MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Physics

Physics 8.01 Spring 2005

WEEKLY QUIZ 10 SOLUTIONS

Quiz Date: Friday, April 15, 2005

Corrected Version, April 21, 2005: Diagram for Problem 1(b) was replaced

FAMILY (LAST) NAME

GIVEN (FIRST) NAME

STUDENT ID NUMBER

Please check (✔) your class ☞

	L01	MTW 10:00	Walter Lewin
	L02	MTW 11:00	Walter Lewin
	L03	MTW 2:00	Min Chen
	L04	MTW 3:00	Min Chen

INSTRUCTIONS:

1. The FORMULA SHEET is in the back of this exam. You may tear it off. There is also an extra BLANK PAGE in case you need it.
2. This is a closed book exam. CALCULATORS, BOOKS, and NOTES are NOT ALLOWED.
3. Unless otherwise stated, to earn full credit you must show a valid DERIVATION and/or EXPLANATION of your answer, and you must express it in terms of the GIVEN VARIABLES.

Problem	**Maximum**	**Score**	**Grader**
1	25		
2	30		
3	20		
4	25		
TOTAL	100		

ANNOUNCEMENTS MADE (AND NOT MADE) AT THE QUIZ

Problem 2: The distances d and h are both measured along the direction of the ladder (i.e., they are not measured vertically).

Problem 3: The moment of inertia I is the moment of inertia for rotation about the axis of symmetry of the cylinder (where the center of mass is located).

Problem 1(b): It was not announced at the quiz, but it should have been specified that the moment of inertia under discussion is that for rotation about the center of the sphere.

Problem 4: The diagram was not drawn correctly. The bottom end of the line indicated by h should have been at ground level, the same level as the bottom of the wheels. On part (c) it should have said to neglect the mass of the wheels, relative to the mass of the car.

Name ______________________

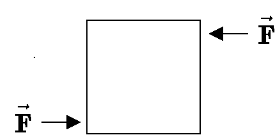
Problem 1: Basic concepts about the rotation of rigid bodies (*25 points*)

Mark your answer by circling it. For these short answer problems you need not show your work, and there will be no partial credit. *Warning: some problems may contain irrelevant information.*

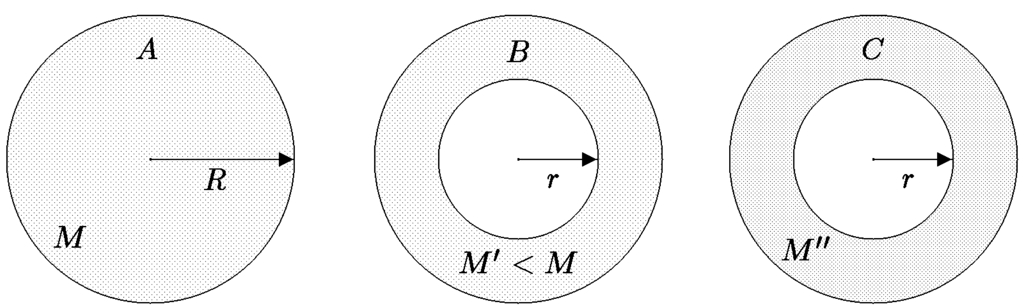
(a) (*5 points*) When is the angular momentum of a system constant? Choose one.

- (i) When the total kinetic energy is constant.
- (ii) When no net external force acts on the system.
- (iii) When the linear momentum and the energy are constant.
- $\boxed{\text{(iv)}}$ When no external torque acts on the system.
- (v) When the moment of inertia is constant.

The only other real candidate for an answer is (ii). However, the inclusion of the adjective "net," a synonym for "total," makes this false. If NO external force acted on the system, then there could be no external torque, and the angular momentum of the system would be conserved. However, if we know only that no NET external force acts, then there is the possibility, for example, that two equal but opposite external forces may act on the system, at different points. If the forces are not directed along the line that joins the two points, then they will create a net torque, even though the forces add to zero.



An example with zero net force but nonzero torque



Same density as A

Same moment of inertia as A

(b) (*5 points*) Consider three spheres, labeled A, B, and C, all with the same radius R. Sphere A is a uniform, solid sphere of mass M. Sphere B is made of the same material, and hence has the same density, but it has a hollow cavity of radius r, where $r < R$. Its mass is M'. Sphere C also has a hollow cavity of radius r, but it is made of a different material so that its mass M'' is just the right value to give sphere C the same moment of inertia as sphere A. List the three masses, M, M', and M'', from largest to smallest, noting if any or all of them are equal.

From largest to smallest, the masses are M, M'', and M'. *To compare M and M'', first consider the possibility that they are equal. If so, then the moment of inertia of sphere C about its center would be larger than that of A, since the mass would be the same but the radius at which the mass is located is larger. So, for the two to have the same moment of inertia, we must have $M'' < M$. To compare M'' and M', note that the moment of inertia of sphere B must be less than that of A, since the sphere B can be constructed from the sphere A by cutting out the central sphere (radius $< r$). (Cutting away mass always reduces the moment of inertia, since it removes the contributions of the affected particles from $I = \sum m_i r_i^2$.) Since B and C are identical except for their density, the fact that C has a larger moment of inertia implies that it must have a larger density, and hence larger mass.*

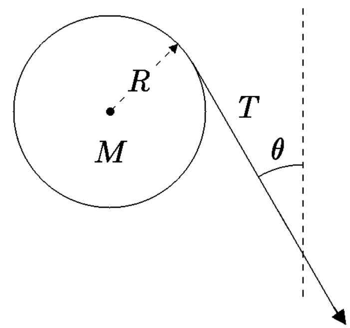
(c) (*5 points*) Suppose that the three spheres in part (b) are all allowed to roll down the same incline, each starting from rest. Assume that they roll without slipping, but that all dissipative forces, such as air friction or rolling friction, are negligible. List the three spheres (A, B, and C) in the order in which they reach the bottom, again noting if any or all of them are equal.

A reaches the bottom first, followed by a tie between B and C. *B and C differ only by their density, which will have no effect on how fast they roll. The moment of inertia for C will be larger than that for B by the same ratio as the masses, M''/M', so the energy conservation equation,*

$$E_{\text{mechanical}} = Mgy + \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2 = const \ ,$$

will be equivalent for the two cases— in the equation for C, every term will be larger than the corresponding term in the B-equation by the factor M''/M'. To compare A with C, note that they have the same moment of inertia and radius, but that A has a larger mass. At a given speed the two spheres will have the same rotational kinetic energy, but for C this energy will be a larger fraction of the total potential energy released by the decrease in height, and hence for C the rotation will be more of an impediment to the linear motion down the hill.

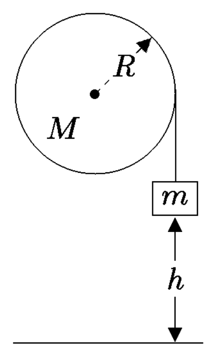
(d) (*5 points*) A wheel, which consists of a solid uniform cylindrical disk of radius R and mass M, is pivoted about a fixed axle through the axis of the cylinder. A massless inextensible rope is wound around the rim of the wheel so that it cannot slip. The rope is pulled with a tension T at angle θ with respect to the vertical, as shown. What is the magnitude of the angular acceleration α of the wheel about the pivot? Neglect all friction.



$\boxed{\text{(i) } \dfrac{2T}{MR}}$ (ii) $\dfrac{T\sin\theta}{\pi MR}$ (iii) $\dfrac{T\cos\theta}{\pi MR}$ (iv) $\dfrac{T}{\pi MR}$ (v) $\dfrac{2T\sin\theta}{MR}$ (vi) $\dfrac{2T\cos\theta}{MR}$

Explanation: The torque about the axis is $\tau = TR$, independent of θ, and the moment of inertia for a solid cylinder is $I = \frac{1}{2}MR^2$. So $\alpha = \tau/I = 2T/MR$.

(e) (*5 points*) A solid uniform cylinder of mass M and radius R is pivoted about a fixed horizontal rod. A massless inextensible string is wrapped around it, and attached to a block of mass m which is initially at a height h above the floor. The acceleration of gravity is g, directed downward. The block is released from rest. By what total angle $\Delta\theta$ (in radians) has the cylinder turned when the block hits the floor?



(i) $\dfrac{M}{m}$ (ii) $\sqrt{1+\dfrac{2M}{m}}$ $\boxed{\text{(iii) } \dfrac{h}{R}}$ (iv) $\sqrt{1+\dfrac{h}{2R}}$

(v) $\dfrac{MR}{mh}$ (vi) $\left(\dfrac{MR}{mh}\right)^2$

Explanation: This is essentially the same as the standard rolling constraint. Every time the cylinder makes a full rotation, with $\Delta\theta = 2\pi$, it unwinds a length of string equal to its circumference, $2\pi R$. Thus the length of unwound string h is equal to $R\,\Delta\theta$, so $\Delta\theta = h/R$.

Name

Problem 2: Will the ladder slip? (*30 points*)

A uniform ladder of mass m and length ℓ rests against a smooth wall, making an angle θ with respect to the ground. Assume that there is negligible friction between the ladder and the wall. A do-it-yourself enthusiast of mass M stands on the ladder a distance d from the bottom. The force of gravity acts downward with acceleration g, where $g > 0$. Assume that the ladder is stationary, i.e., that it does not slip.

(a) (*5 points*) What is the magnitude N_g of the upward normal force that the ground exerts on the ladder?

(b) (*5 points*) Does the frictional force that the ground exerts on the ladder point toward the wall, or away from the wall? Explain briefly.

(c) (*10 points*) Let F denote the magnitude of the force of friction that the ground exerts on the ladder, and let N_w denote the magnitude of the normal force that the wall exerts on the ladder. Write down two equations which can be solved to determine these two variables. The equations should be written in terms of any or all of the variables F, N_w, m, ℓ, θ, M, d, g, and N_g.

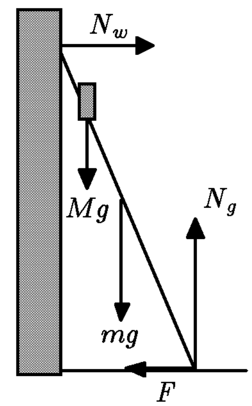
(d) (*5 points*) What is the minimum coefficient of friction between the ladder and the ground that is required in order that the ladder will not slip? Whether or not you have answered parts (a) and (c), you can use any of the variables N_w, N_g, and F in expressing your answer, as well as any of the given variables.

Now suppose that the ladder is not uniform, but instead is tapered so that there is more material near the bottom than near the top. We are not told the detailed geometry, but we are told that the center of mass of the ladder lies a distance h from the bottom. (The mass of the ladder is still m, and its overall length is still ℓ.)

(e) (*5 points*) For this case, write down two equations which can be solved to determine F (the magnitude of the force of friction that the ground exerts on the ladder) and N_w (the magnitude of the normal force that the wall exerts on the ladder). The equations should be written in terms of any or all of the variables F, N_w, m, ℓ, h, θ, M, d, g, and N_g.

(a) Using the force diagram at the right, one can see that vanishing of the total force in the vertical direction implies that

$$N_g - mg - Mg = 0 \quad \Longrightarrow \quad \boxed{N_g = (M+m)g\ .}$$

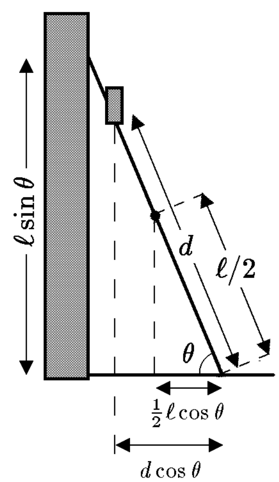


(b) The force of friction that the ground exerts on the ladder must point $\boxed{\text{toward the wall.}}$ The only other horizontal force is the normal force which the wall exerts on the ladder, which must be away from the wall, and the two forces must cancel.

(c) One equation is

$$\boxed{F = N_w\ ,}$$

which is the statement that the total force in the horizontal direction vanishes. The other equation must come from setting the torque equal to zero. The torque must vanish about any origin, so I will choose to take it about the bottom end of the ladder. Using the geometry as shown in the diagram at the right, this torque is given by



$$\tau = \boxed{-N_w \ell \sin\theta + mg\frac{\ell}{2}\cos\theta + Mgd\cos\theta = 0\ .}$$

(d) If the ladder is about to slip, then $F = \mu_s N_g$. Since N_g and F are both acceptable variables to use in the answer, we can write simply

$$\boxed{\mu_s = F/N_g\ .}$$

(e) The equation

$$\boxed{F = N_w\ ,}$$

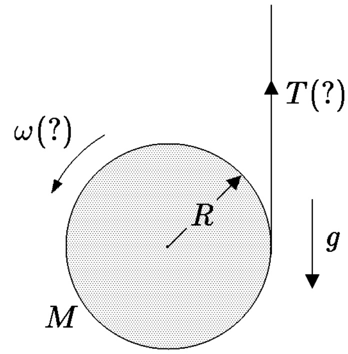
remains valid. The torque due to gravity acting on the ladder is identical to what it would be if the force of gravity acted as a single force on the center of mass, at a distance h from the bottom of the ladder. The torque equation is then modified only by replacing $\ell/2$ by h, so it reads

$$\tau = \boxed{-N_w \ell \sin\theta + mgh\cos\theta + Mgd\cos\theta = 0\ .}$$

Name ______________________________

Problem 3: A cylinder on a string (*20 points*)

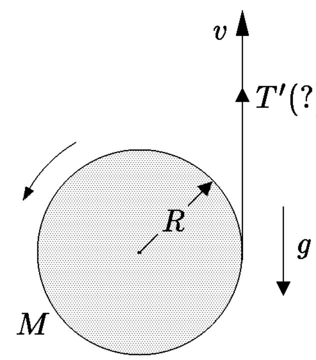
A cylinder of radius R and mass M is wrapped with an inextensible string of negligible mass. The density of the cylinder is not uniform, but depends on the distance from the axis. The moment of inertia I therefore cannot be calculated in terms of M and R, but is to be taken as a given variable. One end of the string is tied to the ceiling, and the cylinder is allowed to fall with its axis horizontal, as the string unrolls. Take the acceleration of gravity as g, downward, with $g > 0$.



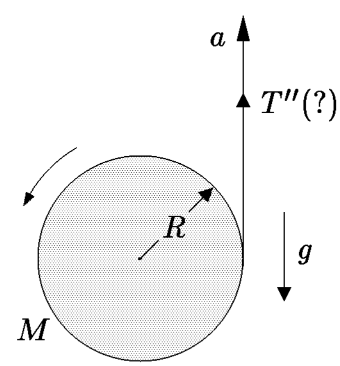
(a) (*5 points*) Find the angular velocity ω of the cylinder after it falls a distance ℓ, starting from rest with the string taut.

(b) (*5 points*) Find the tension T in the string, as the cylinder is falling.

(c) (*5 points*) Now suppose that instead of the string being tied to the ceiling, it is being held by a person who pulls the end upward with a constant speed v. What is the tension T' in the string in this case?



(d) (*5 points*) Now suppose that instead of pulling the string upward with a constant speed, the person pulls the end of the string upward with a constant acceleration a. What is the tension T'' of the string in this case?



(a) This can be solved most easily by using conservation of energy. If we define the gravitational potential energy to be zero at the bottom of the fall, then the initial energy is all potential energy,

$$E_{\text{initial}} = U(\ell) = Mg\ell \ .$$

After falling a distance ℓ the cylinder will be moving at speed v and angular velocity ω, so its energy will be

$$E_{\text{final}} = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2 \ .$$

The string is being unwound from a circle of radius R, so v will be related to ω by $v = R\omega$. So, setting the initial and final energies equal,

$$Mg\ell = \frac{1}{2}MR^2\omega^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}(MR^2 + I)\omega^2 \ ,$$

so

$$\boxed{\omega = \sqrt{\frac{2Mg\ell}{(MR^2 + I)}} \ .}$$

(b) METHOD 1: The tension can be found from the previous answer, adding a little information. The final vertical velocity v_y, after falling the distance ℓ, is given by $v_y = -R\omega = -R\sqrt{2Mg\ell/(MR^2 + I)}$. Here I am defining v_y as being positive when upward, so its value for this problem is negative. For uniform acceleration we know that $v^2 = 2a\ell$, so the cylinder is undergoing uniform acceleration with

$$a_y = -\frac{g}{1 + \frac{I}{MR^2}} \ .$$

The $\vec{\mathbf{F}} = M\vec{\mathbf{a}}$ equation for motion in the vertical (y) direction is $F_y = T - Mg = Ma_y$, so the equation above implies that

$$\boxed{T = \frac{Mg}{1 + \frac{MR^2}{I}} \ .}$$

METHOD 2: Alternatively, we can use the torque and force equations to find the tension directly. Once again, the $\vec{\mathbf{F}} = M\vec{\mathbf{a}}$ equation for vertical motion is

$$F_y = T - Mg = Ma_y \ .$$

The torque about the center of mass is given by $\boldsymbol{\tau} = TR$, and it is positive (counterclockwise). So

$$\boldsymbol{\tau} = TR = I\alpha \ .$$

The rolling constraint for these sign conventions is $v_y = -R\omega$, since a positive (counterclockwise) rotation corresponds to motion of the cylinder downward (negative v_y). Differentiation of the rolling constraint implies that $a_y = -R\alpha$, so the torque equation can be rewritten as

$$TR = -Ia_y/R.$$

The torque equation above then implies $a_y = -TR^2/I$, which can be substituted into the first equation to give

$$T - Mg = -\frac{TMR^2}{I} \ ,$$

leading immediately to

$$\boxed{T = \frac{Mg}{1 + \frac{MR^2}{I}} \ .}$$

(c) The answer is the same as part (b),

$$\boxed{T' = T = \frac{Mg}{1 + \frac{MR^2}{I}} \ .}$$

One way to see this is to work the problem in the frame of reference that is moving upwards at speed v relative to the ground, and then the problem becomes identical to the previous one. If the problem is attacked in the frame of reference of the ground, then one can use Method 2 above, and the only change is in the rolling constraint. Now if ω were zero the cylinder would be moving upward with speed v, so the rolling constraint becomes

$$v_y = -R\omega + v \ .$$

Since v is a constant, differentiation gives $a_y = -R\alpha$ as before, so the derivation goes through without any change.

(d) This time the rolling constraint can be written as

$$v_y = -R\omega + v(t) \ ,$$

where $v(t)$ is the instantaneous velocity of the hand that is pulling the string upwards. Now when we differentiate with respect to time,

$$a_y = -R\alpha + a \ ,$$

where $\mathrm{d}v/\mathrm{d}t = a$, where a is the upward acceleration of the person's hand, as specified in the problem. We still have

$$T''R = I\alpha \ ,$$

so

$$T''R = -I(a_y - a)/R \ .$$

Then $a_y = a - T''R^2/I$, which can be inserted into the $\vec{\mathbf{F}} = M\vec{\mathbf{a}}$ equation to give

$$T'' - Mg = Ma_y = M\left(a - \frac{T''R^2}{I}\right) \ ,$$

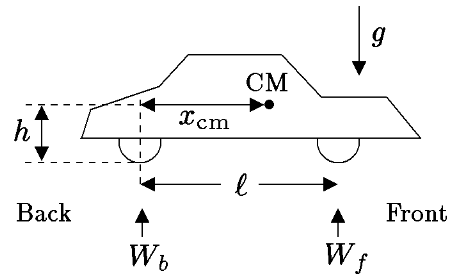
so

$$\boxed{T'' = \frac{M(g+a)}{1 + \frac{MR^2}{I}} \ .}$$

For those who are familiar with working in non-inertial frames, the above answer is no surprise. If we solved the problem in the frame of reference which is accelerating upward with the hand, then the problem reduces to the one solved in part (b), except that we must include a fictitious force associated with the non-inertial nature of the frame. Such fictitious forces are described in the Busza-Cartwright-Guth *Study Guide* (6th Edition) on pp. 206–208. The fictitious force acts on each particle of mass m_i with a force $m_i a$ downward, so it is equivalent to an extra contribution to gravity. The total of the gravitational plus fictitious force on any particle is $m_i(g + a)$, downward, so the only affect of the fictitious force is to replace g by $g + a$.

Problem 4: Distribution of weight on the wheels of a car (*25 points*)

A car of total mass M stands at rest on level ground. The car is symmetric left and right, so the wheels on the left side can be considered equivalent to the wheels on the right side. The distance between the front and back wheels is ℓ, and the center of mass of the car is located a distance h from the ground, and a horizontal distance x_{cm} from a point directly above the contact point of the back wheel, as shown in the diagram. Gravity acts downward, with acceleration g, with $g > 0$. (Note that the diagram shown is correct, different from the diagram on the printed quizzes.)



(a) (*10 points*) The ground exerts a total upward normal force of magnitude W_f on the two front wheels, evenly distributed between the left and the right. The ground exerts a total upward normal force of magnitude W_b on the back wheels, also evenly distributed between the left and the right. Find expressions for W_f and W_b in terms of some or all of the given variables M, g, ℓ, x_{cm}, and h.

(b) (*5 points*) Now suppose that the car is moving forward, on a level road, at a constant speed v. In this case we will let W_f' denote the magnitude of the total upward normal force on the front wheels, and W_b' denote the magnitude of the total upward normal force on the back wheels. Find expressions for W_f' and W_b' in terms of some or all of the variables v, W_f, W_b, M, g, ℓ, x_{cm}, and h. Note that the variables W_f and W_b may appear in your answer, whether or not you answered part (a).

(c) (*10 points*) Now suppose that the car starts from rest at time $t = 0$, and accelerates forward with a uniform acceleration of magnitude a, directed to the right. Assume that the road is level, and that the car remains level as it accelerates. Assume that the wheels roll without slipping, and neglect any dissipative forces, such as air friction or rolling friction. For this part use W_f'' to denote the magnitude of the total upward normal force on the front wheels, and W_b'' denote the magnitude of the total upward normal force on the back wheels. Write down two equations that can be solved to determine W_f'' and W_b'' in terms of any or all of the variables a, W_f, W_b, W_f', W_b', M, g, ℓ, x_{cm}, h, and t.

Name

(a) The total torque must vanish about any point. If we use the contact point between the back wheel and the road as the origin for calculating the torque, then

$$\tau_{\text{back}} = 0 = W_f \ell - Mg x_{\text{cm}} \ ,$$

so

$$\boxed{W_f = Mg \frac{x_{\text{cm}}}{\ell} \ .}$$

The total force in the vertical direction must vanish, so

$$W_b + W_f = Mg \ ,$$

and using the previous expression for W_f gives

$$\boxed{W_b = Mg \frac{\ell - x_{\text{cm}}}{\ell} \ .}$$

(b) If we work in the rest frame of the car, the problem becomes identical to the problem already solved, except that the wheels are spinning. But no step in the calculation in part (a) would be affected by the spinning of the wheels, so the same calculation and the same answer applies:

$$\boxed{W_f' = W_f = Mg \frac{x_{\text{cm}}}{\ell} \ , \qquad W_b' = W_b = Mg \frac{\ell - x_{\text{cm}}}{\ell} \ .}$$

(c) The new addition to the force diagram is a horizontal force to the right, causing the car to accelerate. This force is due to (static) friction, and is divided between the front and back wheels in a way that we do not have enough information to determine. Fortunately all we will need is the total force of friction, which must be equal to Ma, since it is the only horizontal force:

$$F_{\text{tot}} = Ma \ .$$

Working in the non-inertial frame that is accelerating with the car, we can set the torque about the center of mass to zero. (Here we are neglecting the mass, and hence the angular momentum, of the wheels; if we did not neglect the angular momentum of the wheels, the torque would not vanish, but would equal the rate of increase of the angular momentum of the wheels as the car accelerates.) Note that in this non-inertial frame the torque about the center of mass must vanish, although the torque about another origin may not vanish. The torque about the center of mass due to friction is independent of which wheel it acts on, because in either case the perpendicular distance is h. So

$$\boldsymbol{\tau}_{\rm cm} = F_{\rm tot}h + W_f''(\ell - x_{\rm cm}) - W_b'' x_{\rm cm} = 0 \ .$$

Using $F_{\rm tot} = Ma$, one of our equations can be written

$$\boxed{Mah + W_f''(\ell - x_{\rm cm}) - W_b'' x_{\rm cm} = 0 \ .}$$

The other needed equation is the one that says that the total vertical force is zero:

$$\boxed{W_b'' + W_f'' = Mg \ .}$$

You were not asked to solve these equations, but if you did you should have found that

$$W_f'' = M\left[g\frac{x_{\rm cm}}{\ell} - a\frac{h}{\ell}\right]$$

$$W_b'' = M\left[g\frac{\ell - x_{\rm cm}}{\ell} + a\frac{h}{\ell}\right] \ .$$