MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Physics

Physics 8.01 Spring 2005

WEEKLY QUIZ 6 SOLUTIONS

Quiz Date: Friday, March 11, 2005

FAMILY (LAST) NAME

GIVEN (FIRST) NAME

STUDENT ID NUMBER

Please check (✔) your class ☞

	L01	MTW 10:00	Walter Lewin
	L02	MTW 11:00	Walter Lewin
	L03	MTW 2:00	Min Chen
	L04	MTW 3:00	Min Chen

INSTRUCTIONS:

1. The FORMULA SHEET is in the back of this exam. You may tear it off. There is also an extra BLANK PAGE in case you need it.
2. This is a closed book exam. CALCULATORS, BOOKS, and NOTES are NOT ALLOWED.
3. Unless otherwise stated, to earn full credit you must show a valid DERIVATION and/or EXPLANATION of your answer, and you must express it in terms of the GIVEN VARIABLES.

Problem	**Maximum**	**Score**	**Grader**
1	25		
2	40		
3	35		
TOTAL	100		

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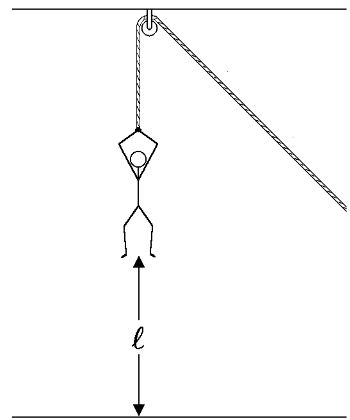


Problem 1: Basic concepts about energy (*25 points*)

Ali, who has mass M, grabs a rope that is wrapped around a pulley attached to the ceiling. He holds the rope tight as several friends pull on the other end, lifting him a distance ℓ above the floor. The acceleration of gravity is denoted by g. Assume that the mass of the rope and all frictional effects can be neglected. *[Note: the explanations included below are for pedagogical purposes only. Students were expected only to circle the correct answers.]*

(a) (*3 points*) From the time he starts at rest on the ground to the time he is hanging at rest at a height ℓ, what is the total work done on Ali by all forces, including gravity?

[(i) 0] (ii) $Mg\ell$ (iii) $-Mg\ell$ (iv) It depends on how fast his friends pulled.

Explanation: The kinetic energy begins and ends at zero, so the change in kinetic energy is zero. By the work-energy theorem, the total work done by all forces is equal to the change in kinetic energy, and hence must vanish.

(b) (*3 points*) During this period, by how much has Ali's gravitational potential energy changed?

(i) no change **[(ii) increase by $Mg\ell$]** (iii) decrease by $Mg\ell$

(iv) It depends on how fast his friends pulled.

Explanation: Near the surface of the Earth, the gravitational potential of an object of mass M is Mgh, where h is the height of the object, measured with respect to an arbitrarily chosen zero point. In this case Ali's height h has increased by ℓ, and hence his gravitational potential energy has increased by $Mg\ell$.

(c) (*3 points*) During this period, how much work was done on Ali by the rope?

(i) 0 **[(ii) $Mg\ell$]** (iii) $-Mg\ell$ (iv) It depends on how fast his friends pulled.

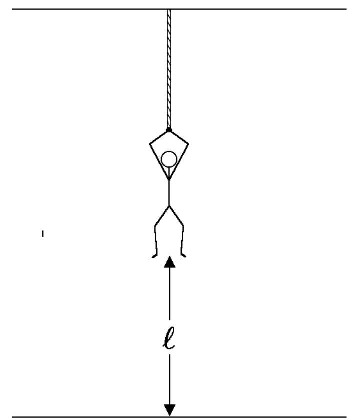
Explanation: The only forces that act on Ali are gravity and the force applied by the rope. The work done by gravity is the negative of the change in gravitational potential energy, or $-Mg\ell$. Since the total work done is zero, the work done by the rope must be $Mg\ell$. Note that this conclusion followed from the work-energy theorem and the calculation of the work done by gravity, so it is not affected by how hard Ali's friends pulled.

(d) (*3 points*) During this period, how much work has Ali done on himself?

[(i) 0] (ii) $Mg\ell$ (iii) $-Mg\ell$ (iv) It depends on how fast his friends pulled.

Explanation: Ali can cling to the rope without moving, so he acts as a rigid object. A rigid object can never do work on itself, so the answer is zero. We actually already assumed this result, when in part (c) we determined the work done by the rope by the use of the work-energy theorem.

Suppose now that the rope is tied directly to the ceiling. Ali starts at rest on the ground, and then climbs the rope hand over hand to the same height ℓ as before.



(e) (*3 points*) From the time he starts at rest on the ground to the time he is hanging at rest at a height ℓ, how much work was done on Ali by the rope?

[(i) 0] (ii) $Mg\ell$ (iii) $-Mg\ell$

(iv) It depends on how fast Ali pulled.

Explanation: This is subtle, because it is the force that the rope applies to Ali that causes him to move upward, against the force of gravity. Nonetheless, as he climbs hand over hand, the hand that is holding the rope is always stationary, while Ali's body and his free hand move upward. Since the force of the rope is applied to Ali's stationary hand, there is no displacement of the object to which the force is applied, and hence no work done. See Study Guide Problem 4D.1 for a more thorough explanation.

(f) (*3 points*) During this period, how much work has Ali done on himself?

(i) 0 **[(ii) $Mg\ell$]** (iii) $-Mg\ell$ (iv) It depends on how fast Ali pulled.

Explanation: As in parts (a)-(d), the total work done on Ali is zero, since his kinetic energy is zero at the start and zero at the finish. The work done by gravity is again $-Mg\ell$, so something else must be contributing work $Mg\ell$, to make the total zero. Since the rope does no work, and there are no other forces acting, it must be Ali who does work $Mg\ell$ on himself.

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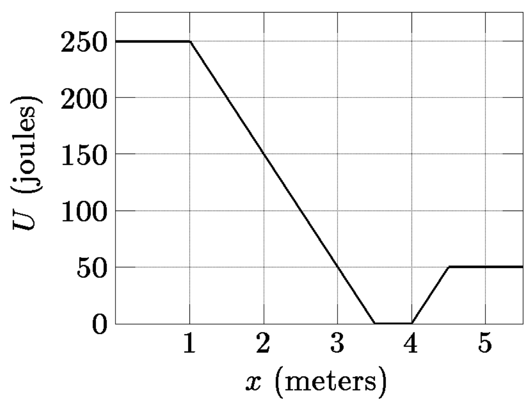
A particle with mass 2 kg moves in one dimension, in the presence of a force that is described by the following potential energy graph.



(g) (*2 points*) If the particle is located at $x = 0.5$ m, what is F_x, the x-component of the force acting on the particle?

[(i) 0] (ii) 25 N (iii) 50 N (iv) 75 N (v) 100 N (vi) 200 N (vii) 400 N

Explanation: The general formula is $F_x = -\partial U/\partial x$. *Here* $U(x)$ *is flat at* $x = 0.5$ *m, so* $-\partial U/\partial x = 0$.

(h) (*2 points*) If the particle is located at $x = 2$ m, what is F_x, the x-component of the force acting on the particle?

(i) 0 (ii) 25 N (iii) 50 N (iv) 75 N **[(v) 100 N]** (vi) 200 N (vii) 400 N

Explanation: Again $F_x = -\partial U/\partial x$. *At* $x = 2$ *m, the slope* $\partial U/\partial x$ *is negative, and equal in magnitude to two vertical boxes per horizontal box, or 100 J/m = 100 N.*

(i) (*3 points*) If the particle is released from rest at $x = 2$ m, what will be its speed when it crosses $x = 5$ m?

(i) 0 (ii) $\sqrt{50}$ m/s **[(iii) 10 m/s]** (iv) $\sqrt{150}$ m/s (v) $\sqrt{200}$ m/s (vi) $\sqrt{250}$ m/s

Explanation: The graph shows that the potential energy U decreases from 150 to 50 J, for a decrease by 100 J. The kinetic energy must therefore increase by that amount, because total mechanical energy (in the absence of friction) is conserved. The initial kinetic energy is zero, so the final must be 100 J. So $\frac{1}{2}Mv^2 = 100J$, *so* $v^2 = 100$ *J/kg = 100* m^2/s^2, *and hence* $v = 10$ *m/s.*

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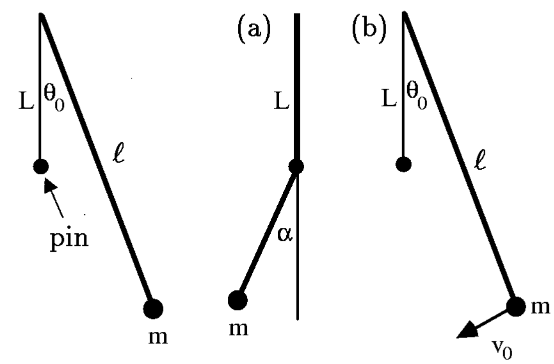
Problem 2: The pin and the pendulum (*40 points*)

A simple pendulum consisting of a mass m attached to a string of length ℓ is released from rest at an angle θ_0. A pin is located at a distance L below the pivot point. When the pendulum swings down, the string hits the pin as shown. Assume that gravity acts downward, with acceleration $g > 0$, and that the mass of the string and any frictional effects are negligible.



(a) (*20 points*) What is the maximum angle α that the string makes with the vertical after hitting the pin?

(b) (*7 points*) If the bob had been released with an initial velocity v_0 as shown, what would be the maximum value of α?

(c) (*3 points*) How would this be affected if v_0 were in the opposite direction?

(d) (*5 points*) At what point in the motion will the tension in the string reach its maximum value? Be sure that your answer is clear enough to specify whether the maximum tension is reached before or after the string makes contact with the pin.

(e) (*5 points*) If the particle is released from rest at an angle θ_0, as in part (a), what is the maximum tension T that the string experiences during this motion?

(a) In this problem mechanical energy (i.e., kinetic energy plus potential energy) is conserved. No work is done when the string hits the pin and partially wraps around it, because the part of the string that experiences a force is the part that is in contact with the pin, and is stationary. Work can be done on an object only if the object moves.

Let us adopt a coordinate system with the y-axis vertical, and with $y = 0$ at the top end of the string. We can also choose the gravitational potential energy at $y = 0$ to be zero, so the potential energy of the pendulum bob is given by $U = mgy$. Since the string is massless, the only energy of the pendulum is that of the bob. Since the bob is initially at rest, its initial mechanical energy is solely its potential energy:

$$E_{\text{initial}} = U(\theta_0) = -mg\ell \cos\theta_0 \ .$$

When the bob reaches its maximum height on the left side it will again be stationary, so again the only mechanical energy will be potential:

$$E_{\text{final}} = U_{\text{pin}}(\alpha_{\text{max}}) = -mg[L + (\ell - L)\cos\alpha_{\text{max}}] \ .$$

(Note that although $U_{\text{pin}}(\alpha_{\text{max}})$ and $U(\theta_0)$ both represent the same physical quantity, the potential energy of the bob, they do so under different circumstances. $U_{\text{pin}}(\alpha_{\text{max}})$ applies when the string hits the pin, and $U(\theta_0)$ applies when it does not. They therefore have different mathematical forms, so they are different mathematical functions. I have chosen to distinguish them by using a subscript for one and no subscript for the other. It would be bad notation to omit the subscript and call them $U(\alpha_{\text{max}})$ and $U(\theta_0)$, because once $U(\theta_0)$ has been defined by $U(\theta_0) = -mg\ell\cos\theta_0$, then the standard conventions of mathematics imply that $U(\alpha_{\text{max}})$ should mean $-mg\ell\cos\alpha_{\text{max}}$.) By conservation of energy,

$$E_{\text{final}} = E_{\text{initial}} \quad \Longrightarrow \quad \ell\cos\theta_0 = L + (\ell - L)\cos\alpha_{\text{max}} \ ,$$

so

$$\boxed{\alpha_{\text{max}} = \cos^{-1}\left\{\frac{\ell\cos\theta_0 - L}{\ell - L}\right\} \ .}$$

(b) If the bob has an initial speed v_0, then its initial mechanical energy includes a kinetic contribution:

$$E_{\text{initial}} = \frac{1}{2}mv_0^2 - mg\ell\cos\theta_0 \ .$$

The expression for E_{final} does not change form, so

$$E_{\text{final}} = E_{\text{initial}} \quad \Longrightarrow \quad \frac{1}{2}mv_0^2 - mg\ell\cos\theta_0 = -mg[L + (\ell - L)\cos\alpha_{\text{max}}] \ ,$$

or

$$\ell \cos\theta_0 - \frac{1}{2}\frac{v_0^2}{g} = L + (\ell - L)\cos\alpha_{\max} ,$$

so

$$\boxed{\alpha_{\max} = \cos^{-1}\left\{\frac{\ell\cos\theta_0 - L - \frac{v_0^2}{2g}}{\ell - L}\right\} .}$$

(c) The initial mechanical energy does not depend on the direction of v_0, so $\boxed{\alpha_{\max} \text{ will not depend on the direction of the initial velocity.}}$

(d) The tension in the string can be calculated from the radial component of $\vec{\mathbf{F}} = m\vec{\mathbf{a}}$. The radial component of $\vec{\mathbf{a}}$ is inward, with magnitude v^2/r. v^2 is maximized at the bottom of the swing, since the gravitational potential energy is the lowest at this point. r is equal to ℓ before the string hits the pin, and $\ell - L$ afterward, so v^2/r is maximized immediately after the string hits the pin. Gravity also contributes to the radial component of $\vec{\mathbf{F}} = m\vec{\mathbf{a}}$, where an outward gravitational force leads to an increase in the tension T. The outward gravitational force is also maximized at the bottom of the swing of the pendulum. So, the tension T is maximum

$$\boxed{\text{at the bottom of the swing, just after the string makes contact with the pin.}}$$

(e) Treating the argument in part (d) quantitatively,

$$F_r = ma_r = -\frac{mv^2}{r} ,$$

where the outward radial direction is defined as positive. At the bottom of the swing, conservation of mechanical energy implies

$$\frac{1}{2}mv^2 - mg\ell = -mg\ell\cos\theta_0 .$$

At the bottom of the swing $F_r = mg - T$, and just after the string hits the pin, $r = \ell - L$. Putting these relations together,

$$\boxed{T = mg + \frac{2mg\ell(1 - \cos\theta_0)}{\ell - L} .}$$

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Problem 3: Force and potential in one dimension (*35 points*)

A particle of mass m is constrained to move in one dimension, described by a coordinate x. The force on the particle depends on its position, and is given by

$$F_x = \begin{cases} 0 & \text{if } x \le 0 \\ -Ax^2 & \text{if } 0 < x < b \\ 0 & \text{if } x \ge b\ , \end{cases}$$

where A is a positive constant.

(a) (*5 points*) Is this force conservative? Give a brief explanation of your answer.

(b) (*5 points*) Choose the zero of potential energy $U(x)$ so that $U(0) = 0$. What is $U(x)$ for $x < 0$?

(c) (*10 points*) What is $U(x)$ in the range $0 < x < b$?

(d) (*5 points*) What is $U(x)$ in the range $x \ge b$?

(e) (*5 points*) Suppose that the particle begins in the region $x < 0$ with a speed v_0 to the right (i.e., to larger values of x). If v_0 is larger than a certain value v_1, then the particle will continue moving to the right forever. Find an expression for v_1 either in terms of m, A, and b, or in terms of m and $U(x)$. *[Note that you can express your answer in terms of the function $U(x)$ even if you have not calculated it. Remember, however, that the symbol $U(x)$ does not represent any particular number, since x can have any value. Your answer should involve something like $U(0)$, $U(b)$, $U(b/2)$, $U(2b)$, etc.]*

(f) (*5 points*) If $v_0 < v_1$, then the particle will reach a maximum value of x, which we will call x_{max}, and then it will move back to the left. Find an expression for x_{max} either in terms of m, v_0, A, and b or in terms of m, v_0, and the function $U(x)$.

(a) Yes. Since F_x depends only on x, and not on time or on the velocity, the work done to get from x_1 to x_2 will depend only on x_1 and x_2, and not on any other property of the trajectory.

(b) From the formula sheet,

$$U(x_p) = U_0 - \int_{x_0}^{x_p} F\,\mathrm{d}x \ ,$$

so we can take $x_p < 0$ and $x_0 = 0$, with $U_0 = U(x_0) = 0$. Then

$$U(x_p) = 0 - \int_0^{x_p} 0\,\mathrm{d}x = \boxed{0\ .}$$

(c) Applying the same reasoning for $0 < x_p < b$, everything is the same except that F_x in the region of integration will be given by $-Ax^2$. So

$$U(x_p) = 0 - \int_0^{x_p} (-Ax^2)\,\mathrm{d}x = \boxed{\frac{1}{3}Ax_p^3\ .}$$

(d) The easiest thing is to use $x = b$ as x_0. From (c), we know that $U(b) = \frac{1}{3}Ab^3$. It is important to realize that the answer in part (c) for $U(x_p)$ holds for $x_p = b$, and not just for $0 < x_p < b$. The reason is that the integral in part (c) does not change if the integrand, F_x, changes suddenly at the endpoint, as it does at $x = b$. The value of a function at a single point is what the mathematicians call a "set of measure zero," and it does not affect the value of the integral.

So

$$U(x_p) = \frac{1}{3}Ab^3 - \int_b^{x_p} F_x\,\mathrm{d}x = \frac{1}{3}Ab^3 - \int_b^{x_p} 0\,\mathrm{d}x = \boxed{\frac{1}{3}Ab^3\ .}$$

Note that even though the force is zero in this region, the potential energy is not zero. The absence of a force implies that the potential energy is constant. The fact that one must do work to move the particle from $x = 0$ to the region $x > b$ implies that the potential energy for $x > b$ must be larger than the potential energy at $x = 0$, which we have defined to be zero.

Some of you may wonder what happens if one tries to apply the general formula from the formula sheet directly, using $x_0 = 0$ as we did in parts (b) and (c). We will get the same answer as above, as long as we remember that to integrate a function that changes form depending on x, as $F_x(x)$ does, one must break up the range of integration into different segments. For $x_p > b$,

$$U(x_p) = 0 - \int_0^{x_p} F_x(x)\,\mathrm{d}x = -\int_0^{b} (-Ax^2)\,\mathrm{d}x - \int_b^{x_p} 0\,\mathrm{d}x = \frac{1}{3}Ab^3\ ,$$

as expected.

(e) The particle will move to the right forever if it can reach the region $x > b$, when there will be no more forces acting on it. This requires that the initial energy be larger than the potential energy in the region $x > b$. The borderline value is when the initial energy is just equal to the potential energy in the region $x > b$, so

$$\frac{1}{2}mv_1^2 = U(b) \ ,$$

or

$$\boxed{v_1 = \sqrt{\frac{2U(b)}{m}} \ .}$$

Using the expression for $U(b)$ from (c) or (d),

$$\boxed{v_1 = \sqrt{\frac{2Ab^3}{3m}} \ .}$$

(f) $x_{\max}$ will be determined by conservation of energy. The final kinetic energy will be zero, since the particle will be instantaneously at rest when it reaches the maximum value of x, so

$$\boxed{\frac{1}{2}mv_0^2 = U(x_{\max}) \ ,}$$

which can be solved for $x_{\max}$. Using the explicit form of $U(x)$ found in part (c),

$$\frac{1}{2}mv_0^2 = \frac{1}{3}Ax_{\max}^3 \ ,$$

so

$$\boxed{x_{\max} = \left(\frac{3mv_0^2}{2A}\right)^{1/3} \ .}$$