MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Physics

Physics 8.01 Spring 2005

WEEKLY QUIZ 8 SOLUTIONS

Quiz Date: Friday, April 1, 2005

FAMILY (LAST) NAME

GIVEN (FIRST) NAME

STUDENT ID NUMBER

Please check (✔) your class ☞

	L01	MTW 10:00	Walter Lewin
	L02	MTW 11:00	Walter Lewin
	L03	MTW 2:00	Min Chen
	L04	MTW 3:00	Min Chen

INSTRUCTIONS:

1. The FORMULA SHEET is in the back of this exam. You may tear it off. There is also an extra BLANK PAGE in case you need it.
2. This is a closed book exam. CALCULATORS, BOOKS, and NOTES are NOT ALLOWED.
3. Unless otherwise stated, to earn full credit you must show a valid DERIVATION and/or EXPLANATION of your answer, and you must express it in terms of the GIVEN VARIABLES.

Problem	**Maximum**	**Score**	**Grader**
1	40		
2	30		
3	30		
TOTAL	100		

Name ______________________

Problem 1: Basic concepts about the rotation of rigid bodies in two dimensions (*40 points*)

Mark your answer by circling it. For these multiple choice questions you need not show your work, and there will be no partial credit. *[For pedagogical purposes, explanations are included with these solutions.] Warning: some problems may contain irrelevant information.*

(a) (*5 points*) A wheel is rotating at 120 rpm (revolutions per minute). What is its angular velocity in radians per second?

(i) 120 (ii) 240π (iii) 2 (iv) π (v) 2π (vi) 3π **[(vii) 4π]**

Explanation:

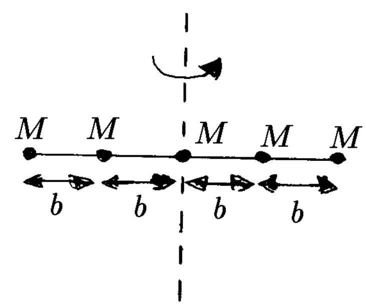
$$120\ \frac{\text{rev}}{\text{min}} \times \frac{1\ \text{min}}{60\ \text{s}} \times \frac{2\pi\ \text{rad}}{\text{rev}} = 4\pi\ \text{rad/s}\ .$$

(b) (*5 points*) At $t = 0$ a flywheel is rotating with angular velocity ω_0. It then undergoes uniform angular acceleration for a time t_1, at the end of which the angular velocity is ω_1. How many revolutions did the flywheel make during this time interval?

(i) $\omega_0 t$ (ii) $\omega_1 t$ (iii) $\frac{1}{2}(\omega_0 + \omega_1)t$ (iv) $\frac{\omega_0 t}{2\pi}$ (v) $\frac{\omega_1 t}{2\pi}$ **[(vi) $\frac{(\omega_0 + \omega_1)t}{4\pi}$]**

Explanation: Since the acceleration is uniform, the average angular velocity is the average of the initial and final angular velocities, so $\omega_{\text{av}} = (\omega_0+\omega_1)/2$. *Then* $\theta = \omega_{\text{av}}\, t$, *and the number of revolutions is* $\theta/2\pi$.

(c) (*5 points*) Five small balls, each of mass M, are attached to a massless rigid rod of length $4b$. One ball is at the center, one ball is at each end, and one ball is a distance b from the center in each direction, as shown. What is the moment of inertia of this object for rotation about an axis through the center of the rod and perpendicular to it?

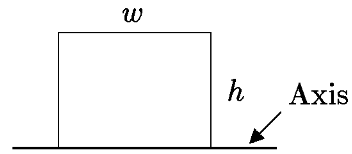


(i) $2Mb^2$ (ii) $5Mb^2$ (iii) $8Mb^2$ **[(iv) $10Mb^2$]** (v) $12Mb^2$

Explanation:

$$I = \sum_i M_i r_i^2 = M(1 \cdot 0^2 + 2 \cdot b^2 + 2 \cdot (2b)^2 = 10Mb^2\ .$$

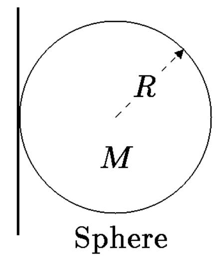
— Problem 1 Continues —



(d) (*5 points*) What is the moment of inertia I of a thin rectangular plate of mass M and dimensions $w \times h$, pivoted about one of the edges of length w? (Remember that there is a table of moments of inertia on the formula sheet.)

(i) $\frac{1}{3}Mw^2$ **(ii) $\frac{1}{3}Mh^2$** (iii) $\frac{1}{2}Mw^2$ (iv) $\frac{1}{2}Mh^2$ (v) $\frac{1}{3}M(w^2+h^2)$

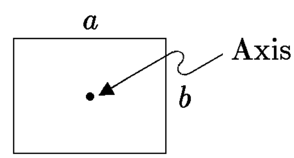
Explanation: This answer comes straight from the formula sheet, except that it labels the rectangle as $a \times b$ instead of $h \times w$.



(e) (*5 points*) What is the moment of inertia I of a sphere of mass M and radius R, pivoted about a rod that is tangent to the surface of the sphere?

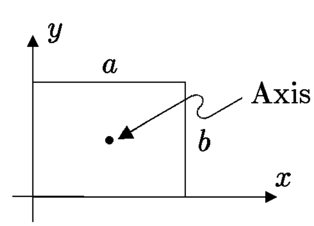
(i) $\frac{2}{5}MR^2$ (ii) $\frac{3}{5}MR^2$ (iii) $\frac{4}{5}MR^2$ (iv) MR^2 (v) $\frac{6}{5}MR^2$ **(vi) $\frac{7}{5}MR^2$**

Explanation: If the axis went through the center of the sphere, the table tells us that $I_{\rm cm} = \frac{2}{5}MR^2$. The axis shown is parallel to such a center of mass axis, so the parallel axis theorem implies that $I_{\parallel} = MR^2 + \frac{2}{5}MR^2 = \frac{7}{5}MR^2$.



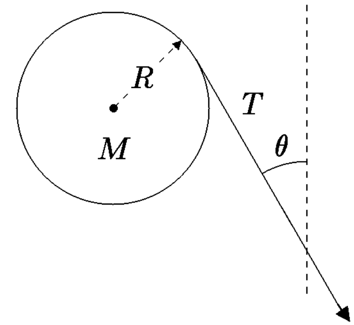
(f) (*5 points*) What is the moment of inertia I of a thin rectangular plate of mass M and dimensions $a \times b$, pivoted about an axis perpendicular to the plate and through its center?

(i) $\frac{1}{2}Mab$ **(ii) $\frac{1}{12}M(a^2+b^2)$** (iii) $\frac{1}{3}M(a^2+b^2)$ (iv) $\frac{1}{2}M(a^2+b^2)$ (v) $\frac{7}{12}M(a^2+b^2)$



Explanation: Using a coordinate system as shown at the right, the values of I_x and I_y, the moments of inertia about the x and y axes respectively, are given in the table as $I_x = \frac{1}{3}Mb^2$ and $I_y = \frac{1}{3}Ma^2$. Then the perpendicular axis theorem implies that $I_z = I_x + I_y = \frac{1}{3}M(a^2+b^2)$. The desired axis is through the center of mass and parallel to the z-axis, so the parallel axis theorem relates the desired moment of inertia $I_{\rm cm}$ to I_z: $I_z = I_{\rm cm} + Md^2$, where d is the distance *from the center of the rectangle to a corner, or* $d^2 = \frac{1}{4}(a^2 + b^2)$. *One solves for* I_{cm} *to obtain the boxed result above.*

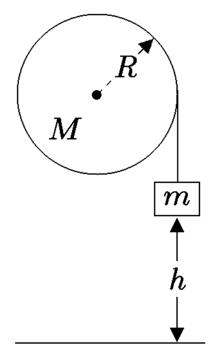
(g) (*5 points*) A wheel, which consists of a solid uniform cylindrical disk of radius R and mass M, is pivoted about a fixed axle through the axis of the cylinder. A massless inextensible rope is wound around the rim of the wheel so that it cannot slip. The rope is pulled with a tension T at angle θ with respect to the vertical, as shown. What is the magnitude of the angular acceleration α of the wheel about the pivot? Neglect all friction.



(i) $\dfrac{T}{\pi MR}$ (ii) $\dfrac{T\sin\theta}{\pi MR}$ (iii) $\dfrac{T\cos\theta}{\pi MR}$ $\boxed{\text{(iv) } \dfrac{2T}{MR}}$ (v) $\dfrac{2T\sin\theta}{MR}$ (vi) $\dfrac{2T\cos\theta}{MR}$

Explanation: The torque about the axis is $\tau = TR$, *independent of* θ, *and the moment of inertia for a solid cylinder is* $I = \frac{1}{2}MR^2$. *So* $\alpha = \tau/I = 2T/MR$.

(h) (*5 points*) A solid uniform cylinder of mass M and radius R is pivoted about a fixed horizontal rod. A massless inextensible string is wrapped around it, and attached to a block of mass m which is initially at a height h above the floor. The acceleration of gravity is g, directed downward. The block is released from rest. By what total angle $\Delta\theta$ (in radians) has the cylinder turned when the block hits the floor?



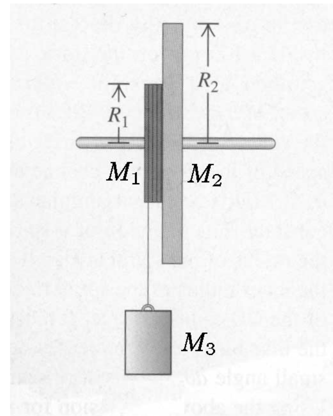
(i) $\dfrac{M}{m}$ (ii) $\sqrt{1 + \dfrac{2M}{m}}$ (iii) $\sqrt{1 + \dfrac{h}{2R}}$ $\boxed{\text{(iv) } \dfrac{h}{R}}$

(v) $\dfrac{MR}{mh}$ (vi) $\left(\dfrac{MR}{mh}\right)^2$

Explanation: This is essentially the same as the standard rolling constraint. Every time the cylinder makes a full rotation, with $\Delta\theta = 2\pi$, *it unwinds a length of string equal to its circumference,* $2\pi R$. *Thus the length of unwound string* h *is equal to* $R\,\Delta\theta$, *so* $\Delta\theta = h/R$.

Problem 2: Two disks welded to a common shaft (*30 points*)

Two uniform disks, one with radius R_1 and mass M_1 and the other with radius $R_2 > R_1$ and mass M_2, are welded together and mounted on a frictionless axle through their common center.



(a) (*6 points*) What is the total moment of inertia I_{tot} of the two-disk system about the axis of rotation?

(b) (*10 points*) A light string is wrapped around the edge of the smaller disk (disk 1), and a block of mass M_3 is suspended from the free end of the string. If the block is released from rest at a distance ℓ above the floor, what is its speed v just before it strikes the floor?

(c) (*4 points*) Now suppose that the string is wrapped around the larger disk, the one of radius R_2, and is again released from rest at a distance ℓ above the floor. In this case, what is the speed v' of the block just before it strikes the floor.

(d) (*5 points*) In which case, (b) or (c), is the final speed of the block the greatest? Explain, in a sentence or two, why this is so.

(e) (*5 points*) In which case, (b) or (c), is the final angular velocity of the axle and the two disks the greatest? Explain, in a sentence or two, why this is so.

(a) The total moment of inertia is just the sum of the moments for each disk, and the moment of inertia for a cylinder can be found in the tables. So,

$$\boxed{I_{\rm tot} = \frac{1}{2}M_1 R_1^2 + \frac{1}{2}M_2 R_2^2 \ .}$$

(b) If we define the zero of gravitational potential energy at the floor, then the block initially has potential energy $U_i = M_3 g\ell$, and the final potential energy is zero. The initial kinetic energies are all zero, and the final kinetic energy includes contributions both from the rotation of the disks and the downward motion of the block. Thus, conservation of mechanical energy,

$$E_{\rm initial} = E_{\rm final} \ ,$$

can be written in detail as

$$M_3 g\ell = \frac{1}{2}I_{\rm tot}\omega^2 + \frac{1}{2}M_3 v^2 \ .$$

v and ω are related, however, because the block can fall only at the speed that the string is unwrapped by the disk of radius R_1. So

$$v = R_1\omega \ ,$$

and the equations can be put together to give

$$M_3 g\ell = \frac{I_{\rm tot}}{R_1^2}v^2 + \frac{1}{2}M_3 v^2 \ .$$

This equation can be solved for v, giving

$$\boxed{v = R_1\sqrt{\frac{2M_3 g\ell}{I_{\rm tot} + M_3 R_1^2}} \ .}$$

(c) If the string is wrapped around the disk of radius R_2, one just replaces R_1 in the previous answer by R_2:

$$\boxed{v' = R_2\sqrt{\frac{2M_3 g\ell}{I_{\rm tot} + M_3 R_2^2}} \ .}$$

(d) It is easier to compare these two answers if they are written as

$$v = \sqrt{\frac{2g\ell}{1 + (I_{\text{tot}}/M_3 R_1^2)}} \qquad \text{and} \qquad v' = \sqrt{\frac{2g\ell}{1 + (I_{\text{tot}}/M_3 R_2^2)}} \ ,$$

which shows that $v' > v$, since the denominator for v' is smaller. This makes sense, because the constraint $v = R\omega$ implies that for a given speed v of the block, the disks will rotate more slowly when the string is wrapped around the larger wheel. That means that less energy is used for the rotation of the disks, so more is available for the downward motion of the block.

(e) Using ω and ω' to denote the final angular velocity of the disks in the two cases, respectively, one has

$$\omega = \frac{v}{R_1} = \sqrt{\frac{2M_3 g\ell}{I_{\text{tot}} + M_3 R_1^2}}$$

and

$$\omega' = \frac{v'}{R_2} = \sqrt{\frac{2M_3 g\ell}{I_{\text{tot}} + M_3 R_2^2}} \ ,$$

so $\boxed{\omega \text{ is larger than } \omega'.}$ The disks rotate faster when the string is wound around the smaller disk. This also makes sense, since mechanical energy is conserved. The potential energy that is released by the falling block is the same in both cases. If the block falls faster when the string is wrapped around the larger disk, then the disks must rotate slower so that the total energy is the same in the two cases.

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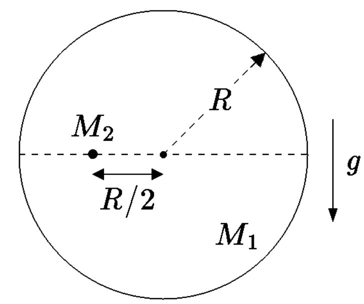
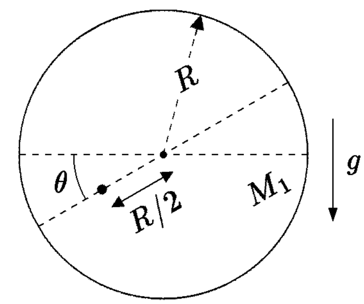
Problem 3: A pivoted disk with attached weights (*30 points*)

A uniform disk of mass M_1 and radius R is pivoted on a frictionless horizontal axle through its center. A small marble of mass M_2 is attached to the disk at radius $R/2$, at the same height as the axle. Assume that the marble is small enough to be treated as a point mass. The acceleration of gravity is downward, with magnitude g.



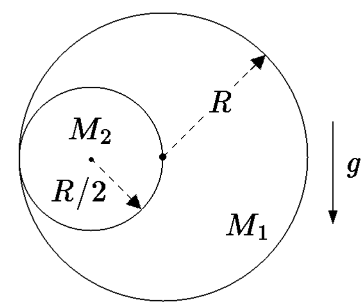
(a) (*8 points*) If this system is released from rest to rotate about the pivot, what will be the angular acceleration α_0 of the disk immediately after it is released?



(b) (*4 points*) After the disk has rotated through an angle θ, what will be the angular acceleration α_1?

(c) (*6 points*) What will be the maximum angular velocity $\omega_{\max}$ that the disk will reach in its subsequent motion?

(d) (*7 points*) Now consider the situation if the marble is replaced by a disk of radius $R/2$, with the same mass M_2, located with its center at the same place where the marble was located in part (a). (When calculating the torque on this disk, you can use the fact that the torque caused by gravity can be calculated as if the force of gravity were a single force acting at the center of mass of the object.) For this case, find the angular acceleration α'_0 immediately after the system is released from rest.



(e) (*5 points*) For the case described in part (d), what will be the maximum angular velocity $\omega'_{\max}$ that the disk will reach in its subsequent motion?

(a) Since the axle goes through the center of mass of the disk of mass M_1, the gravitational force on this disk does not result in any torque about the axle. But there is a torque caused by the gravitational force on M_2, given by

$$\tau = R_\perp F = \frac{1}{2} M_2 g R \ .$$

The moment of inertia of the combined system about the axle is that of the disk M_1 plus the mass M_2, so

$$I = \frac{1}{2} M_1 R^2 + M_2 \left(\frac{R}{2}\right)^2 = \frac{1}{4}(2M_1 + M_2) R^2 \ ,$$

where the moment of inertia of the disk is taken directly from the table in the formula sheets. The angular acceleration immediately after release is therefore

$$\boxed{\alpha = \frac{\tau}{I} = \frac{2M_2 g}{(2M_1 + M_2)R} \ .}$$

(b) The only change is in the calculation of the torque, which in general is given by $\tau = R_\perp F$, where $R_\perp$ is the component of the radius vector perpendicular to the force. In this case $R_\perp$ is the horizontal component of the vector from the pivot to the marble, which is $(R/2)\cos\theta$. Thus α is modified from the answer to part (a) by the insertion of this factor of $\cos\theta$:

$$\boxed{\alpha = \frac{2M_2 g \cos\theta}{(2M_1 + M_2)R} \ .}$$

(c) The maximum angular velocity will be attained when M_2 is at the bottom of its motion. The value of the angular velocity can be determined by using the conservation of energy. The potential energy of the disk M_1 does not change, since its center of mass does not move, so the only potential energy that needs to be considered is that of M_2. This potential energy can be written $U = M_2 g y$, where y is the vertical coordinate, measured from an arbitrary origin. I will take that origin as the height of the axle. Thus $U_{\text{initial}} = 0$, and U_{final} (at the bottom of the motion) is $-M_2 g R/2$. Then

$$E_{\text{initial}} = 0$$

$$E_{\text{final}} = \frac{1}{2} I \omega_f^2 - \frac{1}{2} M_2 g R$$

$$E_{\text{final}} = E_{\text{initial}} \quad \Longrightarrow \quad \boxed{\omega_f = \sqrt{\frac{M_2 g R}{I}} = \sqrt{\frac{4M_2 g}{(2M_1 + M_2)R}} \ .}$$

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(d) The only difference between this case and the previous one is the moment of inertia of the disk of mass M_2. According to the table, the moment of inertia of this disk about its own center is $\frac{1}{2}M_2(R/2)^2$. But we need the moment of inertia about the center of the larger disk, for which we have to use the parallel axis theorem:

$$I_{\|} = I_{\rm cm} + Md^2 = \frac{1}{2}M_2\left(\frac{R}{2}\right)^2 + M_2\left(\frac{R}{2}\right)^2 = \frac{3}{8}M_2R^2 \ .$$

So in this case the total moment of inertia is given by

$$I' = \frac{1}{2}M_1R^2 + \frac{3}{8}M_2R^2 = \frac{1}{8}(4M_1 + 3M_2)R^2 \ .$$

The torque is the same as in part (a), since the torque due to the gravitational force on M_2 can be calculated as if the entire force acted on the center of mass. Thus,

$$\boxed{\alpha_0' = \frac{\tau}{I'} = \frac{4M_2g}{(4M_1 + 3M_2)R} \ .}$$

(e) The maximum angular velocity is also the same as it was in the previous case, except for the modification of the moment of inertia.

$$\boxed{\omega_f' = \sqrt{\frac{M_2gR}{I'}} = \sqrt{\frac{8M_2g}{(4M_1 + 3M_2)R}} \ .}$$