamma m c^2 \ ,$$

one can see immediately that

$$\vec{\mathbf{v}} = \frac{\vec{\mathbf{p}}\, c^2}{E} \ .$$

so

$$u = \boxed{\frac{\vec{\mathbf{P}}_{i,x}\, c^2}{E_i}} = \frac{\gamma m_1 v c^2}{\gamma m_1 c^2 + m_2 c^2} = \boxed{\frac{\gamma m_1 v}{\gamma m_1 + m_2}} \ ,$$

where

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \ .$$

(e) The final particle has energy $E_f = E_i$, and momentum $\vec{\mathbf{P}}_f = \vec{\mathbf{P}}_i$. Knowing the energy and momentum, the rest mass can be determined by the formula

$$E^2 - (pc)^2 = (mc^2)^2 \ ,$$

which is given on the formula sheet. Applying this formula to the notation of the current problem,

$$E_f^2 - c^2 \left| \vec{\mathbf{P}}_f \right|^2 = (Mc^2)^2 \ ,$$

so

$$\boxed{M = \frac{1}{c}\sqrt{E_f^2 - c^2 \left| \vec{\mathbf{P}}_f \right|^2} \ .}$$

You were not asked to express this in terms of the originally given variables, but if you did you should have found

$$M = \sqrt{m_1^2 + m_2^2 + \frac{2 m_1 m_2}{\sqrt{1 - v^2/c^2}}} \ .$$

(f) $\boxed{M > m_1 + m_2.}$ In the rest frame of the final particle, the total energy is Mc^2, which by conservation of energy must be equal to the initial energy in that frame. But the initial energy is $m_1c^2 + m_2c^2 +$ kinetic energy, and since the kinetic energy is always positive, the inequality follows.

(g) $\boxed{\left|\vec{\mathbf{k}}_1\right| = \left|\vec{\mathbf{k}}_2\right|\ .}$ Since this frame moves at the same speed as the final particle, the final particle is at rest in this frame, and hence has zero momentum. Since momentum is conserved in all frames, it must in particular be conserved in this frame, so the initial momentum in this frame must be zero. So $\vec{\mathbf{k}}_1 + \vec{\mathbf{k}}_2 = 0$, which implies that the magnitudes are equal.

Name

Problem 4: Doppler Shift with Two Spaceships and a Station (*15 points*)

Two spaceships, the Excelsior and the Reliant, are each approaching Star Base 12 from exactly opposite directions. Each spaceship is moving at speed v relative to Star Base 12.

(a) (*7 points*) The Excelsior sends a radio signal with wavelength λ. What is the wavelength λ' of the radio signal as it is received by Star Base 12?

(b) (*8 points*) The radio signal continues past Star Base 12, and is soon received by the Reliant. What is the wavelength λ'' of the radio signal as it is received by the Reliant? *Hint: when the radio signal passes Star Base 12, you can imagine that it is received at wavelength λ' and then retransmitted at the same wavelength.*

Name

(a) The radio signal will be Doppler shifted to a higher frequency when it is received at Star Base 12, since the source is approaching. Using the formula from the formula sheet,

$$f_{\text{received}} = \sqrt{\frac{c+v}{c-v}}\, f_{\text{emitted}}\ ,$$

where v is the speed of approach. Wavelength is related to the frequency by

$$\lambda = \frac{c}{f}\ ,$$

so

$$\boxed{\lambda' = \sqrt{\frac{c-v}{c+v}}\,\lambda\ .}$$

The observed wavelength is shorter than the emitted wavelength, corresponding to the higher frequency. This sign is usually referred to as a "blue shift," while a shift to a longer wavelength (and lower frequency) is called a "red shift."

(b) Using the hint, one can imagine that the radio signal is emitted from Star Base 12 at wavelength λ'. Since the Reliant and the star base are approaching each other with the same relative speed v, the radiation received by the Reliant will be blue-shifted by exactly the same factor as in part (a). Thus the blue-shift factor becomes squared, and

$$\boxed{\lambda'' = \frac{c-v}{c+v}\,\lambda\ .}$$

An alternative approach would be to calculate the relative speed u between the Reliant and the Excelsior, which is found by using the relativistic velocity addition formula:

$$u = \frac{v+v}{1+\frac{vv}{c^2}} = \frac{2v}{1+v^2/c^2}\ .$$

This velocity can then be used to calculate the blue-shift between the Excelsior and the Reliant directly:

$$\lambda'' = \sqrt{\frac{c-u}{c+u}}\,\lambda\ .$$

Substituting and simplifying, one finds

$$\lambda'' = \sqrt{\frac{c-\frac{2v}{1+v^2/c2}}{c+\frac{2v}{1+v^2/c2}}}\,\lambda = \sqrt{\frac{c+\frac{v^2}{c}-2v}{c+\frac{v^2}{c}+2v}}\,\lambda$$

$$= \sqrt{\frac{c^2+v^2-2vc}{c^2+v^2+2vc}}\,\lambda = \sqrt{\frac{(c-v)^2}{(c+v)^2}}\,\lambda = \frac{c-v}{c+c}\,\lambda\ .$$

— End of Part I of Final Exam —

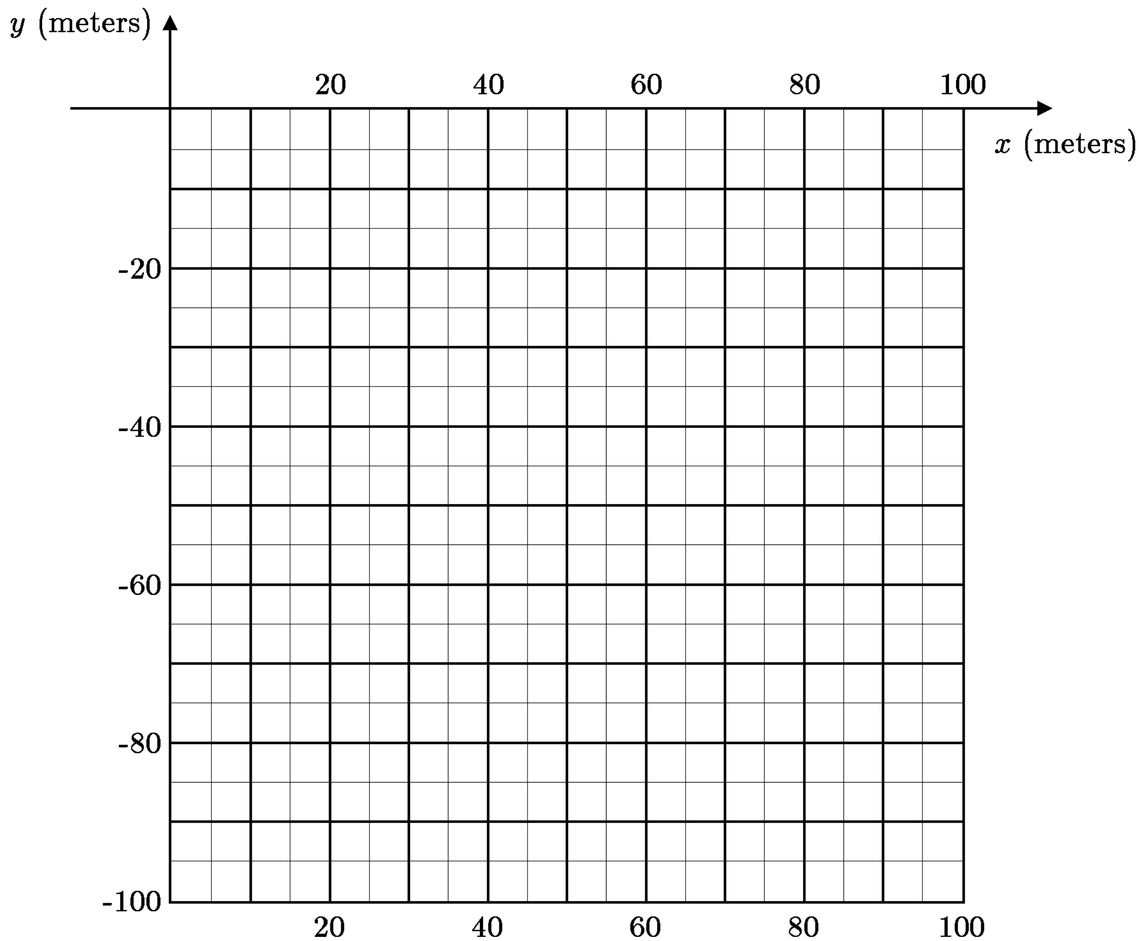
Name

PART II: MAKE-UP

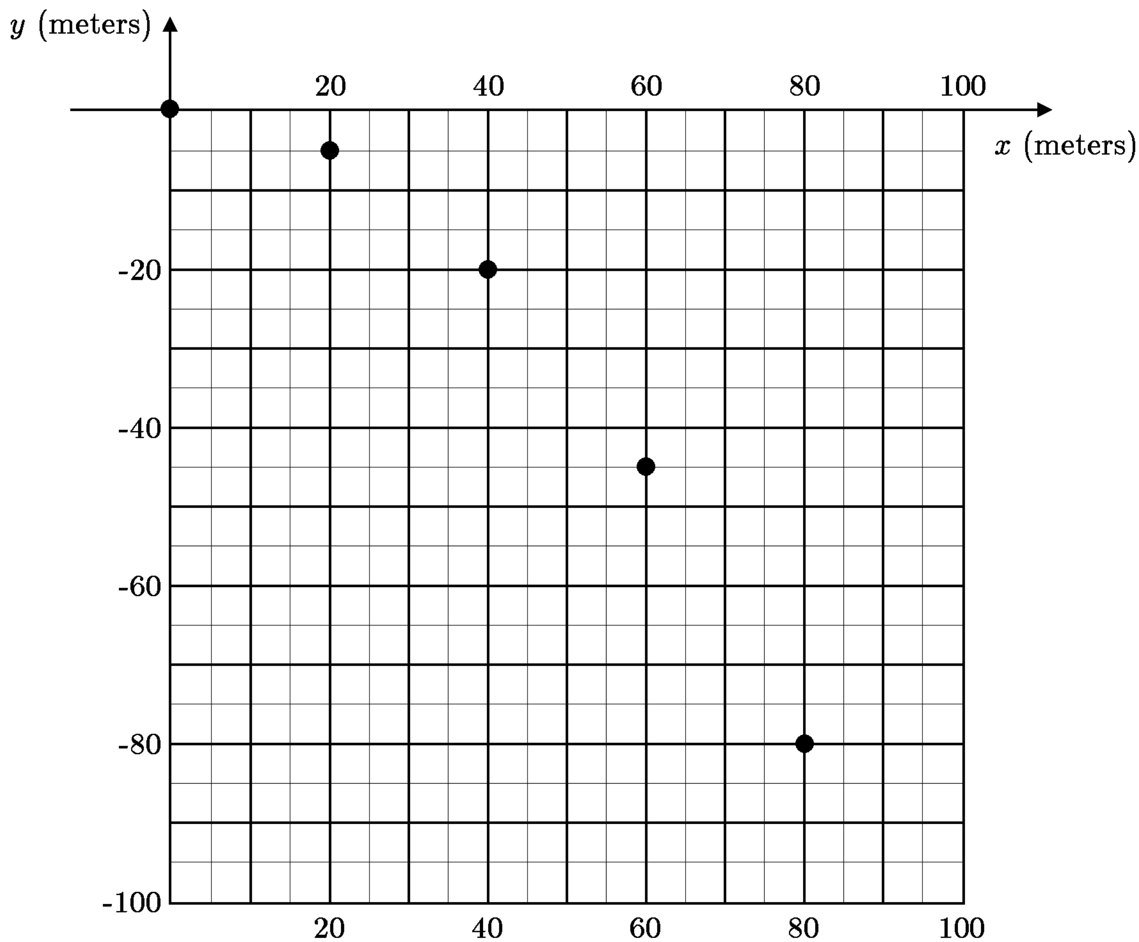
Part II is the second of two parts. It is OPTIONAL, and serves as a make-up quiz for the 12 weekly quizzes that took place during the term. This part can only raise your grade, and cannot lower it. If your average for this make-up is higher than your average for the first 12 weekly quizzes, after dropping one of them, then your average for the first 12 weekly quizzes will be replaced by 0.35 × (Make-up grade) + 0.65 × (Original grade), where all the grades have been scaled to a maximum of 100.

Problem 5: A Simple Trajectory (*20 points*)

A ball is thrown from a cliff, with an initial velocity that is exactly horizontal, with a magnitude of 20 meter/second. It travels under the influence of gravity, and for numerical simplicity we use the approximate value $g = 10$ meter/second^2 for the acceleration of gravity. Use a coordinate system in which x axis is horizontal and the y axis is directed upward. The ball is thrown from [0, 0, 0], and travels initially in the positive x-direction. Ignoring all frictional effects, indicate the trajectory followed by the ball by putting a dot on the following graph at the location of the ball at $t = 1$, 2, 3, and 4 seconds.



Name ______________________



$$x = v_0 t = (20 \text{ m/s})\, t$$

$$y = -\frac{1}{2} g t^2 = -\frac{1}{2}(10 \text{ m/s}^2)\, t^2 = -(5 \text{ m/s}^2)\, t^2 \ .$$

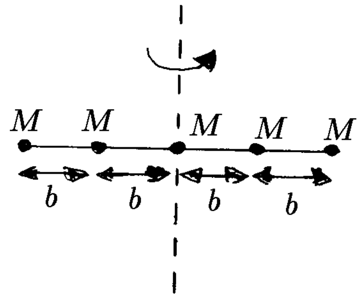
t	x	y
1 s	20 m	-5 m
2 s	40 m	-20 m
3 s	60 m	-45 m
4 s	80 m	-80 m

Name ______________________

Problem 6: Assorted Multiple Choice Problems (*35 points*)

Mark your answer by circling it. For these multiple choice questions you need not show your work, and there will be no partial credit, except as indicated in part (e). *Hint: the right answer is not a leafy vegetable. (Don't say that we don't try to help you out.)*

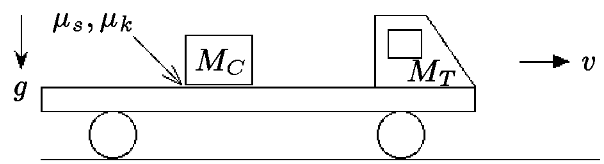
(a) (*6 points*) Five small balls, each of mass M, are attached to a massless rigid rod of length $4b$. One ball is at the center, one ball is at each end, and one ball is a distance b from the center in each direction, as shown. What is the moment of inertia of this object for rotation about an axis through the center of the rod and perpendicular to it?



(i) $2Mb^2$ (ii) $5Mb^2$ (iii) $8Mb^2$ **[(iv) $10Mb^2$]** (v) $12Mb^2$ (vi) asparagus

Explanation: The general formula for the moment of inertia is $I = \sum_i m_i R_i^2$, where m_i is the mass of the ith particle, and R_i is the distance of that particle from the rotation axis. Here $I = M \cdot 0^2 + 2 \cdot M \cdot b^2 + 2 \cdot M \cdot (2b)^2 = 10Mb^2$.

(b) (*6 points*) A crate of mass M_C rests on a flatbed truck of mass M_T. The coefficients of static and kinetic friction between the crate and the truck are μ_s and μ_k, respectively. The acceleration of gravity is g. If the truck is moving in a straight horizontal line at constant speed v, what is the magnitude of the frictional force that the truck exerts on the crate?

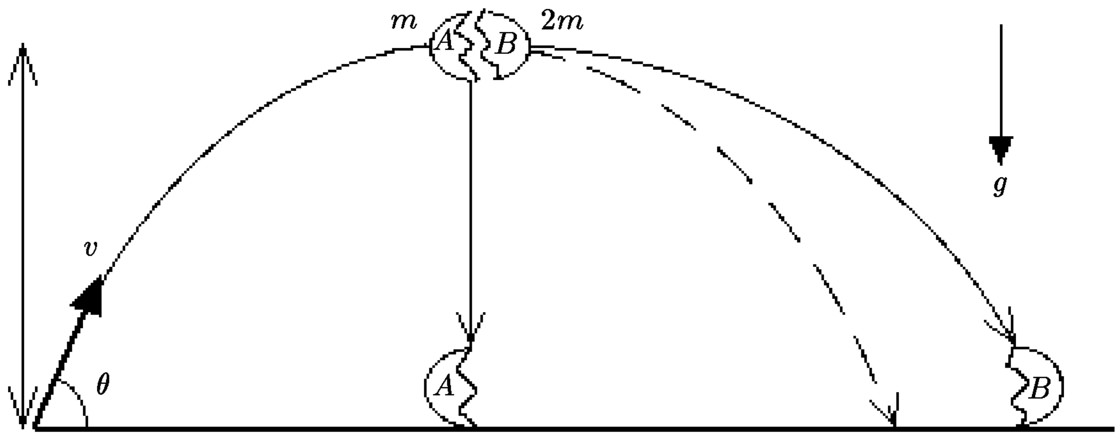


(i) $\mu_k M_C g$ (ii) $\mu_s M_C g$ (iii) $\mu_k M_T g$ (iv) $\mu_s M_T g$ **[(v) zero]** (vi) broccoli

Explanation: Since the crate is not moving relative to the truck, this is an example of static friction. Static friction can have any magnitude up to μ_s times the normal force, where the exact value is determined by the force that is needed to prevent the surfaces from sliding. Since the truck and crate are moving at a constant velocity, there is no net force acting on either, and no frictional force is needed to prevent the crate from sliding.

— *Problem 6 continues on the next page* —

Problem 6 Continued:



(c) (*6 points*) A projectile is launched from the origin with speed v at an angle θ from the horizontal. At the highest point in the trajectory, the projectile breaks into two pieces, A and B, of masses m and $2m$, respectively. Immediately after the breakup piece A is at rest relative to the ground. Neglect air resistance. Which of the following sentences most accurately describes what happens next?

(i) Piece B will hit the ground first, since it is more massive.

(ii) Both pieces have zero vertical velocity immediately after the breakup, and therefore they hit the ground at the same time.

(iii) Piece A will hit the ground first, because it will have a downward velocity immediately after the breakup.

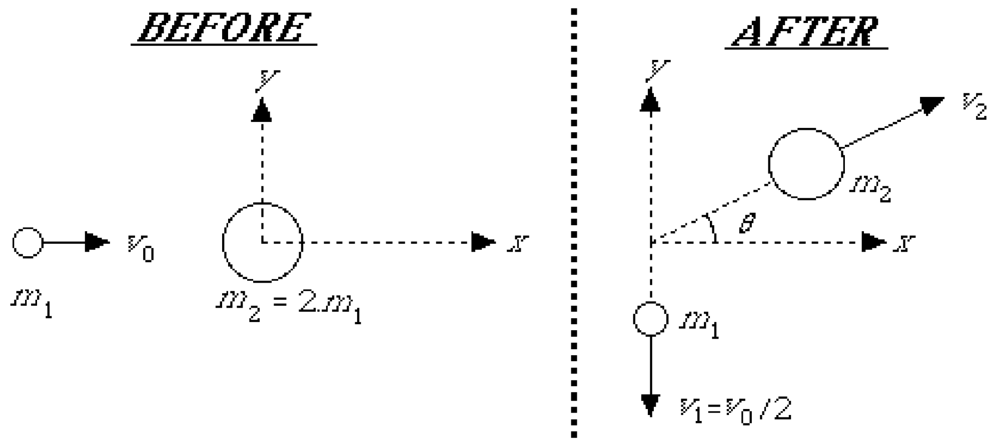
(iv) There is no way of knowing which piece will hit the ground first, because not enough information is given about the breakup.

(v) Spinach.

Explanation: Since the breakup occurs at the highest point, the vertical velocity just before the breakup is zero. The vertical momentum is therefore also zero. After the breakup piece A is at rest, and hence has zero vertical momentum, so by conservation of momentum piece B must also have zero vertical momentum, and therefore zero vertical velocity. Since pieces A and B are falling from the same height with the same initial vertical velocity, they will hit the ground at the same time. The pieces differ only in their masses and in their horizontal velocities, but neither of these quantities affect the time of fall.

— *Problem 6 continues on the next page* —

Problem 6 Continued:



(d) (*7 points*) A particle of mass m_1 with initial speed v_0 in the positive x-direction collides with a particle of mass $m_2 = 2m_1$, which is initially at rest at the origin. After the collision the first particle (m_1) moves off with speed $v_1 = \frac{1}{2}v_0$ in the negative y-direction and the second particle (m_2) moves off with speed v_2 at an angle θ, as shown. Which one of the following equations is a valid conservation law?

- (i) $m_2 v_2 \sin\theta = \frac{1}{2}m_1 v_0$
- (ii) $m_2 v_2 \cos\theta = \frac{1}{2}m_1 v_0$
- (iii) $m_1 v_0 = m_2 v_2 + \frac{1}{2}m_1 v_0$
- (iv) $m_1 v_0 = m_2 v_2 - \frac{1}{2}m_1 v_0$
- (v) $\frac{1}{2}m_1 v_0^2 = \frac{1}{2}m_1\left(v_1^2 + v_2^2\right)$
- (vi) Cabbage

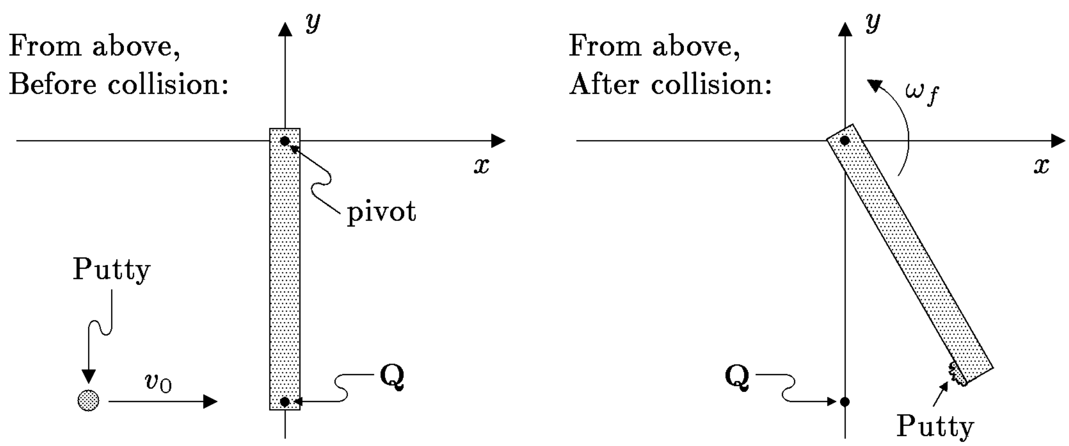
Explanation: $m_2 v_2 \sin\theta$ is the y-component of the final momentum of particle 2 (of mass m_2), while $\frac{1}{2}m_1 v_0$ is the negative of the y-component of the final momentum of particle 1. These must be equal in magnitude, since the total initial y-momentum is zero, so conservation of momentum implies that the final y-momentum must be zero. For choice (ii), the LHS (left-hand side) is the x-component of the final momentum of particle 2, and there is no reason why it should equal the RHS, which is again the negative of the y-component of the final momentum of particle 1. For choice (iii), the LHS is the initial x-component of the total momentum, but there is no reason why it should be equal to the RHS, which is the sum of the magnitudes of the momenta of the two particles. For choice (iv), the LHS is again the initial x-component of the total momentum, but again there is no reason why it should be equal to the RHS, which is the difference of the magnitudes of the momenta of the two particles. For choice (v) the LHS is the total initial kinetic energy, but the expression for the final kinetic energy would be $\frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 = \frac{1}{2}m_1(v_1^2 + 2v_2^2)$, which is not the same as the RHS.

— *Problem 6 continues on the next page* —

Name ______________________

Problem 6 Continued:

(e) (*10 points*) On a frictionless horizontal table a slender rigid rod is attached at one end to a fixed, frictionless pivot. A nonrotating disk of putty is moving with speed v_0 perpendicular to the line of the rod, and collides with it at point Q, at the opposite end from the pivot. The putty sticks to the end of the rod, so that after the collision they move together. Assume that the putty disk is small enough to be treated as a point particle. Circle each of the quantities that are conserved throughout the entire process, from the initial linear motion of the putty through the rotational motion of the rod/putty system at the end. *Partial credit on this question: 4 points off for one error, 8 points off for two errors, no credit for more than two errors.*



(i) Linear momentum of the putty and rod in the x-direction

(ii) Linear momentum of the putty and rod in the y-direction

(iii) Kinetic energy of the putty and rod

(iv) Angular momentum of the putty and rod about the point Q

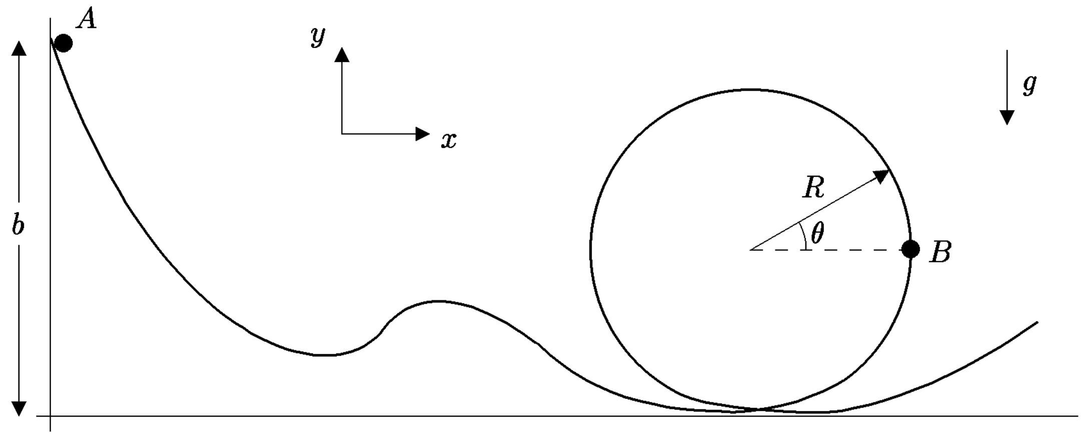
(v) Angular momentum of the putty and rod about the pivot

Explanation: Since the putty sticks to the rod, the collision is inelastic, so kinetic energy is not conserved. The pivot can exert a force on the rod in both the x- and y-directions, so neither component of linear momentum is conserved. Such a force can also result in a torque about Q, so angular momentum about the point Q is not conserved. A frictionless pivot cannot exert a torque about itself, however, since the vector $\vec{\mathbf{r}}$ in $\vec{\boldsymbol{\tau}} = \vec{\mathbf{r}} \times \vec{\mathbf{F}}$ vanishes. Thus, the total angular momentum about the pivot is the only conserved quantity.

Name ______________________________

Problem 7: A Puck Sliding on a Frictionless Track (*30 points*)

A small puck of mass m slides along a frictionless track of the following shape:



It starts from rest at point A, at a height b above the ground level. Let g denote the acceleration of gravity. The track includes a circular loop of radius R, tangential to the ground. The point B lies at the same height as the center of the circle.

(a) (*6 points*) What is the minimum initial height b_B that is needed to allow the puck to reach the point B?

(b) (*6 points*) Assuming that $b > b_B$, what is the speed v_B of the puck as it passes point B? Express your answer in terms of some or all of the variables m, b, R, and g.

(c) (*6 points*) What is the acceleration vector $\vec{\mathbf{a}}_B$ of the puck as it passes point B? You may express your answer in terms of v_B and the given variables.

(d) (*6 points*) If b is large enough, then it will be possible for the puck to complete the full circle without falling off. What is the minimum initial height b_{circle} that would allow this to happen?

(e) (*6 points*) If $b_B < b < b_{\text{circle}}$, then the puck will lose contact with the track at some angle θ, measured from the horizontal, as shown. What is this angle θ? Express your answer in terms of some or all of the variables m, b, R, and g.

(a) If v_B denotes the speed at point B, then conservation of energy implies that

$$\text{Initial energy} = \text{Final energy}$$

$$mgb = mgR + \frac{1}{2}mv_B^2 \ .$$

Since $v_B^2 \geq 0$, it follows that $b \geq R$. Thus the minimum initial height $\boxed{h_B = R.}$

(b) From the conservation of energy equation in the previous part,

$$\boxed{v_B = \sqrt{2g(b-R)} \ .}$$

(c) This is an example of circular motion, but it is not uniform circular motion, since the particle's speed is changing as it goes around the circle. In that case the radial acceleration is v^2/R, directed toward the center, exactly as if it were uniform circular motion. But there is also a tangential component to the acceleration, equal to $\mathrm{d}v/\mathrm{d}t$. When the particle is at point B the tangential component of the acceleration is caused only by gravity, so it has magnitude g, directed downward. So

$$\boxed{\vec{\mathbf{a}}_B = -\frac{v_B^2}{R}\,\hat{\boldsymbol{\imath}} - g\hat{\boldsymbol{\jmath}} \ .}$$

(d) The speed at the top is given by a conservation of energy argument very similar to the way that we found the speed at point B:

$$mgb = 2mgR + \frac{1}{2}mv_{\text{top}}^2 \ .$$

Assuming that it reaches the top, then the downward component (i.e., negative y-component) of $\vec{\mathbf{F}} = m\vec{\mathbf{a}}$ as it passes the top is

$$N + mg = m\frac{v_{\text{top}}^2}{R} \ ,$$

where N is the downward normal force exerted by the track on the puck. The puck will stay pressed against the track as long as $N \geq 0$, and otherwise it will fall off, because the track cannot exert an attractive force. Thus the condition on v_{top} becomes

$$\frac{v_{\text{top}}^2}{R} \geq g \ .$$

Name

Combining this equation with the conservation of energy equation above,

$$mgb \geq 2mgR + \frac{1}{2}mgR = \frac{5}{2}mgR \ .$$

Thus the minimum value is

$$\boxed{b_{\text{circle}} = \frac{5}{2}R \ .}$$

(e) For an arbitrary angle θ, conservation of energy implies

$$mgb = mgR(1 + \sin\theta) + \frac{1}{2}mv^2 \ ,$$

so

$$v^2 = 2g\left[b - R(1 + \sin\theta)\right] \ .$$

Writing the component of $\vec{\mathbf{F}} = m\vec{\mathbf{a}}$ in the radially inward direction, one finds

$$N + mg\sin\theta = m\frac{v^2}{R} \ ,$$

where again N denotes the normal (inward) force that the track exerts on the puck. The puck will start to leave the track exactly where $N = 0$, so

$$mg\sin\theta = m\frac{2g\left[b - R(1 + \sin\theta)\right]}{R} \ ,$$

so

$$R\sin\theta = 2\left[b - R(1 + \sin\theta)\right] \ ,$$

and then

$$3R\sin\theta = 2(b - R) \ ,$$

and finally

$$\boxed{\theta = \sin^{-1}\left[\frac{2(b-R)}{3R}\right] \ .}$$

Name ______________________________

Problem 8: Orbit of the Earth and the Lagrange Point L2 (*35 points*)

The Earth orbits the Sun with a period P_E. (Of course we know that P_E is one year, but please express your answers in terms of P_E, and not its explicit value.) The orbit is actually slightly elliptical, but we will approximate it here as a circle of radius R_E.

(a) (*7 points*) If the mass of the Earth is M_E, what is the magnitude and direction of the force necessary to hold the Earth in its orbit?

(b) (*7 points*) By applying Newton's laws of motion to the Earth's orbit, express the period P_E of the Earth's orbit in terms of the radius R_E, the mass of the Sun M_S, and Newton's gravitational constant G. (No credit will be given for simply writing down the result.)

— *Problem 8 Continues* —

(a) This is uniform circular motion, so the centripetal acceleration has magnitude v^2/R_E and points toward the center of the circle, the Sun.

$$v = \frac{2\pi R_E}{P_E} \ ,$$

so the magnitude of the acceleration is

$$a = \frac{4\pi^2 R_E}{P_E^2} \ ,$$

and therefore the magnitude of the force is

$$\left|\vec{\mathbf{F}}\right| = M_E\, a = \boxed{\frac{4\pi^2 M_E R_E}{P_E^2} \ .}$$

The force is directed $\boxed{\text{towards the center of the Sun.}}$

(b) The gravitational force acting on the Earth is

$$\left|\vec{\mathbf{F}}\right| = \frac{G M_E M_S}{R_E^2} \ , \text{ directed towards the center of the Sun}$$

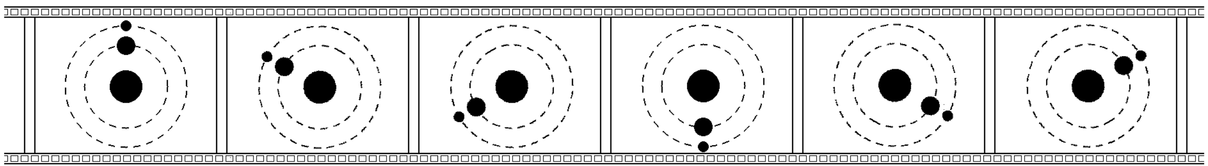
The radial component of $\vec{\mathbf{F}} = M\vec{\mathbf{a}}$ therefore gives

$$\frac{G M_E M_S}{R_E^2} = M_E\, a = \frac{4\pi^2 M_E R_E}{P_E^2} \ .$$

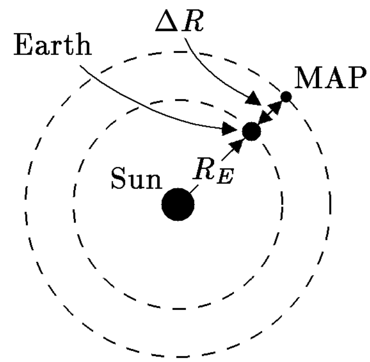
Solving for P_E,

$$\boxed{P_E = 2\pi\sqrt{\frac{R_E^3}{G M_S}} \ .}$$

Problem 8 Continued:



Joseph Lagrange showed in the 1700s that it is possible for a satellite to follow the Earth in its orbit around the Sun, at a slightly larger radius, as shown in the filmstrip above, and in the larger diagram to the right. The satellite would remain a fixed distance ΔR from the center of the Earth, and would always be located along the extension of a radial line from the Sun to the Earth, as shown. This orbit is called L2, and since October 1, 2001, it has been the home of the Wilkinson Microwave Anisotropy Probe (WMAP), a NASA satellite dedicated to high precision measurements of the cosmic microwave background radiation. Note that a satellite at L2 must undergo a slightly larger acceleration than the Earth, but that the extra force can be supplied by the gravitational pull caused by the Earth.



(c) (*7 points*) Using only the variables P_E, R_E, and ΔR, write an expression for the magnitude of the acceleration of the WMAP satellite at L2. What is the direction of the acceleration?

(d) (*7 points*) If WMAP has a mass M_{WMAP}, what is the magnitude and direction of the total force acting on the satellite? Your answer may depend only on the variables R_E, ΔR, M_E, M_S, M_{WMAP}, and G. Note that P_E is **NOT** on this list.

(e) (*7 points*) By applying Newton's laws of motion to the orbit of WMAP, write an equation that must be satisfied by ΔR. The equation might also involve any of the quantities R_E, P_E, M_E, M_S, and G. Do not try to solve this equation for ΔR.

(c) Again it is uniform circular motion, but the speed and the radius are both different from those in part (a). In this case

$$v = \frac{2\pi(R_E + \Delta R)}{P_E} \ ,$$

so

$$|\vec{\mathbf{a}}| = \frac{v^2}{R_E + \Delta R} = \boxed{\frac{4\pi^2(R_E + \Delta R)}{P_E^2} \ .}$$

The direction is again $\boxed{\text{towards the center of the Sun.}}$

(d) The gravitational forces due to the Sun and due to the Earth both point in the same direction, towards the Sun. So the magnitude of the total force is the sum of the magnitudes:

$$\boxed{\left|\vec{\mathbf{F}}\right| = \frac{GM_S M_{\text{WMAP}}}{(R_E + \Delta R)^2} + \frac{GM_E M_{\text{WMAP}}}{\Delta R^2} \ ,}$$

where the force is directed $\boxed{\text{towards the center of the Sun.}}$

(e) The radial component of $\vec{\mathbf{F}} = m\vec{\mathbf{a}}$ in this case gives

$$\frac{GM_S M_{\text{WMAP}}}{(R_E + \Delta R)^2} + \frac{GM_E M_{\text{WMAP}}}{\Delta R^2} = \frac{4\pi^2(R_E + \Delta R)M_{\text{WMAP}}}{P_E^2} \ .$$

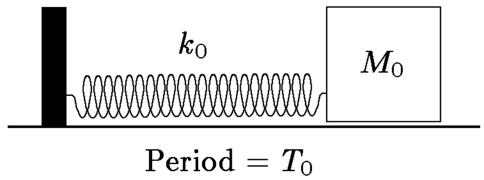
Dividing by M_{WMAP},

$$\boxed{\frac{GM_S}{(R_E + \Delta R)^2} + \frac{GM_E}{\Delta R^2} = \frac{4\pi^2(R_E + \Delta R)}{P_E^2} \ .}$$

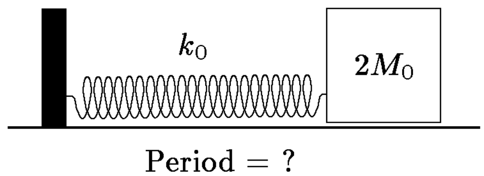
Name ______________________

Problem 9: Blocks and Springs (*30 points*)

A block of mass M_0 slides on a frictionless horizontal plane. It is attached to a spring of spring constant k_0, which is attached at its other end to a fixed support. Assume that the spring obeys Hooke's law. The block is observed to slide back and forth along a line with a period T_0.



(a) (*7 points*) Suppose the block of mass M_0 is replaced by another block of mass $M = 2M_0$. What is the new period?



(i) $4T_0$ (ii) $2\sqrt{2}T_0$ (iii) $2T_0$

(iv) $\boxed{\sqrt{2}T_0}$ (v) T_0 (vi) $\frac{T_0}{\sqrt{2}}$

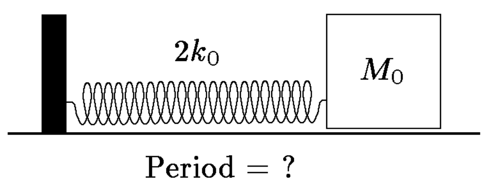
(vii) $\frac{T_0}{2}$ (viii) $\frac{T_0}{2\sqrt{2}}$ (ix) $\frac{T_0}{4}$

Explanation: For a simple spring and mass, the equation of motion is

$$M\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -kx\ ,$$

for which the solution has the general form $x = A\cos(\omega t + \phi)$, *where* $\omega = \sqrt{k/M}$, *and the period* $T = 2\pi/\omega = 2\pi\sqrt{M/k}$. *Since in this part* M *is doubled, the period* T *is multiplied by* $\sqrt{2}$.

(b) (*7 points*) Suppose that the original block of mass M_0 is restored, but the spring is replaced by one of spring constant $2k_0$. What is the new period in this case?



(i) $4T_0$ (ii) $2\sqrt{2}T_0$ (iii) $2T_0$

(iv) $\sqrt{2}T_0$ (v) T_0 (vi) $\boxed{\frac{T_0}{\sqrt{2}}}$

(vii) $\frac{T_0}{2}$ (viii) $\frac{T_0}{2\sqrt{2}}$ (ix) $\frac{T_0}{4}$

Explanation: Here the spring constant k *is doubled, so according to* $T = 2\pi\sqrt{M/k}$, *the period is divided by* $\sqrt{2}$.

— *Problem 9 Continues* —

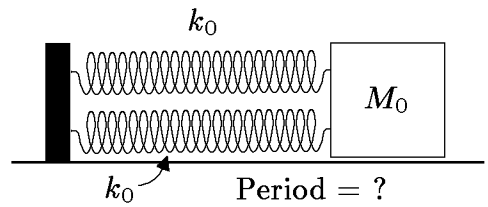
Problem 9 Continued:

(c) (*7 points*) Suppose now that the original spring of spring constant k_0 is replaced, but an additional spring of identical spring constant is attached between the block and the fixed support. What is the new period in this case?

(i) $4T_0$ (ii) $2\sqrt{2}T_0$ (iii) $2T_0$

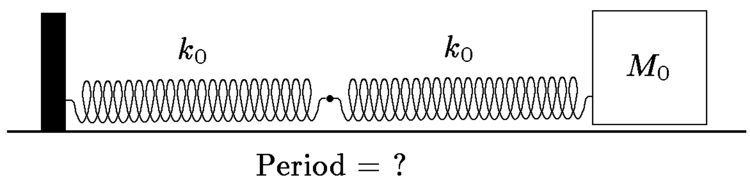
(iv) $\sqrt{2}T_0$ (v) T_0 (vi) $\boxed{\dfrac{T_0}{\sqrt{2}}}$

(vii) $\dfrac{T_0}{2}$ (viii) $\dfrac{T_0}{2\sqrt{2}}$ (ix) $\dfrac{T_0}{4}$



Explanation: The use of two springs in parallel is equivalent to doubling the spring constant. To see this, note that if the block is displaced by a distance x from its equilibrium point, each spring contributes a restoring force $F_x = -kx$, so the total force is $F_x = -2kx$. The answer is therefore the same as in part (b).

(d) (*9 points*) Suppose now that the additional spring is disconnected, but is then placed between the first spring and the block, as shown. What is the new period in this case?



(i) $4T_0$ (ii) $2\sqrt{2}T_0$ (iii) $2T_0$

(iv) $\boxed{\sqrt{2}T_0}$ (v) T_0 (vi) $\dfrac{T_0}{\sqrt{2}}$

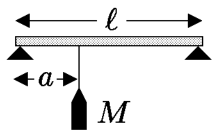
(vii) $\dfrac{T_0}{2}$ (viii) $\dfrac{T_0}{2\sqrt{2}}$ (ix) $\dfrac{T_0}{4}$

Explanation: The use of two identical springs in series is equivalent to halving the spring constant. To see this, note that if the block is displaced by a distance x from its equilibrium point, then each spring stretches by an amount $x/2$. (Since the springs are identical, they stretch by the same amount.) Each spring therefore pulls inward at both ends with a force of magnitude $F = \frac{1}{2}kx$. The forces on the connection point between the two springs cancel. Since the combined spring system pulls inward at both ends with forces of magnitude $F = \frac{1}{2}kx$, it is the same as a spring with spring constant $k/2$. Since $T = 2\pi\sqrt{M/k}$, halving k results in multiplying T by $\sqrt{2}$.

Problem 10: Assorted Multiple Choice Questions on Rotational Physics(*35 points*)

Please mark your answers by circling them. For these multiple choice questions you need not show your work, and there will be no partial credit.

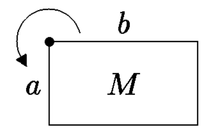
(a) (*7 points*) A horizontal bar of length ℓ and negligible mass is supported at its two ends. A mass M is hung from the bar at a distance a from the left end, as shown. What is the magnitude of the force that the support on the right applies to the bar?



(i) $\frac{1}{2}Mg$ (ii) Mg (iii) $Mg\,\frac{\ell}{a}$ **(iv) $Mg\,\frac{a}{\ell}$** (v) $Mg\,\frac{a}{\ell+a}$ (vi) $Mg\,\frac{\ell}{\ell+a}$

Explanation: Since the bar is static, the torque about any point must vanish. I will calculate the (counterclockwise) torque about the left pivot: $\tau = -Mga + F\ell$, where F is the desired force. Setting $\tau = 0$ gives $F = Mga/\ell$.

(b) (*7 points*) A rectangular slab of mass M has uniform thickness and density, and has height a and width b, as shown. The slab is pivoted about its upper left corner, and rotated in the plane of the paper. What is its moment of inertia about this axis?



(i) $\frac{1}{3}Ma^2$ (ii) $\frac{1}{3}Mb^2$ **(iii) $\frac{1}{3}M(a^2+b^2)$** (iv) $\frac{2}{3}M(a^2+b^2)$ (v) $\frac{1}{3}Mab$ (vi) $\frac{2}{3}Mab$

Explanation: For clarity, adopt a coordinate system with the origin at the dot shown in the diagram, with the x-axis horizontal on the page, the y-axis vertical on the page, and the z-axis out of the page. The formula sheet gives the moment of inertia for a rectangular plate, which has negligible thickness (extent in the z-direction), while here we have a "slab" of unspecified but not necessarily negligible thickness. The thickness, however, does not matter. The moment of inertia about the axis shown, which is the z-axis, is defined by

$$I_z = \sum_i m_i R_i^2 = \sum_i m_i (x_i^2 + y_i^2) \ ,$$

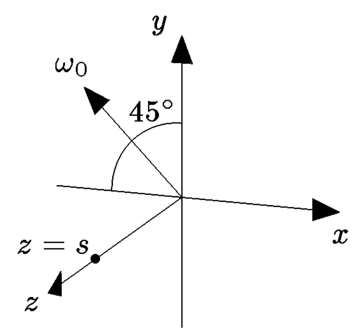
where the sum is over all the infinitesimal masses that make up the object. Since the z-coordinate does not enter, the moment of inertia is the same as it would be if all the masses were relocated in the $z = 0$ plane, forming a thin plate. The other complication is that the formula sheet does not give the moment of inertia for the desired axis. Instead, it tells us that if the object were rotated about the x-axis, which is one of the b edges, then its moment of inertia would be $I_x = \frac{1}{3}Ma^2$. Since the names of the sides are interchangeable, one can also say that if the object were rotated about the y-axis, which is one of the a edges, then its moment of inertia would be $\frac{1}{3}Mb^2$. But the perpendicular axis theorem then implies that the desired moment of inertia $I_z = I_x + I_y$, so $I_z = \frac{1}{3}M(a^2 + b^2)$.

(c) (*7 points*) A skater twirling about a vertical axis pulls her arms inward. Ignoring all frictional effects, which of the following statements are true? You may circle as many as apply. Denote the magnitude of her angular velocity by ω, the magnitude of her angular momentum by L, and her kinetic energy by E_k

(i) L and E_k both increase;	(v) L and ω both increase
(ii) L increases, E_k is constant;	(vi) L increases, ω is constant;
(iii) L is constant, E_k increases;	**(vii) L is constant, ω increases;**
(iv) L and E_k are both constant;	(viii) L and ω are both constant;

Explanation: Since there is no friction there can be no torque about the vertical axis, so the angular momentum about the vertical axis is conserved (constant). As she brings in her arms she is lowering her moment of inertia for rotation about the vertical axis. Since the angular momentum about this axis, $L = I\omega$, must be conserved, ω must increase. The kinetic energy can be expressed as $E = \frac{1}{2}I\omega^2 = L^2/(2I)$, so when I goes down with L fixed, the kinetic energy increases. This may be surprising, but as she brings her arms inward she is behaving as a non-rigid body, so it is possible for her to do work on herself.

(d) (*7 points*) An object which is pivoted at the origin rotates with an angular velocity of magnitude ω_0 directed in the x-y plane at 45° from both the y-axis and the negative x-axis, as shown. What is the velocity $\vec{\mathbf{v}}$ of a point on the rotating object that is located along the positive z-axis at a distance s from the origin?



(i) $\omega_0\, s\, [1, 1, 0]$ (ii) $\omega_0\, s\, [1, -1, 0]$

(iii) $\omega_0\, s\, [-1, 1, 0]$ (iv) $\omega_0\, s\, [-1, -1, 0]$

(v) $\frac{\omega_0\, s}{\sqrt{2}}[1, 1, 0]$ (vi) $\frac{\omega_0\, s}{\sqrt{2}}[-1, 1, 0]$

Explanation: The angular momentum vector is

$$\vec{\boldsymbol{\omega}} = \frac{\omega_0}{\sqrt{2}}[-1, 1, 0] \ ,$$

and the position of the particle is

$$\vec{\mathbf{r}} = [0, 0, s] \ .$$

The velocity of the point is then

$$\begin{aligned}\vec{\mathbf{v}} = \vec{\boldsymbol{\omega}} \times \vec{\mathbf{r}} &= \frac{\omega_0 s}{\sqrt{2}}[-1,1,0] \times [0,0,1] \\ &= \frac{\omega_0 s}{\sqrt{2}}(-\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}) \times \hat{\boldsymbol{k}} \\ &= \frac{\omega_0 s}{\sqrt{2}}(-\hat{\boldsymbol{\imath}} \times \hat{\boldsymbol{k}} + \hat{\boldsymbol{\jmath}} \times \hat{\boldsymbol{k}}) = \frac{\omega_0 s}{\sqrt{2}}(\hat{\boldsymbol{\jmath}} + \hat{\boldsymbol{\imath}}) \\ &= \frac{\omega_0 s}{\sqrt{2}}[1,1,0] \ .\end{aligned}$$

(e) (*7 points*) An unknown comet has a highly elliptical orbit about the Sun; its maximum distance from the center of the Sun is 100 times larger than its minimum distance. If its speed at perihelion (the point of closest approach to the Sun) is 20 km/s, what is its speed at aphelion (the farthest point in its orbit from the Sun)?

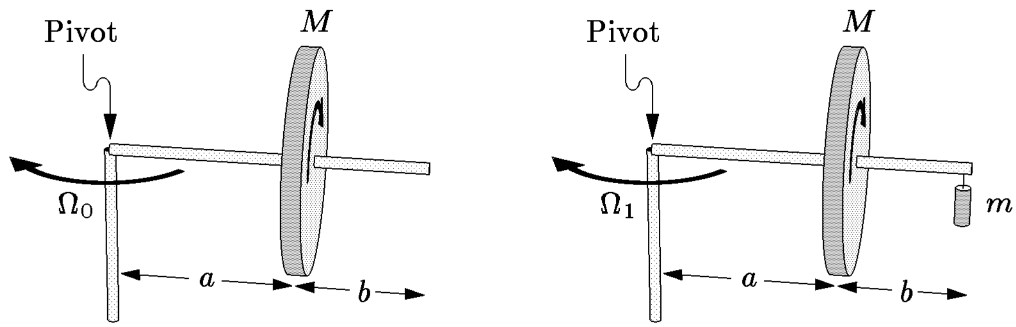
(i) 2,000 km/s (ii) 200 km/s (iii) 20 km/s (iv) 2 km/s **[(v) 0.2 km/s]** (vi) 0.02 km/s

Explanation: The orbital angular momentum about the center of the Sun is conserved, and the magnitude is given by $MRv_\perp$, where M is the mass of the comet, R is its distance from the center of the Sun, and $v_\perp$ is the component of the velocity perpendicular to the radius vector. At perihelion and aphelion there is no radial component to the velocity, so $v_\perp = v$. Thus the quantity Rv must have the same value at perihelion and aphelion. So if v at perihelion is 20 km/s, *then its value at aphelion, when R is 100 times larger, must be 100 times smaller, to conserve the product. Thus* $v_{\text{aphelion}} = 0.2$ km/s.

Problem 11: A Spinning Wheel Gyroscope (*15 points*)

A gyroscope consists of a spinning wheel, rotating freely about a horizontal axle at a distance a from a pivot, as shown. The axle extends a distance b beyond the wheel. Assume that the wheel has a mass M, and the mass of the axle is negligible. The axle is free to rotate freely about the pivot in the horizontal plane. Take the acceleration of gravity as g, with $g > 0$.

The gyroscope is spinning rapidly about the axle, and the axle is observed to precess about the vertical axis with an angular velocity Ω_0. If a small weight of mass m is hung from the tip of the axle, without disturbing the rotational speed of the wheel about its axis, what will be the new angular velocity Ω_1 of the precession?



Answer:

The precession angular velocity of a gyroscope is proportional to the applied torque, a fact which follows from $\vec{\boldsymbol{\tau}} = \mathrm{d}\vec{\mathbf{L}}/\mathrm{d}t$. In this problem, the initial torque is just the torque of the wheel under the influence of the downward pull of gravity, with

$$\left|\vec{\boldsymbol{\tau}}_{\text{Wheel}}\right| = Mga \ .$$

When the weight is attached, it increases the torque by

$$\left|\vec{\boldsymbol{\tau}}_{\text{Weight}}\right| = mg(a+b) \ .$$

Thus

$$\begin{aligned} \frac{\Omega_1}{\Omega_0} &= \frac{\left|\vec{\boldsymbol{\tau}}_{\text{Wheel}}\right| + \left|\vec{\boldsymbol{\tau}}_{\text{Weight}}\right|}{\left|\vec{\boldsymbol{\tau}}_{\text{Wheel}}\right|} \\ &= \frac{Mga + mg(a+b)}{Mga} \\ &= \frac{Ma + m(a+b)}{Ma} \ . \end{aligned}$$

So

$$\boxed{\Omega_1 = \frac{Ma + m(a+b)}{Ma}\,\Omega_0 \ .}$$